Conceptualizing the Learning of Algebraic Technique: Role of Tasks and Technology

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This article is divided into four parts. The first part presents some introductory remarks on the use of Computer Algebra System (CAS) technology in relation to the long-standing dichotomy in algebra between procedures and concepts. The second part explores the technical-conceptual interface in algebraic activity and discusses what is meant by conceptual (theoretical) understanding of algebraic technique – in other words, what it means to render conceptual the technical aspects of algebra. Examples to be touched upon include seeing through symbols, becoming aware of underlying forms, and conceptualizing the equivalence of the factored and expanded forms of algebraic expressions. The ways in which students learned to draw such conceptual aspects from their work with algebraic techniques in technology environments is the focus of the third part of the article. Research studies that have been carried out by my research group[1] with a range of high school algebra students have found evidence for the kinds of theoretical thinking that can be fostered by specific types of technique-oriented tasks within CAS environments. The fourth part of the article then shifts to the perspective of teaching practice and discusses some of the issues that, according to this research, are to be taken into account by teachers when planning for the orchestration of such task-technique-theory activity in technological environments.

Keywords: tasks, technology, technique, theory, algebra at secondary school level, conceptual learning of algebraic technique

1. Introduction

1.1 What is Computer Algebra System (CAS) technology?

A Computer Algebra System (CAS) is a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form (Wikipedia, Sept. 5, 2007). In 1987, Hewlett-Packard introduced the first hand-held CAS calculator with the HP-28 series, and it became possible, for the first time with a calculator, to arrange algebraic expressions, to differentiate, to do limited symbolic integration and Taylor series construction, and to solve algebraic equations. The Texas Instruments company in 1995 released the TI-92 calculator with an advanced CAS, based

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on the software Derive. This calculator, and its successors (including TI-89, Voyage 200, and TI-Nspire), have featured a reasonably capable and relatively inexpensive hand-held Computer Algebra System with symbolic, graphical, and tabular capabilities.

1.2 CAS use in secondary school mathematics classes

Ever since the appearance of computers and calculators enabled with symbol-manipulating capabilities, educators have considered these tools to be quite appropriate for student use in college-level mathematics courses, and in calculus courses offered at some upper-level high schools (see, e.g., Heid, 1988; Shaw, Jean, & Peck, 1997; Zbiek, 2003). However, these tools have generally not been adopted for secondary school mathematics up until quite recently. Many secondary school mathematics teachers have, for several years, tended to stay away from CAS technology in their classrooms, preferring that their students first develop paper-and-pencil skills in algebra (National Council of Teachers of Mathematics, 1999).

However, these attitudes are changing – based both on research findings and on the leadership of interested teachers and mathematics educators, as well as on the greater availability of teacher resources for using this technology at the Grade 9, 10, and 11 levels of secondary school. The result is that student access to this technology is increasing in schools (Hoyles & Lagrange, 2009).

1.3 What does the research have to say?

CAS technology has been found to encourage the use of general mathematical reasoning processes and to improve student attitude, according to research reported during the five-year period from 2003 to 2008 at the annual conferences of the International Group for the Psychology of Mathematics Education (PME):

* *"It allows for generating, testing, and improving conjectures"*
* *"It allows for developing awareness and intuition"*
* *"It leads students to explore their own conjectures"*
* *"It provides non-judgmental feedback"*
* *"It develops the learner's confidence."*

This research has also found that CAS can help develop students' knowledge of algebraic content: their understanding of equivalence (Ball, Pierce, & Stacey, 2003), parameters and variables (Drijvers, 2003), and literal-symbolic algebraic objects in general, without "leading to the atrophy of by-hand symbolic-manipulation skills or to the slower development of these skills" (Heid, Blume, Hollebrands, & Piez, 2002, p. 586).

Since the mid-1990s, in France, when CAS technology started to make its appearance in secondary school mathematics classes, researchers (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of the technical work in algebra learning. However, this emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996). From their observations, the research team of Artigue and her collaborators came to think of techniques as a link between tasks and conceptual reflection, in other words, that the learning of techniques was vital to related conceptual thinking. The implication of these findings, as Michèle Artigue stated in her plenary presentation at this ICME-11 conference (Artigue, 2008), is that the dichotomy between techniques and concepts in algebra is a false one. It is argued not only that the two are complementary, but also that, within appropriate learning environments, techniques and concepts co-emerge and mutually support each other's growth.

1.4 The Task-Technique-Theory framework

Chevallard describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. Chevallard (1999, p. 225) states that *tasks* are normally expressed in terms of verbs, for example, "multiply the given algebraic expression." He defines *technique* as "a way of accomplishing, of carrying out tasks." In his theory, Chevallard separates *technique* from the discourse that justifies/explains/produces it, which he refers to as *technology*. But he also admits that this type of discourse is often integrated into technique, and points out that such technique can be characterized in terms of theoretical progress. According to Chevallard, *theory* takes the form of abstract speculation, a distancing from the empirical. Thus, within the anthropological approach, discourse can be viewed as bridging technique and theory.

Artigue (2002a) and her research collaborators adapted Chevallard's anthropological theory by collapsing *technology* and *theory* into the one term, *theory*. This gave the theoretical component a wider interpretation than is usual in the anthropological approach; it also reserved the use of the term *technology* for digital devices. Furthermore, Artigue (2002a, p. 248) has emphasized that *technique* also has to be given a wider meaning than is usual in educational discourse: "A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work."

Lagrange (2002, p. 163), one of Artigue's collaborators, has expressed the interrelationship of task, technique, and theory as follows:

> Within this dynamic, tasks are first of all problems. Techniques become elaborated relative to tasks, then become hierarchically differentiated. Official techniques emerge and tasks lose their problematic character: tasks become routinized, the means to perfect techniques. The theoretical environment takes techniques into account – their functioning and their limits. Then the techniques themselves become routinized to ensure the production of results useful to mathematical activity. … Thus, technique has a pragmatic role that permits the production of results; but it also plays an epistemic role (Rabardel and Samurçay, 2001) in that it constitutes understanding of objects and is the source of new questions. [my translation]

Elsewhere, Lagrange (2003, p. 271) has further extended this latter idea: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency."

Our research group was intrigued by the theoretical notion that algebra learning at the high school level might be conceptualized in terms of a dynamic among Task-Technique-Theory (T-T-T) within technological environments. And so it came to be that we began a series of studies in 2002, which continue to this day, that explored the relations among task, technique, and theory in the algebra learning (and teaching) of Grades 10, 11, and 12 students (15-18 years of age) in CAS environments. I will be elaborating on aspects of this research in a short while; nevertheless, I summarize briefly here our main findings so as to situate my underlying theme.

As reported in Kieran and Drijvers (2006), technique and theory emerged in mutual interaction. Techniques gave rise to theoretical thinking; and the other way around, theoretical reflections led students to develop and use techniques.

As reported in Kieran and Damboise (2007), a comparative study of a CAS class and non-CAS class involving the same tasks, the CAS class improved much more than the non-CAS class in both technique and theory, but especially in theory; and the sequence of lessons was one where the technical component was clearly in the forefront.

This brings us to the main question to be addressed in this paper: How does the learning of algebraic technique in a CAS environment lead to the emergence of students' theoretical/conceptual growth? In other words, how is technique rendered conceptual? What does it mean to have a conceptual understanding of algebraic technique?

2. The interface between technique and theory in algebra

Note that, within this text, I will be using the terms *conceptual* and *theoretical* interchangeably. I also wish to point out that the context of this article is related to the letter-symbolic aspects of algebra. There are two reasons for this. On the one hand, a great deal of research exists already with respect to the benefits of multi-representational approaches (e.g., graphical representations) in making algebraic objects more meaningful to students (Kieran & Yerushalmy, 2004). On the other hand, algebra involves more than representational activity; symbolic transformational activity lies at its core. However, the amount of research related to the ways in which the literal-symbolic transformational activity of algebra can be viewed as being conceptual is limited, to say the least.

2.1 What is meant by a conceptual understanding of algebraic technique?

I propose that *a conceptual understanding of algebraic technique* includes:

- Being able to see a certain **form** in algebraic expressions and equations, such as a linear or quadratic form;
- Being able to see **relationships**, such as the equivalence between factored and expanded expressions;
- Being able to see through algebraic transformations (the technical aspect) to the underlying changes in form of the algebraic object and being able to explain/justify these changes.

Some classic examples of conceptual understandings in algebra include: (a) the distinctions between variables and parameters, between identities and equations, between mathematical variables and programming variables, and so on; as well as (b) the knowledge of the objects to which the algebraic language refers (generally numbers and the operations on them) and the need to include certain semantic aspects of the mathematical context so as to be able to interpret the objects being treated. But these classic examples deal more with objects than with techniques.

2.2 Some examples of a conceptual understanding of algebraic technique

Example 1. Seeing through symbols to the underlying forms, e.g.,

(a) seeing $x^6 - 1$ as $((x^3)^2 - 1)$ and as $((x^2)^3 - 1)$, and so being able to factor it in two ways.

(b) seeing that x^2+5x+6 and x^4+7x^2+10 are both of the form ax^2+bx+c.

Example 2. Conceptualizing the equivalence of the factored and expanded forms of algebraic expressions, e.g., awareness that the same numerical substitution (not a restricted value) in each step of the transformation process of expanding will yield the same value:

$(x+1)(x+2)$ – factored form –
$= x(x+2) + 1(x+2)$
$= x^2 + 2x + x + 2$
$= x^2 + 3x + 2$ – expanded form –

and so substituting, say 3, into all four expressions produces the same numerical result – in this case, 20 – for each expression.

Example 3. Coordinating the "nature" of equation solution(s) with the equivalence relation between the two expressions that comprise the original equation, e.g., for the following task,

Given the three expressions: $x(x^2\text{-}9)$, $(x+3)(x^2\text{-}3x)\text{-}3x\text{-}3$, $(x^2\text{-}3x)(x+3)$,

(a) determine which of these three expressions are equivalent;
(b) construct an equation using one pair of the above expressions that are not equivalent, and find its solution;
(c) construct an equation from another pair of the above expressions that are not equivalent and, by logical reasoning only, determine its solution.

So, for the three given expressions,

Exp1: $x(x^2\text{-}9)$
Exp2: $(x+3)(x^2\text{-}3x)\text{-}3x\text{-}3$
Exp3: $(x^2\text{-}3x)(x+3)$

(a) Which are equivalent?
Only Exp1 and Exp3 are equivalent.
(b) An equation using a pair of non-equivalent expressions from the three given expressions? And its solution?
One could use Exp1 and Exp2 in the equation: Exp1 = Exp2.
Its solution (with CAS or with paper and pencil): x = -1.
(c) An equation from another pair of non-equivalent expressions from the above three expressions? And its solution (by logical reasoning only)?
This time, one uses Exp3 and Exp2 in the equation: Exp3 = Exp2.
One deduces that the solution has to be the same as in (b): (x = -1).
(A conceptual/theoretical understanding involving substitution of equivalent expressions and transitivity leads to this deduction.)

2.3 The importance of fostering a conceptual understanding of algebraic technique

Having just seen some examples of what is intended by the phrase, *a conceptual understanding of algebraic technique*, I now argue, briefly, for the importance of this aim for algebra instruction.

National and international mathematics assessments during the 1980s and 1990s reported that secondary school students, in order to cover their lack of understanding, resorted to memorizing rules and procedures and that students eventually came to believe that this activity represented the essence of algebra (e.g., Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988).

Although some of the recent reform movements have attempted to make algebra more meaningful for students – at least during the earlier years of high school – by infusing "real-world" problem-solving activities and multiple representations of these problems into algebra curricula, these same curricula have tended to maintain the traditional dichotomy of procedures and concepts when dealing with the transformational activity of algebra in the later years of high school. When students are then faced with the literal-symbolic transformational activity of algebra, it is presented, by and large, as a primarily concept-free domain.

Although Skemp (1976) described "relational understanding" as knowing both the rules and why they work, there has never been much movement in the direction of describing what this might mean for algebra.

The point I wish to make is that this dichotomy between procedures and concepts in algebra is both unnecessary and unproductive for students, and in fact can lead to depriving them of the conceptual insights that can make their work with procedures meaningful. But before looking at how techniques can be approached so that the conceptual component might co-emerge along with the technical, we need first to consider the issue of tasks.

2.4 The role of tasks in the T-T-T triad

At a recent PME Research Forum on "The Significance of Task Design in Mathematics Education", Ainley and Pratt (2005) – the organizers of the Forum – argued that, "We see task design as a crucial element of the learning environment … [and contend that] the nature of the task influences the activity of students." Hoyles (2002) has emphasized that a focus on the design of task situations is at the heart of the "transformative potential of [technological] tools in activities" and that, with this focus, "knowledge and epistemology are brought back to center stage" (p. 284). Lagrange (1999) has suggested that task situations ought to be created in such a way as to "bring about a better comprehension of mathematical content" (p. 63) via the progressive acquisition of techniques in the achievement of a solution to the task. Guin and Trouche (1999) have added that tasks should aim at fostering experimental work (investigation and anticipation).

More specifically, Drijvers (2003) has pointed out that more attention needs to be paid to the role of paper-and-pencil work throughout CAS task activity. For Hitt and Kieran (2009), a main consideration in task design is the nature of the theorizing that is to be elicited by the specific tasks and techniques of a teaching sequence. Artigue (2002b) has suggested that CAS tasks can capitalize on "the surprise effect that can occur when one obtains results that do not conform to expectations and that can destabilize erroneous conceptions, as well as on the multiplicity of results that can be obtained in a short space of time when exploring and trying to understand a certain phenomenon" (p. 344, my translation). Zehavi and Mann (2003) have described how the tasks they developed had the potential to intertwine student work, CAS performance, and student reflection. Ball and Stacey (2003) have argued that students' written task records ought to focus principally on the reasoning that has been evoked.

As is suggested by all of the above studies – research that has involved mathematical activity within technology environments – there is an undeniable importance accorded to the design of tasks, tasks whose goal is to promote conceptual reflection and development, even in technique-oriented work! Absent are task sequences whose main purpose is for students simply to provide answers to procedural questions.

2.5 To sum up

Because of the (a) recent advances in the development of theoretical frameworks, such as that of Task-Technique-Theory, (b) increasing use of technology in schools, for example, CAS at the secondary school level, and (c) attention being paid to the role that the nature of the task/situation plays in students' mathematical learning, we are well poised to make headway in reflecting upon the ways in which technique can be viewed from a conceptual angle in the teaching and learning of algebra and, in fact, how technology can enhance the conceptualizing of technique.

3. How 10th grade students in our project drew conceptual aspects from their work with algebraic techniques in a CAS environment

Two preliminary remarks are in order, the first concerning the tasks, the second concerning the technologies. With respect to the tasks: The tasks went beyond merely asking technique-oriented questions; the tasks also called upon general mathematical processes that included observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying. With respect to the technologies: Both CAS and paper-and-pencil were used, often with requests to coordinate the two; in general, the CAS provided the data upon which students formulated conjectures and arrived at provisional conclusions.

3.1 Conceptualizing that emerged while learning new techniques with the aid of CAS technology

The examples in this section are drawn from Kieran and Drijvers (2006) and Hitt and Kieran (2009). The two-lesson task-sequence was related to factoring (adapted from Mounier & Aldon, 1996). It involved the family of expressions, $x^n - 1$. The aim of the task sequence was to arrive at a general form of factorization for $x^n - 1$ (for integer values of $n \geq 2$) and then to relate this to the complete factorization of particular cases for integer values of n from 2 to 13. Proving one of these cases was part of the sequence, but is not included in this article (for details on the proving component and its unfolding in class, see Kieran & Guzmán, 2010).

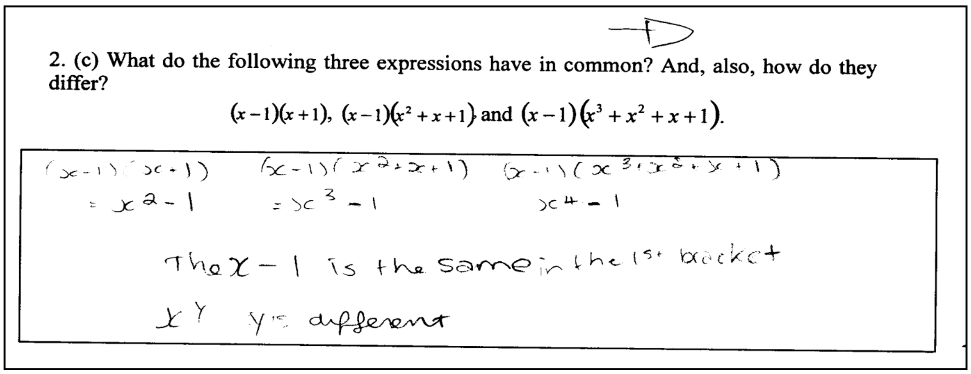
One of the initial tasks of the sequence involved the following questions, which have been compressed for this article into Figure 1.

1. Perform the indicated operations: $(x-1)(x+1)$; $(x-1)(x^2+x+1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product $(x-1)\left(x^3+x^2+x+1\right) =$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ? $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)\left(x^3+x^2+x+1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. Is your explanation valid for the following equality: $(x-1)(x^{134}+x^{133}+x^{132}+\ldots+x^2+x+1) = x^{135}-1$? Explain.

Figure 1. Some of the initial task questions of the x^n - 1 sequence.

After students had worked on these questions, either in groups or individually, the teacher opened up a whole-class discussion and asked students to state their responses to one particular question (Question #4 of Figure 1). Different students noticed different things in the pattern of expressions. The teacher's aim in having the whole-class discussion was to encourage students to learn from what some of their peers had noticed. Figures 2 and 3 provide some samples of their responses to the given question. (As an aside: the issue of what students notice when doing exploratory mathematical work with technology is one that has received little research attention.)

The particular student whose work is shown in Figure 2 focused on the $(x-1)$ in the factored form and on the exponent in the expanded form.



2. (c) What do the following three expressions have in common? And, also, how do they differ?

$(x-1)(x+1)$, $(x-1)(x^2+x+1)$ and $(x-1)(x^3+x^2+x+1)$.

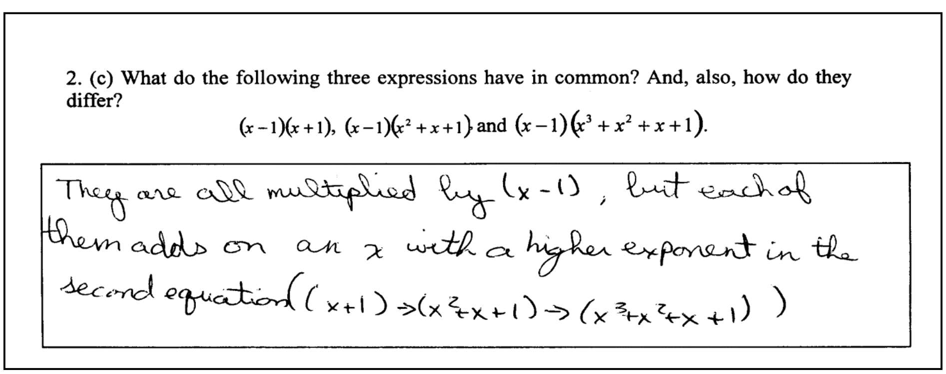
$(x-1)(x+1) = x^2-1$ $(x-1)(x^2+x+1) = x^3-1$ $(x-1)(x^3+x^2+x+1)$ x^4-1

The x − 1 is the same in the 1st bracket

x^y y's different

Figure 2. For this question, this student focused on the (*x*-1) and the exponents.

The student whose work is displayed in Figure 3 helped others to "refine their noticing" when she described during the whole-class discussion what she had focused on. She noticed more than did some other students and was also able to express herself with a certain clarity – even if she misused terminology. Linguistic imprecisions such as this one, where *equation* was used for *factor*, were a common occurrence among the students in the classes we observed.



2. (c) What do the following three expressions have in common? And, also, how do they differ?

$(x-1)(x+1)$, $(x-1)(x^2+x+1)$ and $(x-1)(x^3+x^2+x+1)$.

They are all multiplied by (x-1), but each of them adds on an x with a higher exponent in the second equation ((x+1) → (x²+x+1) → (x³+x²+x+1))

Figure 3. This student helped others in the class to "refine their noticing".

The class then moved on to a general form of factorization for x^n-1 based on the above prior examples: $x^n - 1 = (x-1)(x^{n-1}+x^{n-2} + \ldots x+1)$ (see Sacristán & Kieran, 2006, for student work related to this component of the task sequence). After arriving at this general form, the students worked on the Factorization Task where they were confronted with the completely factored forms produced by the CAS and where they were requested to reconcile their paper-and-pencil (p/p) factorizations with those produced by the CAS. One of the ways in which students attempted to reconcile their expected factorization of, for example, x^4-1 with the CAS factorization is suggested by the work displayed in Figure 4. Here the student multiplied the 2nd and 3rd CAS factors to yield the same second factor that she had obtained with paper and pencil. Other students reconciled their p/p and CAS productions either by factoring more completely their 2nd p/p factor or by asking the CAS to multiply its 2nd and 3rd factors so as to see whether that produced the same polynomial as their 2nd p/p factor.

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

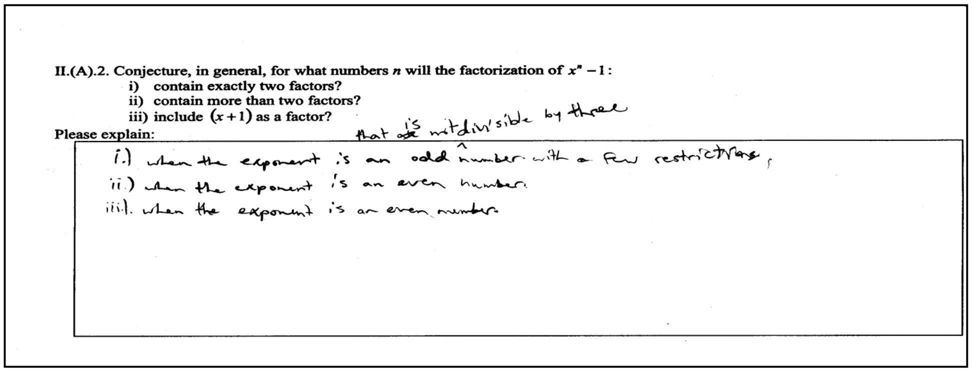
Figure 4. Reconciling paper-and-pencil and CAS factorizations for x^4-1.

After completing the Factorization Task for n = 2 to 6 in x^n - 1, students were presented with the Conjecture Task: “Conjecture, in general, for what numbers n will the factorization of x^n-1: (i) contain exactly two factors? (ii) contain more than two factors? (iii) include (x+1) as a factor? Explain.” The following pair of students, Chris and Peter, incorrectly conjectured that, for all odd ns, the complete factorization of x^n-1 would contain exactly two factors (see Figure 5). The last line of the transcript extract indicates the moment of surprise when their initial conjecture proved false (this extract is drawn from Hitt & Kieran, 2009).

Chris	‘Two factors’ means two separate sets of brackets, right?
Peter	Yeah.
Chris	The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can’t be broken down anymore. ‘Cause it always ends up being all positive. And uh, then, because, it’s sort of hard to explain.
Peter	When the exponent is [pause], when the exponent is an even number you’ll have more than two factors, but when the exponent is not an even number, you’ll have exactly two factors all the time.
Chris	Yeah. [Types Factor (x^7-1) into the CAS] Yeah, because any time you plug in an odd number as the exponent power, it’s uh, the calculator always stays at the most simplified [pause] and [Types in Factor (x^9-1); the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$] And, no!!! [a look of utter surprise on Chris’s face]

Figure 5. The role played by the CAS in disproving the initial false conjecture.

The two students then began to wonder: If it is not the case that all odd ns produce exactly two factors when x^n-1 is completely factored, then which ns will produce only two factors? The CAS allowed them to test a variety of values for n, including the extreme case of n = 99, which led to a first revision of their initial conjecture (see Figure 6).



II.(A).2. Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x+1)$ as a factor?
Please explain:

i.) when the exponent is an odd number that is not divisible by three with a few restrictions,
ii.) when the exponent is an even number.
iii.) when the exponent is an even number.

Figure 6. A first revision of their odd-number conjecture: *Exclude multiples of 3*.

But they had not quite finished with their conjecturing, and testing of conjectures, with the CAS. In addition to eliminating multiples of 3 as possible values for n, they soon were able to eliminate multiples of 5 and 7 as well. Then one of them suggested trying x^{60} - 1 because, as he said, "I think it has to do with how many numbers can go into it." This led to the "eureka" moment: that n had to be a prime number in order for the complete factorization of x^n-1 to contain exactly two factors.

From these samples drawn from Chris and Peter's activity, we have had a glimpse at the role that CAS technology, within a thought-provoking task sequence, can play in supporting algebraic conjecture-making and conjecture-refining – allowing these two students to focus their trials on certain multiples of the exponent, to try out extreme cases, … in short, to arrive at a new conceptualization of the factors for expressions from this family of polynomials – all this within an activity related to technical work on factoring.

3.2 Further evidence for the emergence of theoretical/conceptual ideas arising from work with CAS techniques

The second set of examples to be presented is pulled from a comparison study that we carried out with two classes of weak Grade 10 algebra students (Kieran & Damboise, 2007). Some of the characteristics of the task and test design were as follows:

* A set of tasks was developed on the topic of factoring and expanding.
* Tasks were identical for the two classes except that, where one class was to use p/p only, the other class was to use CAS or a combination of CAS and p/p (see Figures 7 and 8 for an example of the parallel task-sets for each class).
* Some tasks were technique-oriented; others were theory-oriented.
* A pretest and posttest were also created with some questions being technical and others theoretical.

Note that, in both task-sets of Figures 7 and 8, the technical is the focus of the first question; the theoretical is the focus of the second question with its four subparts. Note as well that, in the CAS version of Question 1, students are asked to enter onto their worksheet the output produced by the CAS, while in the non-CAS version they are to record their paper-and-pencil factorizations and expansions. (N.B.: The "dissected" form of the first column was one with which both classes were quite familiar by the time that they encountered this Activity.)

Activity 3 (CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Use the calculator in completing the table below.

Given trinomial (in "dissected" form)	Factored form using FACTOR	Expanded form using EXPAND
(a) $x^2 + (3+4)x + 3 \bullet 4$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why did the calculator not factor the trinomial expressions of 1(d) and 1(e) above?
2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?
2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.
2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 7. One of the task-sets for the CAS class.

Activity 3 (non-CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Complete the table below by following the example at the beginning of the table.

Given trinomial (in "dissected" form)	Factored form	Expanded form
Example: $x^2 + (3+4)x + 3 \bullet 4$	$x^2 + (3+4)x + 3 \bullet 4$ $= x^2 + 3x + 4x + 3 \bullet 4$ $= x(x+3) + 4(x+3)$ $= (x+3)(x+4)$	$x^2 + 7x + 12$
(a) $x^2 + (5+6)x + 5 \bullet 6$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why could you not factor the trinomial expressions in 1(d) and 1(e) above?
2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?
2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.
2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 8. The parallel task-set for the non-CAS class.

In this study, the technology was found to play several roles in the CAS class:

* It provoked discussion;
* It generated exact answers that could be scrutinized for structure and form;
* It helped students to verify their conjectures, as well as their paper-and-pencil responses;

* It motivated the checking of answers; and
* It created a sense of confidence and thus led to increased interest in the algebraic activity.

Of all the roles that the CAS played in this study, the fact that CAS generated exact answers that could be scrutinized for structure and form was found to be crucial to the success of these weak algebra students. It proved to be the main mechanism underlying the evolution in the CAS students' algebraic thinking. Ironically, the importance of this role was first made apparent to us by the voicing of frustration on the part of one of the students in the non-CAS class. This student from the non-CAS class, when faced with Questions 2(c) and 2(d) of the task shown in Figure 8, remarked:

> *"How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our paper-and-pencil factorizations and expansions from Question 1 are right."*

Students in the non-CAS class were at a loss to answer these explanation-oriented questions. They stated emphatically that they were not sure of their paper-and-pencil answers to Question 1, and could hardly use these as a basis for answering, say, Question 2d. In contrast, the students in the CAS class had at their disposal a set of factored and expanded expressions that had been generated by the calculator. They thus had confidence in these responses and could begin to examine them for elements related to structure and form.

This study analyzed the improvements of two classes of weak algebra students in both *technique* (being able to do) and *theory* (i.e., being able to explain why and to note some structural aspects), in the context of tasks that invited technical and theoretical development. At the outset, both the CAS class and the non-CAS class scored at the same levels in a pretest that included technical and theoretical components. However, the CAS class improved more than the non-CAS class on both components, but especially on the theoretical component.

We see this finding as being of some interest. Being able to generate exact answers with the CAS allowed students to examine their CAS work and to see patterns among answers that they were sure were correct. This kind of assurance, which led the CAS students to theorize, was found to be lacking in the uniquely paper-and-pencil environment where students made few theoretical observations. The theoretical observations made by CAS students worked hand-in-hand with improving their technical ability. In other words, *their technique had become theorized*, which in turn led to further improvement in technique.

4. The role of the teacher

Are good tasks and CAS technology all that are needed to render technique conceptual, that is, to develop a conceptual understanding of algebraic technique? It would seem not!

Another deciding factor is the nature of the teacher's orchestration of classroom activity that gives rise to the conceptualizing of technique in technology environments. It is the teacher who is pivotal in encouraging the students to struggle with the task, who asks them key questions at appropriate times, who helps them to see the overarching themes within the tasks, who makes the instrumental genesis converge to a common set of techniques and insights, and who leads the classroom discussions that provoke this convergence through discourse. However, not all of the teachers in our research study proved to be equally successful in orchestrating the co-emergence of technique and theory within their students.

Currently our research group is analyzing teaching practice with the aim of identifying some of the key characteristics of teachers' orchestrations of classroom activity with CAS technology that relate to drawing out the conceptual aspects of technical work in algebra. Some of the characteristics we have begun to identify include the following: (a) importance accorded to the mathematical aspects of the task – both technical and conceptual; (b) emphasis on the mathematical-technological similarities/differences; (c) interest in inquiring into the students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications; and (d) awareness of the many possible roles that the technology can play. These possible roles encompass, for example, creating surprising results, generating results for the purpose of exploration, verifying other results or conjectures, and serving as a computational assistant. However, teachers also need to be able to capitalize on these roles in such a way as to encourage student learning.

Other characteristics of teachers' orchestrations of classroom activity with CAS technology that we have been observing include having a repertoire of tasks that engage a variety of learning approaches and evoke different processes, such as, provoking cognitive conflict and seeking to resolve the conflict; looking for patterns; generalizing; activating general mathematical processes, such as observing, comparing, extrapolating, conjecturing, and predicting; and having considered, before the lesson begins, possible student responses and how to encourage further evolution of their thinking within the ensuing lesson. Promising teacher orchestrations also consider the ways in which to incorporate additional artifacts (e.g., worksheets, paper and pencil, the blackboard (or the equivalent), electronic projection devices, etc.) and the roles they might play, namely guiding the work of pupils and structuring their explorations (worksheets), focusing their attention (blackboard), and leading to a convergence of ideas (blackboard).

In sum, effective teaching practice with CAS would appear to embody planning that takes into account at the very least the following:

1. Starting with a key mathematical idea.
2. Thinking about both the technical and theoretical aspects of the key idea.
3. Trying out, when planning the task, some technical examples on the CAS to see how best to take advantage of the technology (does it produce any surprises that could be integrated into an interesting sequence?)
4. Deciding what role the technological artifact should play in the task (generate examples, create surprises, serve as calculation assistant, …)
5. Deciding on the epistemological processes to be engaged by the task (pattern matching and generalization, conjecturing, seeking connections between representations, resolving cognitive conflict, predicting, …)
6. Reflecting on how to draw out effectively within class discussions the mathematical-technological links.

Last, but not least, our research observations so far suggest that the one aspect of teacher's practice in CAS environments that seems to be most crucial to students' becoming aware of the conceptual aspects of their technical work in algebra is the following: Orchestrating classroom discussion in such a way as to draw out students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications. When such orchestration is accompanied by tasks that (a) go beyond merely asking technique-oriented questions and which (b) call upon mathematical processes that include: observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying, and which (c) require at times that students coordinate CAS techniques with paper-and-pencil techniques, as well as (d) seek consistency between surprising CAS outputs and existing theoretical notions, then algebraic techniques will have a greater likelihood of being rendered conceptual.

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