LEARNING ABOUT EQUIVALENCE, EQUALITY, AND EQUATION IN A CAS ENVIRONMENT: THE INTERACTION OF MACHINE TECHNIQUES, PAPER-AND-PENCIL TECHNIQUES, AND THEORIZING[1]

Carolyn Kieran, Université du Québec à Montréal, Montréal, Canada
Paul Drijvers, Freudenthal Institute, Utrecht, the Netherlands

with the collaboration of
André Boileau, Fernando Hitt, Denis Tanguay,
Luis Saldanha, and José Guzman

ABSTRACT. The study presented in this report is part of a larger project on the intertwining co-emergence of technique and theory within a CAS-based task environment for learning algebra, which also includes paper-and-pencil activity. The theoretical framework consists of the instrumental approach to learning mathematics with technology, in particular Artigue and colleagues' adaptation of Chevallard's anthropological theory. The theme presented herein is that of equivalence, equality, and equation. Two 10^{th} grade classes were taught by the same mathematics teacher during two successive years, using project materials designed by the research team. Classroom observations, student interviews, student activity sheets, and posttest responses were the main data sources used in the analysis. Findings attest to the intertwining of technique and theory in algebra learning in a CAS environment. In addition, the data analysis revealed that probably the most productive learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations, based on their previous theoretical knowledge. Even if such conflicts in applying CAS techniques may seem to be hindrances to students' progress, in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these conflicts to foster learning is their appropriate management in the classroom by the teacher.

CONFERENCE THEME. Learning mathematics with digital technologies
APPROACHES. Theoretical frameworks, Contribution to learning mathematics, Role of the teacher

1. Introduction

School algebra has traditionally been an area where technique and theory collide, with technique usually claiming victory. While a parallel with the terms *skills/procedures* and *concepts* may suggest itself, both *technique* and *theory* are broader in meaning than *procedures* and *concepts* (Artigue, 2002). The notion that school algebra can be an arena for the interaction of both theory and technique has not taken hold until recently. The advent of computer algebra systems (CAS) technology in schools, along with the development of theoretical frameworks for interpreting how such technology becomes an instrument of mathematical thought, have both been contributing factors.

The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). An essential starting point in the instrumental approach is the distinction between an artifact and an instrument. Whereas the artifact is the object that is used as a tool, the instrument involves also the techniques and schemes that the user develops while using it, and that guide both the way the tool is used and the development

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of the user's thinking. The process of an artifact becoming an instrument in the hands of a user -- in our case the student -- is called instrumental genesis. The instrumental approach was recognized by French mathematics education researchers (e.g., Artigue, 1997; Lagrange, 2000; Trouche, 2000; Guin & Trouche, 2002) as a potentially powerful framework in the context of using CAS in mathematics education.

As Monaghan (2005) pointed out, one can distinguish two directions within the instrumental approach. In line with the cognitive ergonomic framework, some researchers (e.g., Trouche, 2000; Drijvers, 2003) see the development of schemes as the heart of instrumental genesis. Although these mental schemes develop in social interaction, they are essentially individual. Within the schemes, conceptual and technical elements are intertwined. More in line with the anthropological approach, other researchers focus on techniques that students develop while using technological tools and in social interaction. The advantage of this focus is that instrumented techniques are visible and can be observed more easily than mental schemes. Still, it is acknowledged that techniques encompass theoretical notions. The focus on techniques is dominant in the work of Artigue (1997, 2002) and Lagrange (2000) in particular.

2. The study

2.1 Theoretical framework: Task-Technique-Theory

Chevallard's anthropological theory of didactics, which incorporates an institutional dimension into the mathematical meaning that students construct, describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. (By *technology,* Chevallard means the discourse that is used to explain and justify techniques; he is not referring to the use of computers or other technological tools.) In their adaptation of Chevallard's anthropological theory, Artigue and her colleagues have collapsed *technology* and *theory* into the one term, *theory*, thereby giving the theoretical component a wider interpretation than is usual in the anthropological approach. Furthermore, Artigue notes that *technique* also has to be given a wider meaning than is usual in educational discourse.

Lagrange (2003, p. 271) has elaborated this latter idea further: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is precisely this epistemic role played by techniques that is a focus of our study, that is, the notion that students' mathematical theorizing develops as their techniques evolve. It is noted, as well, that our perspective on the co-emergence of theory and technique is situated within the context of technological tool use, where the nature of the task plays an equally fundamental role. Thus, the triad Task-Technique-Theory (TTT) served as the framework not only for constructing the tasks of this study, but also for gathering data during the teaching sequences and for analyzing the resulting data.

2.2 Aim of the study

The research study, of which this report is a part, is an ongoing one. It has as a central objective the shedding of further light on the co-emergence of technique and theory within the CAS-based algebraic activity of secondary school students. Because of severe space restrictions, this report will highlight the design and findings from one task set, that of equivalence, equality, and equation.

2.3 Participants

The research involves six intact classes of 10th graders (15-year-olds) in Canada and Mexico, as well as a class of older students in Oregon. Five of the 10th grade classes were observed during the 2004-05 school year; the sixth class, the following year. Two of these 10th grade classes are featured in this report – one from the 2004 study and the other from the 2005 study. Both classes were taught by the same teacher, with five years of experience. He is a teacher who, along with encouraging his pupils to talk about their mathematics in class, believes that it is useful for them to struggle a little with mathematical tasks. He elicits students' thinking, rather than quickly giving them answers. The students in this report had already learned a few basic techniques for solving linear and quadratic equations during their 9th grade mathematics course and had used graphing calculators on a regular basis; however, they had not had any experience with the notion of equivalence, one of the theoretical ideas developed in the project materials, nor with symbol-manipulating calculators (i.e., the TI-92 Plus CAS machines used in this project).

2.4 Data sources

All project classes were observed and videotaped (12-15 class periods for each of the seven project classes). Students were interviewed, alone or in pairs, at several instances -- before, during, and after class. A posttest involving CAS was administered after the task set on equivalence had been completed. All students were pretested. Thus, data sources for the segment of the study presented in this report include the videotapes of all the classroom lessons, videotaped interviews with students, a videotaped interview with the teacher, the activity sheets of all students (these contained their paper-and-pencil responses, a record of CAS displays, and their interpretations of these displays), written pretest and posttest responses, and researcher field notes.

2.5 Task design

The research team created several sets of tasks that aimed at supporting the co-emergence of technique and theory. Because paper-and-pencil techniques were a fundamental part of the algebra program of studies of the schools where the research was carried out, and because we believe in the importance of combining the two media, they too were included in the teaching sequences. Task sets were planned to take from one to five periods. For the task set described in this paper, one class took three periods, and the other, four. Each task set involved student work, either with CAS or paper-and-pencil or both, reflection questions, and classroom discussion of the main issues raised by the tasks. In designing the tasks, we took seriously both the students' background knowledge and the fact that these tasks were to fit into an existing curriculum; but we also did our best to ensure that they would unfold in a particular classroom culture that reflected a certain priority given to discussion of serious mathematical issues. Tasks that asked students to write about how they were interpreting their work and the related CAS displays aimed to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom, and clarifying ideas and distinctions, in ways that simply “doing algebra” may not require.

2.6 The task set on equivalence, equality, and equation

The underlying motive of this task set is the subtle relationship between arithmetic and algebra: on the one hand, the numerical world is the most important motive and model for the world of algebra, on the other hand algebra goes beyond the numerical world, which is in fact part of its power. This two-sided relationship is reflected in the notion of equivalence of algebraic expressions (see Kieran & Saldanha, 2005; Saldanha & Kieran, 2005). Equivalence of two expressions relates to the numeric as it reflects the idea of ‘equal output values for each of an infinite set of input values.’ However, equivalence of two expressions also relates to the algebraic in that the expressions can be rewritten in a common algebraic form.

At the start of the teaching sequence, numerical evaluation of expressions by using CAS and comparison of their resultant values are used as the entry points for discussions on equivalence. One of the core tasks here aims at students’ noticing that some pairs of expressions seem to *always* end up with equal results, and thus evokes the notion of equivalence based on numerical equality. The algebraic expressions included in the task were fairly complex so as not to permit the evaluation of equivalence by purely visual means. The task is followed by a reflection question on what would happen if the table of values were extended to include other values of *x*. The task and the CAS substitution technique lead to the following definition of equivalence of expressions, with deliberate inclusion of the idea of a set of admissible values:

> We specify a set of admissible numbers for *x* (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces *x*, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.

The impossibility of testing all possible numerical substitutions to determine equivalence motivates the use of algebraic manipulation and the explicit search for common forms of expressions in the second part of the task set. Different CAS techniques can be used: Factor, Expand, Automatic Simplification. An additional technique is the “Test of Equality,” which involves entering an equation, followed by the Enter button. In this test, the CAS checks both sides of the equation for equivalence, by means of automatic simplification and other ‘black-box’ means.

The CAS will come up with ‘true’ in cases of equivalence (see Figure 1). Restrictions are ignored, just as with Automatic Simplification. The Test of Equality technique has probably the most ‘black-box’ character, and the output it produces is the most difficult to interpret, especially for cases of non-equivalence. This CAS technique was deliberately introduced in the design of the tasks so as to provoke student questioning of its output.

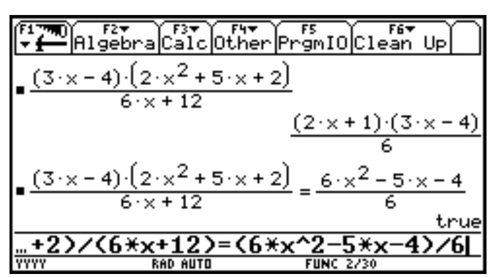

Figure 1. Illustration of the “Test of Equality” and the way that the TI-92 neglects restrictions

In the next part of the task set, the relation between two expressions being equivalent or not, and the corresponding equation having many, some, or no solutions is explored in both CAS (Solve now introduced) and paper-and-pencil tasks. For example, students are asked to generate a pair of equivalent expressions, and, in a similar follow-up task, two non-equivalent expressions. The ensuing reflection question concerns the relation between the nature of an equation's solution(s) and the equivalence or non-equivalence of the expressions that form the equation.

3. Analysis of student activity

Three theoretical elements were found to be intertwined and related to the techniques and tasks of this segment of the data analysis, and thus serve to organize the discussion of results: i) The numeric and algebraic views on equivalence; ii) The issue of restrictions; and iii) Coordination of the notions of equivalence and solution of an equation.

3.1 The numeric and algebraic views on equivalence

Students seemed to have an intuitive idea of equivalence as having always the same numerical value, even if this was sometimes expressed in an informal way. This notion was clearly supported by the CAS substitution technique, which makes numerical substitutions easy to carry out. The repeated substitution with the CAS presented the students with the phenomenon of equal values, which invited algebraic generalization. Still, the relation between the algebraic and the numeric was somewhat vague. The Factor, Expand and Automatic Simplification techniques are on a more algebraic level, but seem to foster the notion of common form as 'simple' form. That is, some students tended to interpret the simplified forms produced by these commands as ordinary or basic or common, and thought that this was what we meant when we asked them to express a pair of expressions in a common form. The Test of Equality technique is probably the most interesting one from the conceptual point of view, as it seems to act at the borderline between the numeric and the algebraic. This technique provides 'true' in cases of equivalence, but just returns the (sometimes transformed) equation in other cases. The latter was difficult to understand for many students, as they would have expected something like 'false'; whereas returning an equation -- so two expressions with an equal sign in between -- unjustly suggested equivalence to them.

The verbatim in Figure 2 illustrates that Suzanne found the output of this CAS test surprising. She tried to interpret it by means of her existing theoretical thinking, but was unable to do so satisfactorily. In spite of the confusion that she expresses in the last line, we appreciate that she takes it as an incentive to rethink about her conceptions. In fact, that is what the tasks and techniques, if dealt with properly by the teacher, can provoke: a rethinking of the theoretical knowledge. The classroom discussion that followed did help to move Suzanne's thinking forward.

Suzanne	Uhm, I entered the problem $(x^2+x-20)(3x^2+2x-1)=(3x-1)(x^2-x-2)(x+5)$ and it gave me pretty much the same problem back, but rearranged, it's the same answer. When you think that the other one said "true," it is kind of puzzling. ... The answer that it gave me. I figure that that's this statement, like the first expression equals the second expression is true. … When I see an equal sign, I figure they are equivalent, the same.

[…]	
Interviewer	How would you now interpret such a display when you enter in two expressions like that?
Suzanne	Uhm, that it can be right sometimes, but isn't always right. With specific numbers, it is correct.
Interviewer	So, when you mean correct?
Suzanne	That you would get the same number in the end on both sides. But only sometimes.
Interviewer	Only for some numbers.
Suzanne	Yah.
Interviewer	So how do you feel about that?
Suzanne	I'm still confused. With the "true"s and the "="s, to me it all has sort of the same meaning. I guess I just have to change my way of thinking.

Figure 2. Confusion about the CAS returning the equation for the case of non-equivalence

A second issue that is related to the role of techniques in the evolution of the students' thinking about equivalence concerned the *coordination of different techniques as a means to check consistency*. In several cases, students used different techniques, both paper-and-pencil and CAS, to verify the consistency of their theorizing. Surprising CAS results in some cases gave rise to conflicts that invited reasoning. For example, at first Andrew was puzzled when the CAS simplified $(2-x)(1-2x)$ as $(x-2)(2x-1)$. After some thinking about this, he found a justification that involved the technique of substitution:

> "I think since it's switching them both that it works out. Let's just say x was represented by 6, -4 times -11, which is 44. And the other one it's 6 – 2, which is 4 times 11, which is also 44. It's just two negatives, since it's switching both of them it's OK."

By the way, this verbatim shows the student's returning to the numerical to check algebraic relations -- not a bad habit of course. Still, when so asked, Andrew indicated that he had several means to check algebraically the equivalence of $(2-x)(1-2x)$ and $(x-2)(2x-1)$, such as entering the corresponding equation or expanding them both. The other students also used these CAS techniques to check their consistency with by-hand results.

3.2 The issue of restrictions

The question of how to deal with restrictions, both with CAS and paper-and-pencil techniques, played a role in the algebraic view on equivalence. It also disclosed limitations in certain students' thinking about *zero in fractions* and *dividing by zero*. This was the case for Andrew when dealing with the equivalence of two particular expressions: $(3x-1)(x^2-x-2)(x+5)$ and $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{x+2}$. At first, he had difficulties with identifying the restriction of $x=-2$. The question to consider the denominator revealed a misconception: "If x were -2 then the denominator would be -2 plus 2, which is zero and anything over zero is equal to zero. One over zero equals to zero," he said. After some intervention, Andrew concluded that the result of division by zero is undefined, but it remained unclear about whether another zero might appear somewhere. To check this out, he substituted $x=-2$ into the first expression and got –84 as a result, clearly not zero. So, he concluded: "Basically, it will work with everything except the -2." Then he substituted $x=-2$ into the expanded form of the first expression, which of course gave -84 once more. This seemed to be a check for consistency, although he was not completely sure about what to expect. Then he wondered about the value of the second expression when $x = -2$ would be substituted. He expected -84, but the calculator displayed 'undefined.' He explained this as follows:

> "That's what I figured out that it should be, undefined, but I didn't think the calculator would show it. Just based on all the other results, just based on the fact that this came out to -84, and this came out to –84. … Well like it substitutes it and then it fills everything in and anything divided by zero is undefined, no matter what the equation is on top, it's still divided by –2 plus 2, so it's undefined."

Andrew's fuzzy thinking about rational expressions, division by zero, and substitution of inadmissible values into the numerator interacted with his expectation of a certain CAS output. He was, however, eventually able to provide a technical interpretation that made sense to him about how the CAS produced 'undef' – as disclosed by his last comment.

3.3 Coordination of the notions of equivalence and solution of an equation

Results concerning the coordination of solutions and equivalent/non-equivalent expressions were mixed. For example, after students had generated a pair of equivalent expressions and were asked what they could say about the solutions of the equation formed from this pair -- but without actually solving the equation -- half the students' responses included *true*, *equal*, *equivalent*, and did not refer explicitly to solutions of the equation. Furthermore, the word *solution* itself seemed problematic.

While some students were able to relate the set of equation solutions to the equivalence of the two expressions involved, this remained unclear for others. The Solve technique in itself was not a problem for the students; but its coordination with the other techniques on equivalence required a change of perspective, which was not easy. Evidence suggests that a language issue is involved here as well: students use the word 'solve' for any operation leading to a result, the result being called the 'solution.'

4. Concluding remarks on equivalence, equality, and equation

If we consider our findings on the task set of equivalence, equality, and equation in retrospect, two main issues come to the fore: the relation between students' theoretical thinking and the techniques they use for solving the proposed tasks, and the specific role of the confrontation of CAS output with students' expectations. To elaborate on the first point, our findings suggest that the relation between Theory and Technique, as it is established while working on appropriate Tasks, can hardly be underestimated. On the one hand, the development of students' theoretical thinking was guided by the techniques that the tasks invited; on the other hand, students' conceptions influenced the development of these techniques. More specifically, students' numerical view on equivalence of expressions was found to be related to three techniques: the numerical substitution technique, and, to a lesser extent, the Test of Equality and the Solve technique. The fact that students seemed to give priority to a numerical view on equivalence was tied to their use of the numerical substitution technique to check equality. In the emergence of an algebraic view on equivalence, the CAS techniques Factor, Expand, Automatic Simplification, and Test of Equality played important roles, even to the extent that the factored and expanded forms were considered as common forms. Finally, with regard to relating the numeric and the algebraic views on equivalence, students had difficulties with the coordination of the Solve technique and the techniques on equivalence; nevertheless, classroom discussion of these techniques turned out to be quite productive.

To elaborate on the second point, the data analysis revealed that probably the most productive learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations, based on their previous theoretical knowledge. The students' seeking for consistency evoked theoretical thinking and further experimentation. Also, the fact that the CAS Automatic Simplification technique and the Test of Equality both neglect restrictions led to an increasing awareness of the importance of these 'exceptions.' Finally, the CAS just returning an equation in cases of non-equivalence struck the students, and gave rise to interesting discussions on the interpretation of the output, as did the interpretation of 'true' and 'false' in cases of numeric or algebraic application of the Test of Equality. Even if such complications in applying CAS techniques may seem to be hindrances (see Drijvers, 2002) to students' progress, in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these complications to foster learning is their appropriate management in the classroom by the teacher.

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