Equivalence of rational expressions: Articulating syntactic and numeric perspectives

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Our study concerns the conceptual mathematical knowledge that emerges during the resolution of tasks on the equivalence of polynomial and rational algebraic expressions, by using CAS and paper-and-pencil techniques. Our findings highlight the mathematical knowledge (technological discourse) constructed in the process of confronting, differentiating, and articulating the several mathematical praxeologies (pertaining to the numeric perspective or the syntactic perspective in algebraic equivalence) that arose in the solution of the designed equivalence tasks.

INTRODUCTION AND BRIEF LITERATURE REVIEW

While a substantial amount of research has been carried out with respect to the equivalence of algebraic expressions, very little of it has dealt explicitly with either the comparison of polynomial and rational expressions or the bridging of syntactic and numeric perspectives. Some researchers have highlighted the importance of being able to work flexibly with algebraic expressions in various forms and of recognizing equivalent expressions; other studies have focused on the difficulties that students encounter with understanding algebraic equivalence (see for example Nicaud et al.; 2004, Sackur et al., 1997; or Ball et al., 2003).

With respect to the different perspectives, Cuoco (2002) distinguishes between polynomial functions and polynomial forms as follows. Polynomial functions involve thinking about the letter in a polynomial as a variable and about the polynomial as an input-output machine that can yield a table or a graph, and has all the attributes of real-valued functions of a real variable. In contrast, polynomial forms are viewed as formal expressions with the letter considered as an indeterminate and which involve operations such as factoring, adding, multiplying, and so on. However, according to Cuoco "the distinctions between polynomial forms and functions tend to be ignored in school mathematics" (2002, p. 297). Cerulli and Mariotti (2002) make similar distinctions in what they refer to as functional and axiomatic definitions of equivalence. They point out that since, for polynomials in *n* variables, the functional and the axiomatic definitions are equivalent, they do not go into the particularities of the equivalence of these definitions with their learners.

Artigue (2002) has drawn on students' work involving the passage from one given form of algebraic expression to another to illustrate the difficulties that students experience with equivalence problems, difficulties that come to the fore when they use Computer Algebra Systems (CAS) (e.g., Artigue, 2002; Guin & Trouche, 1999; Lagrange, 2000). She asserts that the CAS forces students to confront issues of equivalence and simplification in ways that are not so easily achieved in more traditional, paper-and-pencil, treatments. Following Artigue´s considerations, Kieran and Drijvers (2006) and Kieran and Saldanha (2008) have investigated the learning of the technical and theoretical aspects of various topics in high school algebra within CAS environments, including the topic of equivalence. They have found that "students linked the notion of restrictions [within the rational expressions] to the numerical view on equivalence" (Kieran & Drijvers, 2006, pp. 227-231). While the distinctions between form and function are particularly important when students use CAS technology to solve equivalence tasks because the polynomial-form perspective underlies CAS – even if it also deals with polynomials as functions – many crucial questions regarding the articulation of the syntactic and numeric perspectives could not be answered by the classroom-based study reported by Kieran and Drijvers. The present article deals exactly with such articulation, within the methodological frame of individual-based student interviews.

THEORETICAL FRAMEWORK

As in previous studies (e.g., Kieran & Drijvers, 2006), we adopt the Anthropological Theory of Didactics (ATD) developed by Chevallard (1999). As per this theory, the objects of mathematical knowledge emerge from systems of practices whose norms and manners of use define the ways of knowing and understanding these objects and their way of living in specific institutions. These systems are named *mathematical praxeologies* in Chevallard's theory and they are described by: the *types of tasks* in which the objects of knowledge are immersed; the *techniques* or ways of solution of these tasks; the discourse that explains and justifies the techniques, named *technology*; and the *theory* that provides the structural basis of the technological discourse and that can be seen as the "technology of the technology" (Artigue, 2002, p. 248).

In this study we focus on the co-emergence of techniques and conceptual mathematical knowledge when solving equivalence tasks. We share the point of view of Lagrange (2000, p. 16), who affirms that techniques develop mathematical meaning in a double relationship with, on the one hand, the tasks they permit the user to solve, and, on the other, the theorizations they promote. In accordance with Lagrange, Artigue (2002) states that techniques are usually recognized by their *pragmatic value* for task solution, in other words, in terms of their efficacy, cost, and validity domain. However, techniques also have an essential *epistemic value* as they contribute to the understanding of the objects they handle.

METHODOLOGICAL CONSIDERATIONS

The study presented herein is part of a larger program of research whose central objective was to shed light on the co-emergence of algebraic technique and theory within an environment involving novel tasks and a combination of Computer Algebra System (CAS) and paper-and-pencil (PP) media (see Kieran & Drijvers, 2006). With this objective and our theoretical perspective in mind, for this study our research team developed a series of task sequences on equivalence, within an environment involving both paper-and-pencil and CAS (the TI-92 Plus handheld calculator), that would encourage both technical and theoretical development (Chevallard, 1999, Artigue, 2002, and Lagrange, 2000) in 10th grade algebra students.

For the design of tasks, we consider algebraic expressions that are polynomials and polynomial quotients in one indeterminate with coefficients on the set of real numbers R. If we see algebraic expressions as polynomials or polynomial quotients, we will say that two expressions are *equivalent from the syntactic perspective* when they have a *common algebraic rewriting* by applying the properties of the algebraic properties of polynomial and polynomial quotients operations. If we see algebraic expressions as polynomial functions, we will say that two algebraic expressions f and g are *equivalent from the numeric perspective* when have the same values for all x in the *common domain.* The two perspectives of equivalence emphasize different aspects. For the numeric perspective, it is essential to include the study of the characteristics of domains and images for the corresponding algebraic expressions being compared. For the syntactic perspective, the rewriting of expressions plays a central role. On the basis of the algebraic properties of the operations of the ring of polynomials and the field of quotients of polynomials, the rewriting of expressions allows for verifying equivalence and for obtaining equivalent expressions.

The research program includes data collection of classroom lessons and videotaped interviews with students. The analysis presented in this article is based on the first of three interviews carried out with one Grade 10 algebra student, Andrew. At the moment of the interview, Andrew had already learned the four basic operations with polynomials, and a few techniques for factoring certain binomials and trinomials, and for solving linear and quadratic equations. He had not yet had any formal school experience with the notion of equivalence, nor with rational algebraic expressions, but he had studied the introductory topics of domain and range, dependence, relation versus function, and modes of representation. Andrew had also been introduced to CAS technology.

In Table 1 we present the expressions used in the interview of this study.

Expression A:	$(x^2+x-20)(3x^2+2x-1)$
Expression B:	$(3x-1)(x^2-x-2)(x+5)$
Expression C:	$\dfrac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$

Table 1: Designed expressions for this study

ANALYSIS

The first part of the interview consisted of evaluating the given algebraic expressions and producing a conjecture regarding the numerical relationships among the obtained values. Andrew evaluated the expressions for x = 1/3, -5, 6, 7, and conjectured: *The results for Expressions B and C would continue to be equal to each other*. In our theoretical terms, we can say that he conjectured the numeric equivalence of Expressions B and C.

When asked to justify his conjecture "for all numbers", he resorted to syntactic techniques: he *expanded* (with paper and pencil and with CAS) and *factored* the expressions so as to obtain "forms" that he could compare. So, if the rewriting of the expressions (obtained by expansion or by factorization) is the same, then they take on the same values for any x (in our theoretical terms: if they are syntactically equivalent, then they are numerically equivalent). Table 2 shows the expanded and factored forms obtained by Andrew just as he wrote them.

Original expression	Expanded form	Factored form
$(x^2+x-20)(3x^2+2x-1)$	$3x^4+5x^3-59x^2-41x+20$	$((x+5)(x-4))((x+1)(3x-1))$
$(3x-1)(x^2-x-2)(x+5)$	$3x^4+11x^3-25x^2-23x+10$	$(3x-1)((x-2)(x+1)(x+5))$

$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	$3x^4+11x^3-25x^2-23x+10$	$(x-2)(x+5)(3x-1)(x+1)$

Table 2: Andrew's expanded and factored forms

It is important to say that just a few students of the whole research program used syntactic techniques for justifying their conjectures for the numeric equivalence of expressions. Most of them "felt unsure about algebra providing certainty about numerical values, even if their algebraic skills were good" (Kieran & Drijvers, 2006, p. 222).

The interview continued, with Andrew being asked to find the *domain of definition* for Expression C:

Interviewer: Is there any value of x that would not be permissible as a replacement value for x in Expressions B and C?

Andrew concluded that Expression C was not defined for x = -2. Then he was asked about the consequences of this non-definition with respect to the equivalence of Expressions B and C. Andrew answered as follows:

Andrew: Well just, it [the factorized form for Expression C, once the common factors are cancelled: $(x-2)(x+5)(3x-1)(x+1)$] is, like, another form of the expression, which is, I guess, once the expression is factored out, and then they're still equal to each other. So, that makes sense [small laugh]. Basically, Expressions B and C, when they are fully factored, at least to my capability, they're equal to each other, still. So, it just supports my conjecture that, with any x value, excluding negative 2, they would be equal to each other.

In this first contact with the differences between the two perspectives on algebraic equivalence, Andrew incorporated the restriction as an exception to the equality of the values of the expressions: although these expressions are syntactically equivalent (they can be rewritten as the same expression), there is a value of x for which they are not equal.

Straightaway, Andrew spontaneously proceeded to evaluate at x = -2 the expressions syntactically equivalent to Expression C. Table 3 shows the results that he obtained.

Expression		Value at x = -2
Expression C:	$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	Undefined
Expression B:	$(3x-1)(x^2-x-2)(x+5)$	-84

Expanded form of expressions B and C, obtained by using CAS:	$3x^4 + 11x^3 - 25x^2 - 23x + 10$	-84
Factored form of expressions B and C, obtained by using PP:	$(x-2)(x+5)(3x-1)(x+1)$	(-84) Andrew considered this evaluation, but did not explicitly carry it out during the interview.

Table 3: CAS evaluation of the expressions equivalent to Expression C (at x = -2)

This evaluation performed by Andrew was not considered in the original design of the interview. It was a spontaneous confrontation of the different facts that support the numeric equivalence of Expressions B and C versus the fact that these expressions take on different values at the restriction. At the same time, this evaluation was an exploration of the possibility that the restriction is inherited by the syntactically equivalent forms and, thus, if the equality of the values could be kept for x = -2. In Andrew's words "it is very possible that if I work this [expression B] out, with minus two incorporated into it, that that would equal zero too, which is based on the fact that they [expressions B and C] have always been equivalent".

The interview continued with some tasks that required Andrew to confront the differences between the two perspectives on algebraic equivalence. For the solution of these tasks we introduced two CAS techniques: the *equivalence test* and the *numeric equality test*. These techniques allowed him to determine the equivalence of two expressions. In the case of expressions that are syntactically equivalent but which have a restriction (like Expressions B and C), the results obtained by applying the *equivalence* and the *numeric equality tests* are "contradictory". For the restriction (x = -2), when using the numeric equality test, the result that is obtained is FALSE; whereas when using the equivalence test, the result is TRUE.

The contradictory results obtained for Expressions B and C confirmed what Andrew had already obtained by applying the factoring and expanding techniques, with and without CAS. As Andrew said, the CAS was not considering the domain restriction of Expression C. Andrew explained the results as follows:

Andrew: Yeah, at first [he is referring to the result of the test of equivalence] it's saying that any value of x would be true, that any value of x can be substituted and they would be equivalent. But, like this just proves, that when minus two is incorporated that it's not true [he is referring to the result of the test of numeric equality for x = -2], in

> this form at least [original expression C]. Because once it's expanded, it [the calculator] saw they were still equivalent, and it didn't. I guess in different forms it's not true, but in this particular form [original expression C] it is.

Throughout the interview Andrew managed to construct an articulation between the differences and contradictions of the results obtained through the syntactic and numeric techniques: the numeric equivalence of two algebraic expressions could be established in a general manner (for every value of x, not just for a finite set of values) by means of syntactic techniques, both using paper and pencil and CAS. For example, through *expanding* and *factoring techniques*, Expressions B and C could be rewritten as the same expression. However, numeric equivalence requires considering the restrictions. At the domain restriction for Expression C, these expressions do not have the same value. Table 4 presents a theoretical analysis of the techniques and the technological discourses articulated by Andrew.

	Numeric perspective	Syntactic perspective
Type of task	Establishing the equivalence of rational expressions (with remainder equal to zero) and polynomial expressions.	
Technique	Establish the common domain of the expressions, i.e., determine the restrictions. Compare the expressions over the common domain. Evaluate the expressions for several values of the common domain.	Rewrite the expressions in a common algebraic form, in a factorized or expanded form. Compare the rewritten expressions (term by term or factor by factor).
Technology	If the expressions take on the same values for a set of values, they can take on the same values for any value of the common domain (their *numeric equivalence* is conjectured).	If the expressions can be rewritten as the same expression (in a factorized or expanded form), they are *syntactically equivalent*.

Table 4: Numeric and syntactic perspectives for determining the equivalence of rational expressions (with remainder equal to zero) and polynomial expressions

DISCUSSION

Andrew resorted to techniques belonging to different mathematical praxeologies and to different perspectives on algebraic equivalence. Syntactic techniques and technologies used by Andrew belong to two welldistinguishable praxeologies: the *Expansion praxeology* and the *Factorization praxeology*. The third praxeology involved in Andrew's solutions is determined by the *evaluation technique*, which corresponds to the numerical perspective of equivalence. We call it the *Evaluation praxeology*.

For Andrew the applied techniques do not belong exclusively to a unique perspective. In fact, for him, differentiated perspectives on equivalence do not exist. They are just mathematical knowledge and resources for solving the same type of tasks. Andrew articulated the differences and contradictions by establishing distinctions between the numeric and the syntactic techniques, as well as between their corresponding conceptual elements (technologies): numeric equivalence of algebraic expressions can be "proved" by rewriting them and showing that they are the same (syntactic equivalence). However, at the restrictions (numerical values of x where the rational expressions are not defined), numeric equivalence does not correspond to the syntactic equivalence of the expressions; numerical evaluation is necessary in this case.

The use of CAS was central to the exploration of the values of the expressions; its *pragmatic value* (Lagrange, 2000) was given relevance through this use. CAS was central also for making explicit the differences and their conciliation; it thereby acquired an *epistemic value* through the constitution of the technological discourses for explaining and conciliating the "contradictions" that emerged during the solution of the tasks.

Clearly, the articulation of the two perspectives on algebraic equivalence was not completed by the theoretical and technological explanations generated through the solution of the set of tasks presented in this study. For instance, certain issues related to restrictions, zeros, and factors of polynomials were not explained nor even dealt with. However, the design of tasks that involved confronting and differentiating perspectives of equivalence allowed for the creation of a mathematical arena where mathematical knowledge about algebraic expressions and their equivalence (the mathematical praxeologies involved) is articulated and, in this way, constructed.

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