The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra

Carolyn Kieran, Université du Québec à Montréal, Montréal, Canada
Paul Drijvers, Freudenthal Institute, Utrecht, the Netherlands

with the collaboration of
André Boileau, Fernando Hitt, Denis Tanguay,
Luis Saldanha, and José Guzmán

Appearing in: *International Journal of Computers for Mathematical Learning*,
11, 205-263 (October 2006)

CAROLYN KIERAN and PAUL DRIJVERS

with the collaboration of
André Boileau, Fernando Hitt, Denis Tanguay,
Luis Saldanha, and José Guzman

THE CO-EMERGENCE OF MACHINE TECHNIQUES, PAPER-AND-PENCIL TECHNIQUES, AND THEORETICAL REFLECTION: A STUDY OF CAS USE IN SECONDARY SCHOOL ALGEBRA

ABSTRACT. This paper addresses the dialectical relation between theoretical thinking and technique, as they co-emerge in a combined computer algebra and paper-and-pencil environment. The theoretical framework in this ongoing study consists of the instrumental approach to tool use and an adaptation of Chevallard's anthropological theory. The main aim is to unravel the subtle intertwining of students' theoretical thinking and the techniques they use in both media, within the process of instrumental genesis. Two grade 10 teaching experiments are described, the first one on equivalence, equality and equation, and the second one on generalizing and proving within factoring. Even though the two topics are quite different, findings indicate the importance of the co-emergence of theory and technique in both cases. Some further extensions of the theoretical framework are suggested, focusing on the relation between paper-and-pencil techniques and computer algebra techniques, and on the issue of language and discourse in the learning process.

KEY WORDS. algebra, computer algebra, instrumentation, technique in algebra, technology, theoretical thinking in algebra

1. Introduction to the topic and overview

The integration of new technologies in mathematics education has been an ongoing issue for the last two decades. Nowadays, access to technology has drastically increased, not the least because of the availability of hand-held devices such as graphing calculators and symbolic calculators. Still, confusion on the role of technology in the teaching and learning of mathematics is considerable. From the students' perspective, it is often not clear how the use of technological tools relates to the required paper-and-pencil skills. Teachers and teacher educators are struggling with the same questions and are searching for guidelines that foster successful integration of new media into teaching practice. Researchers address these issues from a scientific perspective, but have difficulty in providing evidence of improved learning with technological means, as well as in understanding the influence of technology on learning. In all, the original optimism regarding the benefits of technology, which would allow a focus on conceptual understanding at the expense of calculation techniques, has become quite nuanced:

> Actually, the view of the technological environment as one that imbalances the relationship between technical and conceptual work, by means of saving time on the technical work left to the machine, and concentrating on the conceptual work, was not supported by our observations. (Artigue, 1997, p. 164, translation by the authors)

The issue of technology changing the relation between technical skills and conceptual understanding is particularly pertinent in algebra education. Dedicated pedagogical environments for developing specific algebraic techniques, theorems, or models have been created, such as *Algebrista* (Cerulli and Mariotti, 2002), *Aplusix* (Nicaud, Bouhineau and Chaachoua, 2004) and *AlgebraArrows* (Boon and Drijvers, 2005). Scenarios have been developed for using more general technological tools such as spreadsheet software to foster algebraic thinking (Haspekian, 2005). In this paper, however, we focus on the powerful computer algebra systems (CAS), which nowadays offer broad, general-purpose environments for carrying out all types of algebraic procedures. The question is how the use of these tools interacts with the paper-and-pencil skills and the conceptual understanding of 10th grade students.

To approach this question, several perspectives can be taken. As a point of departure, we see a CAS as a mathematical tool, and consider CAS use as a particular case of tool use in general. In that sense, the work of Vygotsky on tool use is at the base of our work (Vygotsky, 1930/1985). To paraphrase the notion of a tool as an extension of the body, the CAS being a cognitive tool can be seen as an extension of the mind. In recent years, the work of Vygotsky and, to a lesser extent, Piaget, has been elaborated into the so-called instrumental approach to tool use. The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon and Rabardel, 1995; Rabardel, 2002) and the anthropological theory of didactics (Chevallard, 1999). An essential starting point in the instrumental approach is the distinction between an artifact and an instrument. Whereas the artifact is the – often physical – object that is used as a tool, the instrument involves also the techniques and schemes that the user develops while using it, and that guide both the way the tool is used and the development of the user's thinking. The process of an artifact becoming an instrument in the hands of a user – in our case the student – is called instrumental genesis. One of the main characteristics of the instrumental approach, as we see it, is that it stresses the effort and time that the non-trivial process of instrumental genesis requires. A second important aspect of this approach is the importance of the bilateral relationship between the artifact and the user: while the student's knowledge guides the way the tool is used and in a sense shapes the tool (this is called instrumentalization), the affordances and the constraints of the tool influence the student's problem-solving strategies and the corresponding emergent conceptions (this is called instrumentation). In short, the student's thinking is shaped by the artifact, but also shapes the artifact (Hoyles and Noss, 2003).

The instrumental approach to tool use was recognized by French mathematics education researchers as a potentially powerful framework in the context of using CAS in mathematics education. Many publications show how valuable this approach is for the understanding of student-CAS interactions and their influence on teaching and learning (Artigue, 1997, 2002; Lagrange, 2000, 2005; Trouche, 2000, 2004a, 2004b; Guin, Ruthven and Trouche, 2004). It has not only been applied to the integration of CAS into the learning of mathematics, but also to the use of spreadsheets (Haspekian, 2005) and dynamic geometry systems (Falcade, 2003).

As Monaghan (2005) pointed out very clearly, one can distinguish two directions within the instrumental approach, which link up with the two background frameworks. In line with the cognitive ergonomic framework, some researchers see the development of schemes as the heart of instrumental genesis. Although these mental schemes develop in social interaction, they are essentially individual. In the work of Trouche (2000), Drijvers (2003), and Drijvers and Trouche (in press), these so-called schemes of instrumented action play a dominant role, whereas techniques are considered as the visible parts of these schemes. Within the schemes, conceptual and technical elements are intertwined.

More in line with the anthropological approach, other researchers focus on techniques that students develop while using technological tools and in social interaction. The advantage of this focus is that instrumented techniques are visible and can be observed more easily than mental schemes. Furthermore, this approach takes into account the importance of techniques, which tends to be underestimated in discussions on the integration of technology. Still, it is acknowledged that techniques encompass theoretical notions. The focus on techniques is dominant in the work of Artigue (2002) and Lagrange (2000) in particular. The latter view, with techniques as an important factor in Chevallard's anthropological theory, forms the main framework of our study, and will be addressed in more detail in the next section, along with the main research question.

After some information on the methodological components of the study, two themes are addressed: the first on equivalence, equality, and equation; the second on generalizing and proving within factoring. A concluding discussion brings the article to an end. As the study is ongoing, we point out that the results presented here are only part of the project findings.

2. The study

2.1 Theoretical framework: Task-Technique-Theory

In his anthropological theory of didactics, Chevallard (1999, p. 225) describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. He notes that *tasks* are normally expressed in terms of verbs, for example, *multiply* the given algebraic expression. He goes on to define *technique* as "a way of accomplishing, of carrying out tasks" and points out that a technique "is not necessarily algorithmic or quasi algorithmic." In his theory, he separates *technique* from the discourse that justifies/explains/produces it, which he refers to as *technology*[1]. But he also admits that this type of discourse is often integrated into technique, and points out that such technique can be characterized in terms of theoretical progress. According to Chevallard, *theory* takes the form of abstract speculation, a distancing from the empirical. Thus, within the anthropological approach, discourse can be viewed as bridging technique and theory.

In their adaptation of Chevallard's anthropological theory, Artigue and her colleagues (see, e.g., Artigue, 2002) have collapsed Chevallard's *technology* and *theory* into the one term, *theory*, thereby giving the theoretical component a wider interpretation than is usual in the anthropological approach. Furthermore, Artigue (2002, p. 248) notes that *technique* also has to be given a wider meaning than is usual in educational discourse: "A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work."

---

[1] This is in contrast to our use of the term *technology*, which refers to the use of computers and other technological tools.

Lagrange (2002, p. 163) has expressed his perspective on the interrelationship of task, technique, and theory as follows:

> Within this dynamic, tasks are first of all problems. Techniques become elaborated relative to tasks, then become hierarchically differentiated. Official techniques emerge and tasks lose their problematic character: tasks become routinized, the means to perfect techniques. The theoretical environment takes techniques into account – their functioning and their limits. Then the techniques themselves become routinized to ensure the production of results useful to mathematical activity. … Thus, technique has a pragmatic role that permits the production of results; but it also plays an epistemic role (Rabardel and Samurçay, 2001) in that it constitutes understanding of objects and is the source of new questions. [translation by the authors]

Elsewhere, Lagrange (2003, p. 271) has elaborated this latter idea further: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is precisely this epistemic role played by techniques that is a focus of our study, that is, the notion that students' mathematical theorizing develops as their techniques evolve. It is also noted that, within our perspective on the co-emergence of theory and technique, the nature of the task is considered to play an equally fundamental role. However, the importance of the tasks goes beyond the situating of this study within the context of instrumental genesis. Hoyles (2001, p. 284), for example, has drawn attention to the significant role played by the "design of activities and the design or choice of the tools introduced to foster mathematics learning, not that design will lead to outcomes in a deterministic way, but at least this focus would allow investigation of the transformative potential of tools in activities … and bring knowledge and epistemology back into centre stage." Thus, the triad Task-Technique-Theory (TTT) serves as the framework not only for gathering the data during the teaching experiments, and for analyzing that data, but also for constructing the tasks of this study.

2.2 Aim and methodological considerations of the study

The research study had as a central objective the shedding of further light on the co-emergence of technique and theory within the CAS-based symbol manipulation activity of secondary school students. However, because paper-and-pencil techniques were a fundamental part of the algebra program of studies of the schools where the research was carried out, and because we believe in the importance of combining the two media, they too were included in the teaching sequences. Thus, the research question that we attempt to answer in this article is the following:

> *In which ways does the interaction between technique and theory foster students' algebraic thinking when working in a combined CAS/paper-and-pencil environment?*

The first year of the project, 2003, focused on designing the tasks and sequences to be used during the classroom segment of the study. The research team created several sets of activities that aimed at supporting the co-emergence of technique and theory. Two of these sets of activities are included in this article, one dealing with the theme of equivalence, equality, and equation; the other, with the theme of generalizing and proving within factoring. These two were chosen for this article for very specific reasons. First, they deal with rather different underlying concepts, one having a more theoretical focus and the other a more technical focus. Second, the nature of the CAS techniques in each theme is quite different -- one involving a larger and more diverse set of CAS approaches; the other, fewer CAS commands, yet seemingly closer to the corresponding paper-and-pencil techniques than might appear to be the case for the other theme. Lastly, the two themes differ with respect to the nature of the global mathematical activity involved -- the first entailing making connections and seeing relationships; the second, pattern seeking, noticing structure, and proving. We felt that having two quite different sets of activities featured in this article would better illustrate the ways in which the task-technique-theory framework can be applied to a variety of algebra situations at the high school level.

The manner in which the triad task-technique-theory served in designing these two sets of activities is elaborated later within each thematic section. It is noted here, however, that all of our activity sets, which were planned to take anywhere from one to five class periods, involved work with CAS, with paper and pencil (P&P), and questions of a reflective nature. In designing these tasks, we took seriously both the students' background knowledge and the fact that these tasks were to fit into an existing curriculum; but we also did our best to ensure that they would unfold in a particular classroom culture that reflected a certain priority given to discussion of substantive mathematical issues. Tasks that asked students to write about how they were interpreting their work and the related CAS displays aimed to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom, and clarifying ideas and distinctions, in ways that simply "doing algebra" may not require.

In conceptualizing the design of the tasks, an additional factor played an important role. A great deal of research evidence exists already with respect to the benefits of multi-representational approaches (e.g., graphical representations) in making algebraic objects such as variables, expressions, and equations more meaningful to students (see, e.g., Heid, 1996; Kieran and Yerushalmy, 2004). However, algebra involves more than representational activity (Kieran, 2004, 2006); symbolic transformational activity lies at its heart. In view of the limited amount of research that exists regarding the use of CAS tools with the purely symbolic aspects of algebra learning at the secondary school level, a deliberate choice was made to restrict the tasks of this study to those involving the letter-symbolic form.

The second year of the project, 2004, as well as the first segment of the third year, 2005, was devoted to the classroom part of the research. Two of the six participating 10^{th} grade classes (15-year-old students) are featured in this article – one from 2004 and the other from 2005. The 2004 class consisted of 7 girls and 10 boys, all of them considered by their teacher to be of upper-middle mathematical ability. The 2005 class contained 6 girls and 11 boys, all of them high achievers in mathematics and following an enriched program. Both classes were taught by the same teacher, in a private school in Montreal. This teacher, whose undergraduate degree and teacher training had been done in the U.K., had been teaching mathematics for five years, but had not had a great deal of experience with technology use in mathematics teaching, except for the graphing calculator. He was a teacher who, along with encouraging his pupils to talk about their mathematics in class, thought that it was important for them to struggle a little with mathematical tasks. He liked to take the time needed to elicit students' thinking, rather than quickly give them the answers. He, along with the other teachers who took part in the study, followed a program of training prior to their teaching with the new materials. This training program involved some sessions with the CAS technology and others devoted to the pedagogical aspects of the teaching materials. Some of the teachers also provided feedback to the research team on the content of the tasks, at each cycle of task development.

Students in these two classes had learned a few basic techniques of factoring (for the difference of squares and for factorable trinomials) and the solving of linear and quadratic equations during their 9th grade mathematics course and had used graphing calculators on a regular basis; however, they had not had any experience with the notion of equivalence, one of the theoretical ideas developed in the research materials, or with symbol-manipulating calculators. It is noted that these students were quite skilled in algebraic manipulation, as was borne out by the results of a pretest we administered at the outset of the study. It was during the algebra part of their 10th grade mathematics course, which extended from the month of September to the end of January, when the activities designed by the research team, accompanied by CAS technology (TI-92 Plus calculators), were integrated into the students' regular program of mathematics and taught by the classroom mathematics teacher.

For each class that participated, data were collected that focused on the students and the classroom situation at large. Two video cameras were set up in the classrooms, one in the front and a second one in the rear. One or two researchers took field notes during each class period. Students from the 2004 classes were interviewed (and videotaped), individually or in pairs, at several instances – before, during, and after class. In addition to the classroom videotaping, audiotaped mini interviews were also conducted with the 2005 students during class time, particularly in those areas of the tasks where it was thought that further questioning of students might prove helpful to our data analysis. A posttest involving CAS use was administered after the set of activities on equivalence had been completed. All students were pretested; however, the pretest for the 2005 students differed somewhat from the one administered to the 2004 cohort. Thus, data sources for the segment of the study that is presented in this article include the videotapes of all the classroom lessons dealing with the two sets of activities, videotaped individual and pair-wise interviews with students outside of class time, audio-taped mini interviews with individual students during class time, videotaped view-screen displays of student and teacher activity with the CAS, a videotaped interview with the teacher of the two classes featured in this article, the activity sheets of all students (these contained not only their paper-and-pencil responses but also a record of CAS displays and their interpretations of these displays), written pretest and posttest responses, and researcher field notes.

The third year of the study centered on data analysis. As will be illustrated in this article, the analysis combined both qualitative and quantitative approaches. Guided by the TTT foundations of the study, we developed a priori descriptions of the techniques and theories that we considered might emerge among students while working on the given tasks. These descriptions provided the lens for gathering and analyzing the data drawn from the classrooms, from student work, and from student interactions occurring in the second and third years of the study. The structure of this article in fact makes explicit these a priori descriptions of technique and theory that we generated, as well as the way in which they served to focus our analyses.

3. The theme of equivalence, equality, and equation

3.1 Aim of the teaching sequence

The underlying motive of this 3-to-5-lesson teaching sequence is the subtle relationship between arithmetic and algebra: on the one hand, the numerical world is the most important motive and model for the world of algebra, on the other hand algebra goes beyond the numerical world, which in fact is part of its power.

This two-sided relationship is reflected in the notion of equivalence of algebraic expressions. On the one hand, equivalence of two expressions relates to the numeric as it reflects the idea of 'equal output values for all input values'. On the other hand, the notion of equivalence of expressions from an algebraic perspective means that the expressions can be rewritten in a common algebraic form. Therefore, the intended conceptual progression of this 10^{th} grade teaching sequence was to have students develop an integrated understanding of equivalence of expressions, in which the numeric and algebraic perspectives are coordinated. Consultations with teachers and perusal of mathematics textbooks indicated that the 10^{th} grade students in our study had had no explicit previous encounter with the concept of equivalence and its relation to numeric equality, algebraic transformation, and equation. Thus, we set out to design tasks that would encompass these notions.

3.2 Task

Figure 1 shows the outline of the content of the teaching materials for this sequence, which included also a pretest and a posttest.

Activity 1	Equivalence of Expressions	Tools
Part I	Comparing expressions by numerical evaluation	CAS
Part II	Comparing expressions by algebraic manipulation	P&P
Part III	Testing for equivalence by re-expressing the form of an expression – using the Expand command	CAS
Part IV	Testing for equivalence without re-expressing the form of an expression – using a test of equality	CAS
Part V	Testing for equivalence – using either CAS method	CAS
Activity 2	Continuation of Equivalence of Expressions	
Part I	Exploring and interpreting the effects of the Enter button, and the Expand and Factor commands	CAS/P&P
Part II	Showing equivalence of expressions by using various CAS approaches	CAS
	Homework	CAS
Activity 3	Transition from Expressions to Equations	
Part I	Introduction to the use of the SOLVE command	CAS
Part II	Expressions revisited, and their subsequent integration into equations	CAS
Part III	Constructing equations and identities	P&P
Part IV	Synthesis of various equation types	CAS

Figure 1. Outline of the teaching unit

At the start of the teaching sequence, numerical evaluation of expressions by using CAS and comparison of their resultant values were used as the entry points for discussions on equivalence. One of the core tasks here was the Numerical Substitution Task (Figure 2). It aimed at students' noticing that some pairs of expressions seem *always* to end up with equal results, and thus evokes the notion of equivalence based on numerical equality. It is noted that the algebraic expressions included in the task were fairly complex so as not to permit the evaluation of equivalence by purely visual means. The task was followed by a reflection question (which was one of the characteristics of the teaching unit) on what would happen if the table were extended to include other values of x.

For $x =$	1/3	-5		
Expression	Result	Result	Result	Result
1. $(x-3)(4x-3)$				
2. $(x^2+x-20)(3x^2+2x-1)$				
3. $(3x-1)(x^2-x-2)(x+5)$				
4. $(-x+3)^2+x(3x-9)$				
5. $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{x+2}$				

Figure 2. Numerical Substitution Task

The task and the CAS substitution technique led to the following definition of equivalence of expressions:

> We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.

The stress on the set of admissible numbers was made deliberately by the designers, so as to make students aware of the attention that one has to pay to chaining equivalent expressions with possible restrictions. Expression 5 (expr5) in Figure 2 was a first example of this.

The impossibility of testing all possible numerical substitutions to determine equivalence motivated the use of algebraic manipulation and the explicit search for common forms of expressions in the second activity. Different CAS techniques could be used. An example of a core task was the CAS Technique Task (Figure 3). Students could carry out different CAS techniques and compare the results in the light of their understanding of equivalence. This task aimed at developing the notion of equivalence as involving a common algebraic form. The reflection questions that followed concerned the identification of equivalent expressions, including a justification and the consideration of possible restrictions.

Given expression	Result produced by the Enter button	Result produced by Factor	Result produced by Expand
1. $\frac{6x^2-5x-4}{6}$			
2. $\frac{(x-2)^2+(7x-2)(x-2)}{4}$			
3. $(2-x)(1-2x)$			
4. $\frac{(3x-4)(2x^2+5x+2)}{6x+12}$			

Figure 3. CAS Technique Task

After this, paper and pencil were used to 'verify' the CAS results and to reconcile the techniques in the two media. Then attention was drawn towards restrictions on equivalence, and the way the CAS that we used neglects them, whereas students should be aware of them while working with paper and pencil and with CAS.

In the third activity, the relation between two expressions being equivalent or not, and the corresponding equation having many, some, or no solutions was explored in both CAS and paper-and-pencil tasks. The following Construction Task (Figure 4) addressed this issue. Students were asked to construct a pair of equivalent expressions, and, in a similar follow-up task, two non-equivalent expressions. The reflection question that was raised after that concerned the relation between the nature of an equation's solution(s) and the equivalence or non-equivalence of the expressions that form the equation. Once more, CAS technique and theory were interacting.

1. Construct an equation made from <u>two equivalent expressions</u> of your own choosing.
2. Explain your reasons for choosing these two particular expressions.
3. Without solving it, what can you say about the solutions of this equation?
4. How would you use the CAS to support your response to Question 3 just above?

Figure 4. Construction Task

The posttest contained similar construction tasks. A Posttest Task (Figure 5) aimed, once again, at students' expressing the relation between the solutions of an equation and the equivalence of the expressions that form the equation. As before, the coordination of theoretical notions and CAS technique was involved.

Q.5 The following equation has $x = 2$ and $x = 2/3$ as solutions:

$$x(2x-4)+(-x+2)^2 = -3x^2+8x-4$$

(i) Precisely what does it mean to say that "the values 2 and 2/3 are solutions of this equation"?

(ii) Use the CAS to show that: (a) the two values above are indeed solutions, <u>and</u>
(b) there are no other solutions.

What I entered into the CAS	What the CAS displays <u>and</u> my interpretation of it

(iii) Are the expressions on the left- and right-hand sides of this equation equivalent? Please explain.

Figure 5. Posttest Task

3.3 Technique

What are the main techniques that students could develop while working at the tasks presented in the previous section? Figure 6 shows an inventory of techniques that involve the notion of equivalence, using both CAS and paper and pencil.

Technique	**CAS variant (using TI-92 Plus)**	**Paper-and-Pencil variant**
1. Substituting numerical values	With-operator '|', followed by automatic evaluation	Substitution, followed by evaluation by hand
2. Common form - by factoring	Factor command, usually complete factorization	Factor by hand, often incomplete

3. Common form - by expanding	Expand command	Expand by hand, sensitive to errors
4. Common form - by automatic simplification	Automatic simplification after Entering	Manipulation by hand only to a limited extent
5. Test of equality	Enter equation + Enter	Manipulation by hand only to a limited extent
6. Solving equations	Solve command	Limited to a specific set of types of equations

Figure 6. Different techniques in two media

Let us briefly comment on these techniques. First, we should stress that the CAS-techniques and the paper-and-pencil techniques are different. The comparison of the same approach in the two different media, however, is interesting and the reconciliation of the two is not self-evident. Within the tasks set for this theme, CAS techniques dominate.

Technique 1 concerns numerical substitution, which provides the numerical basis for algebraic equivalence. A numerical substitution that gives different results for two expressions proves their non-equivalence, whereas obtaining the same results does not guarantee equivalence. For the paper-and-pencil variant, the difficulty for the students is not the substitution itself, but the evaluation and simplification of the results.

CAS technique 2, finding a common form by factorizing, has two variants, one in which both forms that are tested for equivalence are to be factored, and a second in which one of the two expressions is already factored, so that in fact the non-factored form is put into the form of the other by means of factoring. The difference is that in the latter case a third expression is not involved, whereas in the regular variant the two expressions are rewritten into a third, factorized form that is really 'common'. The same holds for CAS technique 3, finding a common form by expanding. The CAS factor technique usually gives fully factored forms, whereas with paper and pencil students might end up with only partially factored forms. For the Expand technique, the paper-and-pencil variant is close to the CAS variant, though more sensitive to mistakes in more complex cases.

CAS technique 4 comes down to using the CAS for Automatic Simplification. We should note, however, that it is not always clear what is 'simple'. For this CAS technique, similar variants exist as for techniques 2 and 3 with respect to the obtaining of a common form with or without creating a third expression. The CAS Automatic Simplification does not inform on restrictions, which students have to consider themselves. For example, $(x^2 + 3x + 2)/(x + 1)$ will be automatically simplified as $x + 2$. In the definition of equivalence, this is addressed by defining equivalence on a set of admissible values. While working with paper and pencil, automatic simplification as such does not exist, and furthermore requires the ability to look globally at expressions and 'see' possible simplification maneuvers, such as grouping similar terms or canceling out common factors.

CAS technique 5 comes down to the CAS checking both sides of the equation for equivalence, by means of automatic simplification and other 'black-box' means. The CAS will come up with 'true' in cases of equivalence. Once more, restrictions are ignored. This technique has probably the most 'black-box' character, and the output it produces is the most difficult to interpret. This CAS technique was deliberately introduced into the design of the tasks so as to provoke student questioning of its output. Concerning the paper-and-pencil variant, the same remark as was made for technique 4 is applicable here.

CAS technique 6, solving the corresponding equation, involves a change of perspective. The notion of an equation solution is related to the issue of equivalence of expressions. If the equation expr1 = expr2 has infinitely many solutions, expr1 and expr2 are equivalent, whereas only ‘a few’ solutions indicates non-equivalence.

3.4 Theory

From an a priori perspective, we now ask how theory is involved in the tasks and the techniques presented thus far. We distinguish the following four main theoretical elements, which are intertwined and related to the techniques and the tasks.

1. *(Dis-)connecting the numerical and the algebraic*
   This issue concerns the differences and connections between the numerical world and the world of algebra. The algebraic world is rooted in the numerical world, but exceeds it and develops into a world of its own. This is a difficult issue throughout the teaching unit, which emerges initially while using technique 1, and reemerges with techniques 5 and 6.

2. *The notion of equivalence*
   Two notions of the equivalence of two expressions can be distinguished: an algebraic view as having a common form, and a numerical view on equivalence as having – always, in most cases, or even just in some cases – the same numerical output values. The latter view is related to the previous item, and is reflected in the language issue related to the words *equivalent* and *equal*. The algebraic view on equivalence appears in techniques 2, 3, and 4, and in a somewhat more complex way in technique 5.

3. *The issue of restrictions*
   The issue of restrictions on equivalence is an important theoretical aspect of the concept of equivalence. It involves both the particularities of the way the CAS deals with restrictions, and the somewhat strange definition – at least possibly strange in the eyes of the students – of equivalence involving a set of admissible values. The issue emerges particularly when techniques 2 to 6 are applied to expressions with restricted domains.

4. *Coordination of solving an equation and the notion of equivalence*
   The relation between solving an equation and the notion of equivalence of expressions, and between restrictions on equivalence and solutions of the equation, could be confusing for students. Both restrictions and solutions have a sense of ‘exceptions’, but in a kind of complementary way. This issue needs coordination, particularly of technique 6 with techniques 2 to 5.

3.5 Analysis of student data

In this section we present the results of the data analysis, which are organized according to the four theoretical elements mentioned above. As well, this analysis globally describes the development of students’ thinking over time. In particular, we illustrate the ways in which, during the process of developing techniques related to exploring the equivalence of expressions, students grappled with theoretical issues along the way. The reader is reminded that the data in this section are drawn from two grade 10 classes, one from the 2004 segment of the study and the other from 2005 (note, however, that students’ names have been changed).

3.5.1 (Dis-)connecting the numerical and the algebraic

We first encounter the issue of (dis-)connecting the numerical and the algebraic in the Numerical Substitution Task and its reflection question. The application of CAS technique 1, the numerical substitution using the 'with'-operator, was no problem for the students. In the two right-hand columns of the table (Figure 2), students themselves chose some values for substitution. This led to some surprising observations.

- Jacob deliberately chose –2, as he saw this as a problem for expr5. He seemed to want to test the calculator for the case of restrictions. Indeed, he got 'undef' as the answer. This suggests that some students find a challenge in exploring the 'borders' of the CAS capacities. We conjecture that only a minority of the students have this attitude; still, it is interesting to exploit it.
- John chose $x = \pi$, which resulted in 'nearly symbolic' answers such as $(\pi - 3) \cdot (4 \cdot \pi - 3)$ for the fourth expression $(-x+3)^2 + x(3x-9)$. This output bridges the gap between the numerical and the algebraic worlds. A next time, we might encourage students to make such substitutions.

These observations show that the CAS substitution in combination with the task led to interesting thoughts on numerical substitution.

After filling in the table, the following reflection task was posed: Based on your observations with regard to the results in the table (shown in Figure 2), what do you conjecture would happen if you extended the table to include other values of x? All students predicted the same values to appear for the equivalent expressions in case of a new substitution. Out of 32 (two absent), 13 mentioned the restriction $x = -2$ for expr5. The work of Léonie, shown in Figure 7, is representative of these latter responses.

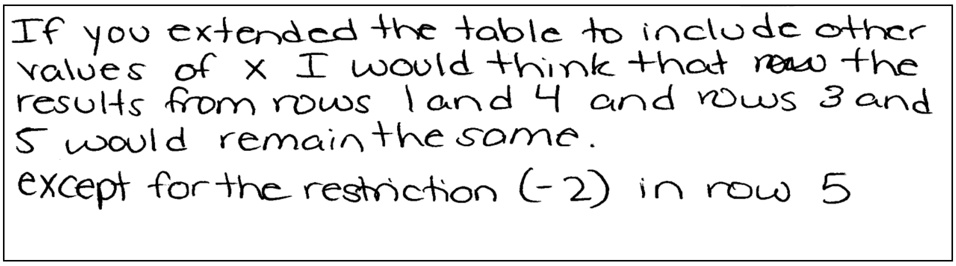

If you extended the table to include other values of x I would think that row the results from rows 1 and 4 and rows 3 and 5 would remain the same.
except for the restriction (-2) in row 5

Figure 7. Léonie's ideas on expanding the table

Some of the student worksheet formulations showed words like 'similar', 'equal', or 'even'; so students had some difficulty in using an appropriate terminology. We see this as a combination of a language issue and a not yet completely sharp view of the underlying conceptions.

A minority of the students seemed to have a sense of what is behind the equal numerical values and were moving towards the notion of algebraic equivalence. For example, Michel wrote down:

> "I believe that if I were to extend the table, #1 and #4 would give the same set of results, as well as #3 and #5 would give the same results. This is probably because when the expressions are simplified, they will come out to the same simplified expression."

We see the notion of common form emerging, particularly in the phrase "the same simplified expression". In spite of this feeling for the relation between the numeric and the algebraic, some students felt unsure about algebra providing certainty about numerical values, even if their algebraic skills were good. In fact, all through the teaching sequence, students referred to the numerical to check their algebraic work.

In all, students seemed to have an intuitive basis for the idea of equivalence as having always the same numerical value, even if this was sometimes expressed in an informal way. This theoretical notion was clearly supported by the CAS substitution technique, which makes numerical substitutions 'cheap' to carry out. The repeated substitution with the CAS confronted the students with a phenomenon of equal values, which invited algebraic generalization. Still, the relation between the algebraic and the numeric was somewhat vague. Even if the students were aware of the impossibility of checking all numerical values, the connection with the algebraic was not yet fully established. Let us have a closer look at these two sides of the notion of equivalence.

3.5.2 The notion of equivalence

The substitution of numerical values and the prediction of what would happen if new columns were added seemed to lead to the numerical view on equivalence of expressions as 'having the same output values for each of an infinite set of input values'. However, the protocol of a classroom discussion in the 2004 class shows that the *algebraic view* of 'having the same form' also came up (Figure 8).

Mark	The expressions are the same thing as the other ones, just in a different format.
Teacher	So you're saying that these pairs of expressions are exactly the same.
Mark	Equivalent representations of the same thing.
Teacher	What are you meaning by what you said?
Mark	They represent the same thing, they give you…like if you substitute an x, like it will come out to the same answer.
Teacher	Why is that the case?
Mark	Because they're just a different form, like they're an unfactored of a, uh, multiplication of two binomials, or something like that.

Figure 8. The notion of algebraic equivalence emerging

While the student refers to substitution of values when he has difficulty in expressing himself, it seems clear that he relates the notion of equivalence to having a similar algebraic form. However, the idea of *common form* turned out to be somewhat problematic in two senses. First, some of the students considered common form as a basic, simplified, or even 'simplest' form (Figure 9).

Interviewer	What does that mean to you "re-expressed in a common form"?
Andrew	Uuh, I think it's just a simplified common form.
Interviewer	This is a common form for what?
Andrew	For this expression.

Figure 9. Common form of one expression instead of two

The following extract (Figure 10) indicates that for many of those students, this simplified or basic form was the expanded form. Some considered the factored form to be common as well.

William	They're all different forms of the same basic expression.
Interviewer	Which one is the basic expression?
William	The expanded form.

Figure 10. Common form being the expanded form

In fact, the CAS Technique Task (Figure 3), with its Factor and Expand commands, may have partially provoked this notion. Also, the Automatic Simplification makes it particularly easy to get a transformed simplified expression even without explicitly asking for it. Furthermore, we conjecture that this difficulty is caused by the *ambiguous meaning* of the word 'common', which can refer both to ordinary or basic – which is not what we meant – and to shared. So we see here an interplay of a language issue and CAS techniques, which influences – and, compared to our teaching aims, can work against – the notion of algebraic equivalence.

A second complication in the notion of common form resulted from the first one. If 'common' is taken as 'basic', then indeed one algebraic expression can have *several common forms*, such as the factored and the expanded one, instead of our idea of two expressions being expressed in a common form. The last line of the above verbatim of Andrew (Figure 9) indicates that he perceived a common form to be common for one single expression, instead of a pair of expressions. Later in the teaching sequence, he changed his interpretation towards the idea of a shared form; however, arriving at this notion took quite some time, with much confusion about *common* being *ordinary* or *simplified*, and with uncertainty about the interchangeability of common form within three equivalent expressions.

Let us now consider in more detail how the available techniques linked up with the conceptual understanding of equivalence of expressions. The first technique on numerical substitution, of course, stresses the 'equal values' view of equivalence. The Factor, Expand and Automatic Simplification techniques are on a more algebraic level, but seem to foster the notion of common form as being a 'simple' form. The *Test of Equality* technique is probably the most interesting one from the conceptual point of view, as it seems to act at the borderline between the numeric and the algebraic. This technique provides 'true' in cases of equivalence, but just returns the (sometimes transformed) equation in other cases. The latter was difficult to understand for many students, as they would have expected something like 'false'; whereas returning an equation – two expressions with an equal sign in between – unjustly suggested equivalence to them (Figure 11).

Suzanne	Uhm, I entered the problem $(x^2+x-20)(3x^2+2x-1)=(3x-1)(x^2-x-2)(x+5)$ and it gave me pretty much the same problem back, but rearranged, it's the same answer. When you think that the other one said "true," it is kind of puzzling. ... The answer that it gave me. I figure that that's this statement, like the first expression equals the second expression is true. … When I see an equal sign, I figure they are equivalent, the same.
[…]	
Interviewer	How would you now interpret such a display when you enter in two expressions like that?
Suzanne	Uhm, that it can be right sometimes, but isn't always right. With specific numbers, it is correct.
Interviewer	So, when you mean correct?
Suzanne	That you would get the same number in the end on both sides. But only sometimes.
Interviewer	Only for some numbers.
Suzanne	Yah.
Interviewer	So how do you feel about that?
Suzanne	I'm still confused. With the "true"s and the "="s, to me it all has sort of the same meaning. I guess I just have to change my way of thinking.

Figure 11. Confusion about the CAS returning the equation for the case of non-equivalence

In spite of the confusion that Suzanne expresses in the last line, we appreciate that she takes it as an incentive to rethink her conceptions. In fact, that is what the tasks and techniques, if dealt with properly by the teacher, can provoke: a rethinking of the theoretical knowledge. The fact that the CAS just returned the equation for the case of non-equivalence enhanced classroom discussion.

A second issue that was related to the role of techniques in the evolution of students' thinking about equivalence concerned the *coordination of different techniques as a means to check consistency*. In several cases, students used different techniques, both paper-and-pencil and CAS, to verify the consistency of their theorizing. Surprising CAS results in some cases gave rise to conflicts that invited reasoning. For example, at first Andrew was puzzled when the CAS simplified $(2-x)(1-2x)$ as $(x-2)(2x-1)$. After some thinking about this, he found an explanation:

> "I think since it's switching them both that it works out. Let's just say *x* was represented by 6, -4 times -11, which is 44. And the other one it's 6 - 2, which is 4 times 11, which is also 44. It's just two negatives, since it's switching both of them it's OK."

By the way, this verbatim once more shows the students' returning to the numerical to check algebraic relations, which is not a bad habit of course. Still, when so asked, Andrew indicated that he had several means to check algebraically the equivalence of $(2-x)(1-2x)$ and $(x-2)(2x-1)$, such as entering the corresponding equation or expanding them both. The students also used these CAS techniques to check their consistency with by-hand results.

To provide a more global view on the meaning of equivalence for the students of both classes, we look at one of the posttest items: What does it mean to say 'two algebraic expressions are equivalent?' Table 1 summarizes the results. The student responses are coded according to whether they referred to either numerical substitution and/or common form. This does not mean that all responses were fully correct; particularly, some students expressed their ideas in a somewhat vague manner. As an example of that, Bryan's answer was:

> "When a number replaces the *x*, both expressions will end up having the same answer making them equivalent."

In his answer, it is not clear if he means 'all numbers' or 'one number' when he writes 'a number'. Furthermore, he does not mention restrictions, and it is not clear if he thinks of the output as being one and the same number for all input values, or only the same for each input value.

However, most answers were well formulated. The answer given by Andrew, who referred both to common form and numerical substitution, taking into account the restrictions, is exemplary of many of the correct responses:

> "This means that if the variable (e.g., *x*) is replaced by any value the equation would be true. There will always be a common form for these 2 expressions. However when division is involved there is a possibility of restrictions."

The posttest results indicate that the numerical substitution view on equivalence dominates. We conjecture that students chose 'the safer way', which is the numerical view, as the algebraic view suffered from the complications of both the idea of common form and some peculiarities in the CAS techniques explained above.

Table 1. Posttest results on the meaning of equivalence

		Refer to numerical substitution?				
		YES		NO		
		2004	2005	2004	2005	
Refer to common form?	YES	3	0	4	7	**14**
	NO	8	10	1	0	**19**
		21		**12**		**33**

In all, we conclude that the combination of tasks and CAS techniques fostered the extension of the notion of equivalence to include an algebraic view. Still, some slippage between both views on equivalence was noted. In the posttest, students mostly referred to the numerical view on equivalence, probably because this was the first notion encountered, or because of the complications with common form and the Test of Equality technique. However, we would like to stress that these complications should not be seen as resulting from CAS anomalies, but rather as conflicts that arise while coordinating CAS techniques and theoretical thinking within the intended design of the tasks. The confrontation of theoretical expectations with CAS output turned out to be productive of further reflection. An additional complication, however, while using the CAS techniques, was the neglecting of restrictions, which is the topic of the next section.

3.5.3 The issue of restrictions

The question of how to deal with restrictions, both with CAS and paper-and-pencil techniques, played a role in the algebraic view on equivalence. The definition of equivalence, provided in the teaching materials, speaks about equivalence on a set of admissible values. This raises the issue of restrictions.

The reflection question after the CAS Technique Task (Figure 3) was: Is this equivalence subject to any constraints on admissible values of x? This led to the following discussion in the 2004 class, which, of course, focused on expr1 and expr4 being equivalent, with restriction $x = -2$ (Figure 12).

Teacher	So what happened is that it seems that you are in agreement that the expressions are equivalent, but is there something else we should say?
Maureen	There's a restriction on the second equation, so it wouldn't work. If you plug in -2 on the first equation, it would come out with an answer, a value, but it wouldn't work for the second one.

Figure 12. Restriction related to numerical substitution

This indicates that the students linked the notion of restrictions to the numerical view on equivalence, which makes sense. They became aware that restrictions should be taken care of, and the next question was how to identify them. The following fragments from a classroom discussion indicate that the students were able to formulate this (Figure 13).

Teacher	What is the restriction, what does it mean?
Alex	x can't equal –2.
Teacher	What does it mean, why is that a restriction?
Alex	Because you can't divide by zero.

Figure 13. Why does a restriction occur?

Still, individual students struggled with the restrictions. A first problem some of them encountered concerned *dividing by zero*. This was particularly the case for Andrew when he was thinking about the equivalence of the third and fifth expressions in the Numerical Substitution Task (Figure 2). Let us look at his approach in more detail. At first, Andrew had difficulties with identifying the restriction of $x = -2$. The question to consider the denominator revealed a misconception concerning dividing by zero:

> "If x were -2 then the denominator would be -2 plus 2, which is zero and anything over zero is equal to zero. One over zero equals to zero."

The interviewer went on with this, which led to the conclusion that the result of a division by zero is undefined. In the next lesson, the equivalence of expr3 and expr5 (Figure 2) was considered once more. A new doubt arises in Andrew:

> "But it could be that somehow if –2 is incorporated here [expr3], this is going to be zero too, so it could be that my rule isn't necessarily correct, that the -2 wouldn't work."

Andrew was now puzzled about the possibility of another zero appearing somewhere. To check this out, he substituted $x = -2$ into expr3 and got –84 as a result, clearly not zero. So, he concluded: "Basically, it will work with everything except the –2." Then he substituted $x = -2$ into the expanded form of expr3, which of course gave -84 once more. This seemed to be a check for consistency, although he was not completely sure about what to expect. Then Andrew wondered about the value of expr5 when $x = -2$ would be substituted. He expected -84, but the calculator displayed 'undefined'. He explained this as follows:

> "That's what I figured out that it should be, undefined, but I didn't think the calculator would show it. Just based on all the other results, just based on the fact that this came out to -84, and this came out to –84."
> (…)
> "Well like it substitutes it and then it fills everything in and anything divided by zero is undefined, no matter what the equation is on top, it's still divided by –2 plus 2, so it's undefined."

Andrew's behavior provides a good illustration of the difficulties some students had with dividing by zero, and with zero divided by zero. The latter part of the verbatim extract also shows how Andrew was trying to coordinate his theoretical thinking with the way CAS deals with the restrictions.

This brings us to the second problem concerning restrictions: as Figure 14 shows, the *CAS equivalence techniques neglect restrictions*. This is something most students did not appreciate (Figure 15).

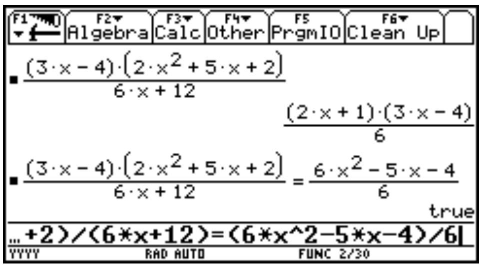


Figure 14. The TI-92 neglecting restrictions

Carey	It [the calculator] doesn't seem to account for any restrictions.
Interviewer	OK, so are we happy with that?
Carey	No, because then it is misleading, because you think there's no restriction and for any value they're equal to each other.

Figure 15. Unhappy with the CAS neglecting restrictions

The idea that not showing restrictions was a constraint of the CAS was quite persistent among the students. Meanwhile, this limitation urged the students to take care of restrictions themselves. In fact, this is similar to working with paper and pencil, where there is no red light flashing when, for example, one multiplies by a factor that might be zero.

The complications of restrictions led most students to *avoid* them. For example, in the Construction Task (Figure 4) students were asked to construct an equation made from two equivalent expressions. Out of the 26 students of both classes (some were absent), 24 came up with examples in which restrictions did not play a role. Only two examples contained restrictions:

a. $\frac{1}{x} = x^{-1}$

b. and a quite complicated one: $(3x+7)(2x^2+x-15) = \frac{(6x^2-x-35)(x^2+7x+12)}{x+4}$

A similar observation is made for the posttest. All students but one out of 33 were able to construct an equation formed from two equivalent expressions without restrictions. Examples of equations formed from two equivalent expressions with exactly one restriction were much harder to generate: only 19 out of 33 were able to provide a correct pair of expressions. Among the correct answers some were complicated, such as the one written down by Laura:

$$\frac{11}{x-2} + 4 = \frac{4x+3}{(x-2)} \qquad x \neq 2$$

Among the incorrect answers, we find equations that had more than one restriction, such as $\frac{1}{x+1} = \frac{x-1}{x^2-1}$, and answers that do involve one domain restriction, but where the equivalence is not taken care of, such as $x^2 - 25 = \frac{x-2}{x+2}$.

In all, we notice that the notion of restrictions in relation to equivalence was not easy to grasp. The confusion was evoked by the tasks, which involved expressions with restrictions, by the definition of equivalence, which spoke about the set of admissible values, and by the fact that the CAS techniques neglect the restrictions. In particular, the Test of Equality technique was confusing: if there were restrictions, it provided 'true', whereas the numerical substitution of the restriction provided 'false'. Furthermore, in the case of non-equivalence, the CAS did not reply 'false' but just returned the equation, as we discussed in the previous section. Thus, once again, the task-technique combination of the teaching sequence and theoretical thinking were confronted with each other, with a growing awareness of the issue of restrictions as a result. As a side effect, the notion of dividing by zero was revisited. The issue of restrictions was encountered once more in the final part of the teaching sequence, when the relation with equation solutions was investigated.

3.5.4 Coordination of solving an equation and the notion of equivalence

Early in the 2005 experiment, while discussing the definition of equivalence, one of the students came up with the idea of equivalence meaning equality for *some* values, namely the solutions of the corresponding equation (Figure 16).

Ron	I'd define it [equivalence] as an equation where values of x exist that will make both sides equal to each other.
Teacher	How many values of x?
Ron	At least one, one or more.
As an example, Ron suggests x+2 and x/2.	
Teacher	x+2 and x/2 are equivalent?
Ron	Could be.
Teacher	Who agrees that these two are equivalent?

Judy	Only for some values.
Teacher	And does that make them equivalent?
Ron	I think yes.
Daniel	I think equivalence is more like all values of *x* work except certain restrictions rather than no values work except certain restrictions.
Teacher	OK, I think that's where we're moving towards.

Figure 16. Equivalence defined on a finite (!) set of admissible values

It should be noted that Ron's suggestion is not a bad one at all, if one takes equivalence as 'having equal output values on a set of input values'. Still, this interpretation is on a numerical and not on an algebraic level. In the classroom discussion, Daniel's comment at the end of the fragment was decisive. Interestingly, he used 'restrictions' in two senses – the first in regard to restrictions on the equivalence; the second in regard to exceptions to the non-equality of the two expressions, that is, the solution(s). Speaking in general, even if students had a correct understanding of equivalence, they often mixed up the words 'equal' and 'equivalent'.

The Construction Task (Figure 4) addressed the coordination of equivalence with the solution(s) to an equation in more detail. After the students constructed pairs of equivalent expressions, the following question was posed: Without solving it, what can you say about the solutions of this equation? Table 2 summarizes the results.

Table 2. What to say about the solution of the equation formed from equivalent expressions?

Type of answer n=25	2004 class	2005 class	Total
All values for *x* are solutions	4	11	**15**
All values for *x* can be substituted	2	0	**2**
True / equal / equivalent	6	2	**8**
Total	**12**	**13**	**25**

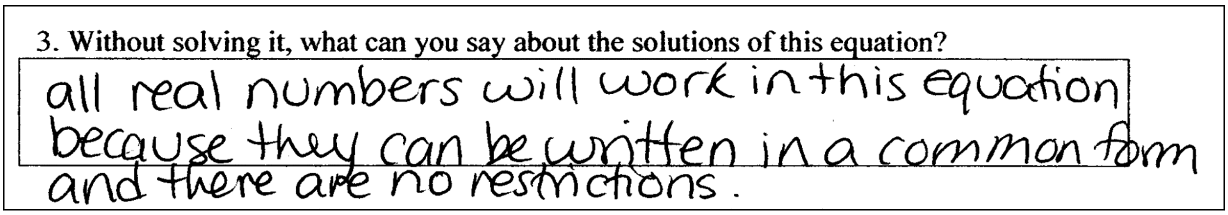
In the first category of answers, students claim that all values of *x* are solutions, or all real numbers, or R. A representative example is Léonie's answer, displayed in Figure 17. In the second category, students refer to numerical substitution and not to solving an equation. Still, the responses suggest insight into what is happening. An example here is Laura's answer:

"Plug in any value for *x* so solutions will be equal."

In the third category, we find a diversity of answers using words like true, equal, equivalent, without explicit reference to solutions of the equation. For example, John wrote:

"They are equal except when $x = -4$ because the restrictions are $x = -4$"

Even if John probably had a good understanding of equivalence, his way of expressing the relationship with the solution of equations is not very clear. We conjecture that by, "they are equal," he means that both sides are equal for all values of *x*, so all values are solutions; but that is an optimistic interpretation.


3. Without solving it, what can you say about the solutions of this equation?

all real numbers will work in this equation because they can be written in a common form and there are no restrictions.

Figure 17. Léonie's answer concerning solutions to equations formed from equivalent expressions

The Construction Task results indicate that the coordination of the notion of equivalence with solving the corresponding equation was taking time to be well established for half of the students of those present from the 2004 class. Complicating the issue of the theoretical coordination of these ideas was the additional issue of students' *struggling with language,* especially with the words *solve* and *solution*. For example, in response to the question of what it means to say that the two given values are solutions of the equation, a representative answer was:

> "It means that when the two values are substituted for *x*, both sides of the equation when solved will be the same"

This student probably means by 'solved' something like 'evaluated' or 'calculated'. The following quote from Andrew, referring to simplified and non-simplified expressions reveals the same conception:

> "Well this one, this one isn't solved yet, they're both not solved yet. Yeah, just the last one, which is a common form."

The word 'solution' was equally problematic. In the extract provided in Figure 18, Andrew answers questions on solutions with arguments on equivalence. At the end, however, he comes to see which sense of *solution* is intended and responds that the number of solutions is infinite for the case of equivalence.

Interviewer	How does that help you answer this question, to find the solution?
Andrew	When they're factored they're, they look the same, they are equivalent.
Interviewer	OK, so you just said something about equivalence, but how does that help you find the solutions to that equation is what I'm asking.
Andrew	Oh, I don't understand.
Interviewer	I'm probably confusing you. What does it mean for a number to be a solution to an equation?
Andrew	That if *x* is replaced by any value, or that certain value, it will make the expressions equal.
Interviewer	So to relate that back to what you just described a few minutes ago.
Andrew	Oh, so once they're factored out they are equal to each other.
Interviewer	I see. How many solutions are there to this equation?
Andrew	Uh, an infinite number.

Figure 18. Struggling with the word 'solution'

Still, most students seemed able to derive enough contextual clues so as to know whether equation solutions or some other solutions were being referred to in a given question.

Despite students' struggles with language issues, the combination of tasks and techniques provoked the development of theoretical links among equivalent/non-equivalent expressions, equations, and equation solutions for a majority of the students, even if not for all of them. In the Posttest Task Q5 (Figure 5), which began with the question regarding the meaning of the two given solutions for the provided equation, all but one of the 33 students writing the posttest gave a correct answer; 6 of these students (all from the 2004 class) also indicated that this did not imply equivalence. As for item (ii) of Q5 – use of a CAS technique to check whether there were solutions other than the two that were proposed – 19 out of 33 used the Solve technique; the others used less appropriate techniques such as numerical substitution – which does not show that there are no other solutions – and the Test of Equality, or did not answer at all (3 students). Item (iii) of the Posttest Task Q5 concerned the relation between solutions of equations and equivalence of expressions. Out of 33 students, 24 correctly indicated that the equality was only true for the two solutions, which in 18 cases led to the correct answer, namely non-equivalence of the left- and right-hand expressions of the equation. Six others were struggling with the idea of 'equivalence for some values', stating that the expressions were equivalent for the two solutions and not for other values. From the 24 more or less correct answers, 5 students referred only to common form. Some of the students referred to both common form and solutions in their answers: "They are not equivalent as only when 2 and 2/3 are plugged in as values of *x* are the expressions equal. They cannot be put into common form."

To conclude this discussion of the issue of coordinating solutions and equivalent expressions, we recognize that the results were, to a certain extent, mixed. The combination of different CAS techniques as they were proposed in the tasks confronted the students with theoretical issues. Some students were really able to relate the set of solutions of the equation to the notion of equivalence of the two expressions involved, whereas this remained fuzzy for others. While the Solve technique in itself was not a problem for the students, its coordination with techniques 2 to 5 on equivalence required a change of perspective, which was not easy for them. The fact that some of the students did not refer to the solutions in the third item of the Posttest Task Q5 suggests that they preferred to stick to their numerical and/or common form views of equivalence. Evidence suggests that a language issue was involved here as well -- particularly the fact that students use the word 'solve' for any operation leading to a result, the result being called the 'solution.'

3.6 Synthesis on the theme of equivalence, equality, and equation

If we consider our findings on the theme of equivalence, equality, and equation in retrospective, two main issues come to the fore: the relation between the students' theoretical thinking and the techniques they used for solving the proposed tasks, and the specific role of the confrontation of CAS output with the students' expectations.

To elaborate on the first point, our findings suggest that the importance of the relation between Theory and Technique, as it is established while working on appropriate Tasks, can hardly be overestimated. On the one hand, the development of the students' theoretical thinking was guided by the techniques that the tasks invited; on the other hand, the students' conceptions influenced the development of these techniques. More specifically in this theme, the students' numerical view on equivalence of expressions was found to be related to three techniques: the numerical substitution technique, and, to a lesser extent, the Test of Equality and the Solve technique. The fact that students seemed to consider the numerical view of equivalence as the more important one (Table 1) links up with their use of the numerical substitution technique to check equality. In the emergence of the algebraic view on equivalence, the CAS techniques Factor, Expand, Automatic Simplification and Test of Equality played an important role, even to such an extent that the factored and expanded forms seemed to be considered as common forms (Figures 9 and 10). Finally, for the coordination of the numeric and the algebraic views on equivalence, the Solve technique turned out to be provocative. While students had difficulties with the coordination of the Solve technique and the techniques on equivalence, the discussion of these techniques turned out to be quite productive for the development of students' theoretical thinking (Figure 17). In short, all through the teaching sequence the co-emergence of techniques and theoretical understanding was observed as being an important issue in the learning process.

To elaborate on the second point, the data analysis revealed that a most productive form of learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations. The students' seeking for consistency evoked theoretical thinking and further experimentation. This phenomenon primarily concerned the algebraic view of equivalence. For example, the fact that the available CAS techniques easily provide factored and expanded forms may have contributed to the idea that common form meant simplified, basic form, which came down to factored or expanded form (Figures 9 and 10). This conflicted with the notion of common form as it was introduced in the student materials. Also, the fact that the CAS Automatic Simplification technique and the Test of Equality both neglect restrictions led to an increasing awareness of the importance of these 'exceptions' (Figures 12, 13 and 15). Finally, the CAS just returning an equation in cases of non-equivalence struck the students, and gave rise to interesting discussions on the interpretation of the output, as did the interpretation of 'true' and 'false' in cases of numeric or algebraic application of the Test of Equality (Figure11). Even if such complications in applying CAS techniques may seem to be hindrances to students' progress (see also Drijvers, 2002), in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these complications to foster learning is their appropriate management in the classroom by the teacher.

All in all, the teaching sequence revealed a strong interaction among task, technique, and theory, in which the CAS use – sometimes in combination with language issues – resulted in extra complications, which led to interesting and enriching thoughts and discussions, all of which fostered conceptual growth.

4 The theme of generalizing and proving within factoring

4.1 Aims of the teaching sequence

The activity that exemplifies this theme is inspired by the work of Mounier and Aldon (1996) who presented to their classes of 16- to 18-year-old students the task of conjecturing and proving general factorizations[2] of $x^n - 1$. While access to CAS improved students' explorations of various factorizations for given integral values of *n*, they had difficulty in moving toward an awareness of the general regularities envisaged in the task. Lagrange (2000) has pointed out that the fact that the CAS produces complete factorizations was working at odds with the mathematical aims of Mounier and Aldon. In contrast, our research group decided to use this particular CAS phenomenon to provoke in the 15-year-old student-participants of our study a confrontation with their existing, but limited, theoretical thinking on factoring. The intended overall aim of the two-lesson teaching sequence was thus to develop in students not only the notion of a general form of factorization for $x^n - 1$, but also its relation to the complete factorization of particular cases, as well as to initiate them to the proving of one of these cases.

4.2 Task

We set about designing the activity in three parts (see Figure 19 for an overview). The first part, which involved CAS as well as paper and pencil (P&P), linked students' past experience with factoring to the generalization that they would be working towards regarding the factoring of $x^n - 1$. The initial set of tasks was oriented towards noticing a particular regularity in the

[2] Some general factorizations of $x^n - 1$ are the following: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + x + 1)$ for positive integers n; $x^n - 1 = (x-1)(x+1)(x^2+1)...(x^{n/2}+1)$ for every n that is a power of 2.

factored examples of the $x^n - 1$ family of polynomials for positive integral values of *n* and the justification of the form of these products. As is illustrated by the sample questions provided in Figure 20 (for the complete set of tasks, see Kieran and Saldanha, in press), the task also aimed at promoting an awareness of the presence of the factor $x - 1$ in the given factored forms of the expressions $x^2 - 1$, $x^3 - 1$, $x^4 - 1$, $x^5 - 1$, and $x^{16} - 1$. To promote *generalization* of the form $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + x + 1)$, students were then asked their opinion concerning the validity of the equality presented in Question 6. After students began to conjecture a general rule for the factorization of the $x^n - 1$ family, they were requested to reflect on how they might express this conjecture by means of symbolic notation, using the symbol *n* for the exponent, rather than specific integers.

	Activity on Factoring	Tools
Part I	Seeing patterns in factors and moving toward a generalization	CAS/P&P
Part II	Refining a generalization – with conjecturing and reconciling	CAS/P&P
Part III	Proving	P&P (mostly)/CAS

Figure 19. Outline of the teaching unit

1. Perform the indicated operations: $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product
   $(x-1)\left(x^3 + x^2 + x + 1\right) =$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ?
   $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x-1)\left(x^3 + x^2 + x + 1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. Is your explanation valid for the following equality:
   $(x-1)(x^{134} + x^{133} + x^{132} + ... + x^2 + x + 1) = x^{135} - 1$? Explain.

Figure 20. Some of the initial tasks of the activity

The next section of the activity engaged students in *confronting* the paper-and-pencil factorizations that they produced for *n* from 2 to 13 with the completely factored forms produced by the CAS, and in *reconciling* these two factorizations (see Figure 21; note that the expressions for which *n* varies from 7 to 13 are not shown). We conjectured that students would begin this part of the activity by applying the general rule that they had just formulated. The confrontation of their paper-and-pencil factoring with the factors produced by the CAS was intended to encourage students to reflect upon their existing notions of factoring with respect to, among others: (i) complete factoring, and (ii) the impact of the nature of the exponent on the ways in which the factoring process can be approached, as well as on the final form of the factors.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary

$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 21. Task in which students confront the completely factored forms produced by the CAS

In order to have students reflect upon the relations between particular expressions of the $x^n - 1$ family and their completely factored forms – as suggested by the entries in the table they had just been completing – they were asked to generate conjectures regarding the nature of some of these relations (Figure 22).

> Conjecture, in general, for what numbers *n* will the factorization of $x^n - 1$:
> i) contain exactly two factors?
> ii) contain more than two factors?
> iii) include $(x+1)$ as a factor?
> Please explain.

Figure 22. Task in which students examine more closely the nature of the factors produced by the CAS

The final part of the activity centered on students' further technical and theoretical development with a task on *proving* the conjecture that $x+1$ is always a factor of $x^n - 1$ for even values of *n*. Access to the students' reflections was facilitated by having them present their proofs at the board, and by encouraging classroom discussion, query, and reaction.

4.3 Technique

Figure 23 provides an inventory of the main techniques that could be used – both with CAS and with paper and pencil – on the above tasks of generalizing and proving within factoring.

Technique	**CAS variant (using TI-92 Plus)**	**Paper-and-Pencil variant**
1. Expanding an expression completely	Expand command	Expanding all of the expression by hand, combining like terms, and ordering final terms
2. Expanding a sub-expression	Expand command, using as argument the desired part of the expression	Expanding, by hand, usually two factors of the given expression
3. Factoring completely an expression (if factorable)	Factor command	Factoring by hand, often with a choice of several possible methods
4. Factoring a sub-expression	Factor command, using as argument the desired part of the expression. The CAS may not always succeed in this regard.	Factoring, by hand, a particular factor of a given expression, often with a choice of methods possible
5. Using symbolic notation to express the general factored form of an expression	No command exists that will factor an expression whose exponents are expressed in general symbolic form.	Factoring the general form, by hand, with appropriate symbolism for the exponents

Figure 23. CAS and paper-and-pencil techniques for this activity

Some comments are in order regarding these techniques. First, the relation between the above CAS techniques and the corresponding paper-and-pencil techniques would appear to be very close in most cases. But, indeed, they are not that close, as will be seen shortly in the student work dealing with the reconciling of the results of both. While the CAS uses a black-box method, several variants can be available for paper-and-pencil methods. For example, in those expressions of the $x^n - 1$ family of polynomials where the exponent has three or more divisors, an assortment of paper-and-pencil factoring methods can be used to achieve complete factorization; for instance, $x^6 - 1$ can be approached as a difference of squares $(x^3)^2 - 1$, a difference of cubes $(x^2)^3 - 1$, or according to the general rule $(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$ followed by further factoring of the terms in the second factor by grouping.[3]

Second, the task of reconciling paper-and-pencil and CAS factors can involve a combination of the techniques mentioned in Figure 23:

i) Using the CAS command Expand (or paper and pencil) to multiply various CAS factors in order to obtain some of the factors that were derived from an initial paper-and-pencil technique that involved the general rule;

ii) Refactoring with paper and pencil the given expression to obtain the CAS factors (e.g., treating the given expression as a difference of squares if the exponent is divisible by 2, or as a difference of cubes if the exponent is divisible by 3);

iii) Refactoring with paper and pencil (or CAS) the not-yet-completely factored part of the paper-and-pencil work in order to obtain the factors produced by the CAS (e.g., ‘factoring by grouping’).

Third, although the CAS (in our case, the TI-92 Plus) can be used to expand any given partially-factored expression, it cannot necessarily undo this expansion by factoring. For example, the CAS factors completely the expression $x^{10} - 1$ as

$$(x-1)(x+1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1).$$

However, if one factors with paper and pencil the expression $x^{10} - 1$ as

$$(x-1)(x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1),$$

the CAS cannot refactor completely the expression $(x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$. It simply produces $(x+1)(x^8 + x^6 + x^4 + x^2 + 1)$ as is illustrated in Figure 24.

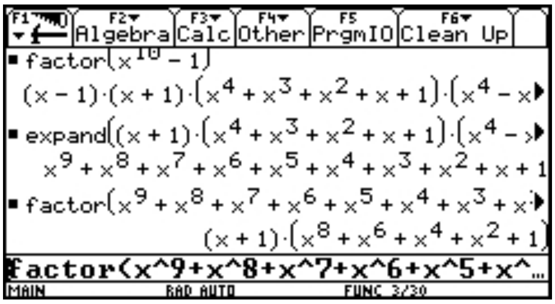


Figure 24. Factoring $x^{10} - 1$ with the TI-92 Plus

---

[3] Factoring a polynomial expression such as $x^5 + x^4 + x^3 + x^2 + x + 1$ by ‘grouping’ involves arranging the terms in groups so as to enable factoring: e.g., $(x^5 + x^4) + (x^3 + x^2) + (x + 1) = x^4(x+1) + x^2(x+1) + 1(x+1) = (x+1)(x^4 + x^2 + 1) = (x+1)(x^2 + x + 1)(x^2 - x + 1)$. There may be more than one way to group terms.

Lastly, we wish to point out that this teaching sequence calls upon a kind of symbol sense (Arcavi, 1994) and global, meta-level processes (Kieran, 1996) that do not involve techniques as such, and for which a given CAS or paper-and-pencil technique cannot be specified. Examples of such processes include looking for patterns, perceiving different ways of structuring a given expression, conjecturing, predicting, and so on. However, CAS and paper-and-pencil techniques can be used to test the objects of these mental processes, be they conjectures, or predictions, or others, by for example generating a multiplicity of examples.

4.4 Theory

According to the TTT framework that is the integrating thread of this research, a student's mathematical theorizing is deemed to be intertwined with the techniques that are used with, and that co-emerge within, the given tasks. Thus, for the tasks and techniques presented above, we distinguish a priori the following three theoretical elements.

1. *Patterns in the factors of $x^n - 1$: Seeing a general form and expressing it symbolically*
   While techniques 1 and 3 (Figure 23) are used in testing the early conjectures related to factoring and expanding a few expressions of the $x^n - 1$ family of polynomials, the technical work is superceded by the processes used in recognizing patterns and in generalizing them. However, expressing a general factorization for $x^n - 1$ with symbolic notation goes beyond the use of these processes. Technique 5, a technique for representing the general factorization $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + x + 1)$ involves knowing, at least, how to express decreasing powers of the exponent *n*, that these decreasing powers continue to $n - n$, that x^{n-n} is equivalent to x^0, that x^0 is equivalent to 1, and that the ellipsis symbol ('…') is used to indicate the undefined middle terms of the second factor.

2. *Complete factorization: Developing awareness of the role played by the exponent in $x^n - 1$*
   The notion of complete factorization can come to the fore as soon as students attempt to factor an expression with a non-prime even exponent, such as $x^4 - 1$, according to the general rule, and are confronted with a CAS factorization that they do not anticipate. The technical work of reconciling paper-and-pencil factors with CAS factors is considered by us to be an important part of the process of extending students' theoretical views on factoring. The refinement of student thinking with respect to the factoring of the $x^n - 1$ family includes a focus on the nature of the exponent and how it bears on the form of the completely factored expression. Another issue concerns the developing awareness that certain exponents can be viewed structurally in different ways and that the expression can thus be factored in different ways, but that the final CAS factorization is complete. Reflection on these issues is related to techniques 1 to 4.

3. *Proving: Theorizing more deeply on the factorization of $x^n - 1$*
   The notion that one can actually go about proving some of the empirical observations made during this factoring activity is an important theoretical development in students' mathematical thinking. Provision for this notion is offered by the task on proving that $x + 1$ is always a factor of $x^n - 1$ for even values of *n*, and is supported by techniques 1 to 5. Proving this conjecture involves theorizing about and coordinating several ideas,

including those related to odd, even, and prime exponents, as well as those that concern the forms of the completely factored expressions of the $x^n - 1$ family of polynomials.

4.5 Analysis of student data

While the three theoretical elements outlined above provide the framework for this section, the analysis is presented in such a way as to permit a view of the emergence of students' notions as they evolved over time, in interaction with the techniques invited by the tasks. It is noted that the analysis in this section is drawn from data collected in one of the 10th grade classes featured in the previous section, that of the 2004 class.

4.5.1 Patterns in the factors of $x^n - 1$: Seeing a general form and expressing it symbolically

Students' prior work with factoring expressions related to $x^n - 1$ had focused almost exclusively on the difference of squares, $a^2 - b^2 = (a+b)(a-b)$, and the sum and difference of cubes, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$. At the outset of the activity, they experienced little difficulty in performing the indicated operations for $(x-1)(x+1)$ and for $(x-1)(x^2+x+1)$, and in using the patterns suggested by these factored forms to anticipate the product of $(x-1)(x^3+x^2+x+1)$. The latter was verified with both paper and pencil and CAS.

The students described the patterns that they were noticing in the factored expressions, $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$, using language such as, " The first brackets all consist of $x-1$; the second brackets though increase by an x with one more power than the previous x." They also justified the fact that the expansion of all of these factored expressions produced a binomial: "Because the middle terms cancel out which creates binomials" -- such justifications resulting from their paper-and-pencil expanding techniques. The fact that they were beginning to see a general rule as a result of this patterning work was quite evident from their prediction of the factorization of $x^5 - 1$ as $(x-1)(x^4+x^3+x^2+x+1)$.

However, when asked to extend their reasoning to expressions containing higher exponents, such as $x^{135} - 1$, some students expressed a need to check their generalization with the CAS (Figure 25).

Laura	Can we check it [the factorization of x^{135} - 1] with the calculator?
Teacher	You could do it, but you don't need to.
Laura	But how are we supposed to know if it's valid or not? We can assume it is, but we don't know for sure.
Teacher	We can, by just the same reasoning as before.

Figure 25. For $x^{135} - 1$, this student wanted to check her generalization by using the CAS

While students had experienced little difficulty in arriving at a generalization of the factoring pattern with which they were working, a significant obstacle presented itself when the teacher initiated a discussion on the symbolic representation of this general form of factorization for the expression $x^n - 1$ with the question: "Can we now find a way of expressing a general factorization of the expression $x^n - 1$ for integral values of n?"

When the teacher wrote $x^n - 1$ on the board, and reiterated his question with: "Can we factor that?", students immediately wondered about the use of the variable *n* rather than a specific integer (Figure 26).

Ellen	Is that an *n*?
Teacher	That's an *n*.
Some students	No, we do not know what *n* is.
Teacher	Well, *n* is any integer, any positive integer.

Figure 26. An obstacle in dealing with a general formulation of the pattern: the use of *n* for the exponent

As students began to consider how they might express the pattern they had been observing with various numerical values for the exponent in $x^n - 1$, one student offered: "$(x-1)(x^{n-1} + x^{n-2}$", but then suddenly stopped and remarked, "But I don't know how far to go." This was echoed by several others in the class. The teacher then suggested: "Go down to $x+1$, as with the others," while he continued to write at the board: $\ldots + x + 1)$. This provoked immediate confusion in the class, with several students wanting to speak at once (see Figures 27 and 28).

Ellen	Shouldn't it be *n* plus one for the first one? 'Cause you know you're multiplying the *x*
Teacher	For this one? [Points to first term of second factor: x^{n-1}]
Ellen	Yeah.
Teacher	Let's just look at how we do this. We're doing *x* [Draws a red line starting under the *x* and linking it with the x^{n-1} in the second bracket, see next Figure]. We do that, what do we get?
Ellen	*x* to the *n* minus one.
Teacher	So, *x* to the *n* [Writes on board: x^n]. *x* times *x* to the *n* minus one is *x* to the *n*. [Pauses] Yeah?
Class	[Expressions like:] Yeah, Oooh, Ahhh, Yeah.
Teacher	You should add the exponents, *n* minus one, plus one is *n*. So you get that. Now look at the next one, if I do this [Draws a red dotted curved line linking the *x* with x^{n-2}, see next Figure].

Figure 27. Students' difficulty with using general notation for the factoring of $x^n - 1$

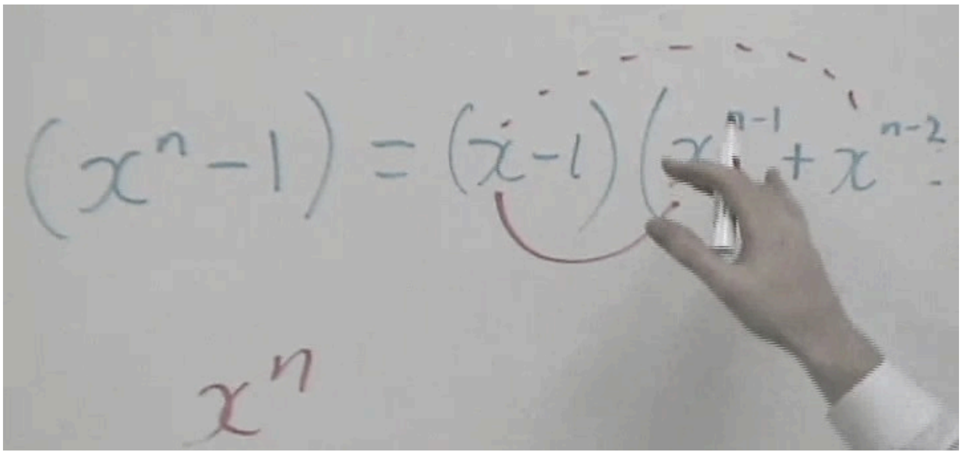


Figure 28. Teacher at the board, explaining the mechanics of the general factored form for $x^n - 1$

On examining the general factored form for $x^n - 1$, which the teacher had been writing and explaining at the board, another student asked once again: "If it's decreasing, how far down do you go?" He was also somewhat mystified by the use of the ellipsis symbol in the same second factor, as suggested by his remark, "I don't like the dots either; I don't think it's a real answer."

All in all, the students seemed to arrive quite easily at a generalization regarding the pattern suggested by the factoring examples of this first set of tasks. Their CAS use served to confirm their initial conjectures and provide support, whenever needed, for those expressions involving quite large exponents. However, the hardships the students experienced with making sense of the symbolic formulation for the general factorization for $x^n - 1$ point to the difficulties that are inherent in such polynomial notation involving non-numerical exponents for students of this age range and algebraic background.

4.5.2 Complete factorization: Developing awareness of the role played by the exponent in $x^n - 1$

After students had encountered the formulation of a general factorization for $x^n - 1$ for integral values of *n*, they moved into Part II of the Activity, which led to a confrontation with their existing ideas on factoring. Their first surprise arrived when they entered Factor $(x^4 - 1)$ into their CAS, which yielded $(x-1)(x+1)(x^2+1)$, in contrast with $(x-1)\left(x^3+x^2+x+1\right)$, which all of them had written for their paper-and-pencil version. It did not take long before students could be heard commenting, “it can be factored further,” “it’s not completely factored,” “it gives you all the factors,” and so on.

To reconcile the CAS factors with their own paper-and-pencil factors for $x^4 - 1$, students did the following:

i) Multiplied the second and third CAS factors to produce their second paper-and-pencil factor (Figure 29a, half the students),

ii) Factored by ‘grouping’ the second paper-and-pencil factor to produce the second and third CAS factors (Figure 29b, a little fewer than half the students), and

iii) Refactored the given $x^4 - 1$ as a difference of squares (Figure 29c, one student).

For some students, the first of the three methods of reconciliation had initially been carried out with the CAS (using Expand) and then transferred to paper. However, most of the reconciliation work was done with paper and pencil, as had been requested by the teacher.

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1= (x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1= (x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

Figure 29a. Reconciling by multiplying the second and third CAS factors of $x^4 - 1$ to produce the second paper-and-pencil factor

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x^2(x+1)+1(x+1))$ $(x-1)(x^2+1)(x+1)$

Figure 29b. Reconciling by 'grouping' the second paper-and-pencil factor to produce the second and third CAS factors of x^4-1

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$x^4-1=(x^2-1)(x^2+1)$ $(x-1)(x+1)(x^2+1)$

Figure 29c. Reconciling by refactoring the given x^4-1 as a difference of squares

In the class discussion that followed the completion of the first set of examples for *n* from 2 to 6 in the factoring of x^n-1, some clarification of the notion of complete factorization took place, which included the teacher's comment that: "Sometimes, they can be factored further. What we did initially is not wrong, it's just not complete." The discussion also permitted students to learn how others in the class were approaching the task of reconciling their paper-and-pencil factors with the CAS factors. The notion that expressions with even exponents greater than 2 could also be regarded as a difference of squares was not obvious for some students, as suggested by the remark uttered by one student: "I can't get [the factors of] x^4-1." Furthermore, while it was mentioned by a few students that x^6-1 could be treated either as a difference of squares, $(x^3)^2-1$, or as a difference of cubes, $(x^2)^3-1$, the upcoming task which involved the factoring of x^9-1 was to provide evidence that *seeing* a difference of cubes was even more difficult for some students than seeing a difference of squares.

Before continuing the next part of the task for x^n-1, with values of *n* from 7 to 13, the students were asked whether they had observed any new patterns emerging from their factoring: Were there some exponents for which the general rule was providing a complete factorization and others for which this was not the case? Based on their limited set of examples thus far, it was inevitable that most students would generate the conjecture that, for odd values of *n*, the general rule seemed to be holding. In other words, they thought that the complete factorization of x^n-1 had exactly two factors for odd ns; while for even values of n, it contained more than two factors, one of which was $(x+1)$. The conversation between two students, which is presented in Figure 30, illustrates how the CAS helped them realize that their conjecture regarding odd ns was incorrect. They followed up on this new awareness with further odd-number replacements for *n,* which led them to notice the multi-factor effect of "certain odd numbers that are divisible by 3, 5, 7," and eventually to hit upon the idea of prime numbers for the exponent.

Chris	'Two factors' means two separate sets of brackets, right?
Peter	Yeah.
Chris	The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can't be broken down anymore. 'Cause it always ends up being all positive. And uh, then, because, it's sort of hard to explain.
Peter	When the exponent is [pause], when the exponent is an even number you'll have more than two factors, but when the exponent is not an even number, you'll have exactly two factors all the time.
Chris	Yeah. [Types Factor $(x^7 - 1)$ into the CAS] Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] and [Types in Factor $(x^9 - 1)$; the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$] And, no!!! [a look of utter surprise on Chris's face]

Figure 30. Testing examples with the CAS led these students to realize that their initial conjecture was incorrect

In light of the classroom discussion related to confronting paper-and-pencil factors with CAS factors for the expressions from $x^2 - 1$ to $x^6 - 1$, some students began to adjust their paper-and-pencil factoring techniques with the aim of "playing a game" with the CAS. They tried to anticipate what it would produce as its factored form, for the expressions from $x^7 - 1$ to $x^{13} - 1$, and thereby to reduce the amount of reconciliation that would need to be done. However, certain values of *n* – in particular, 9 and 10 – proved more difficult than others for the students.

In fact, the expression $x^9 - 1$ pushed a significant number of them to the limits of their current thinking on factoring. A few erroneously handled the expression as if it were a difference of squares, $(x^3+1)(x^3-1)$, or as a "sort-of difference of squares," $(x^3-1)(x^6+1)$. Others used the general rule. When they compared their paper-and-pencil factors with the CAS factors, they came to the realization that the CAS had produced a factored form that they were unable to obtain themselves. Even those who had used the general rule for $x^n - 1$ and who could reconcile their factorization with the factors produced by the CAS – by multiplying all the CAS factors except $(x-1)$ to produce their second paper-and-pencil factor – were still not satisfied. They insisted on knowing how to factor $x^9 - 1$ themselves, and explicitly requested such help from the teacher: "How do you get those factors?" The teacher suggested that they to try to "see" x^9 as $(x^3)^3$, and thus $x^9 - 1$ as $(x^3)^3 - 1$, which could then be treated as a difference of cubes, which they supposedly knew how to factor.

Within this part of the task where students were confronting their paper-and-pencil factors with the CAS factors, the CAS played a role that was quite different from that which it had played in other parts of the activity. The CAS technique of Factor, with its accompanying output, disclosed to the students that there were certain factoring techniques that they were missing from their repertoire. As a consequence, they wanted to learn these techniques. This need to understand the factored CAS outputs and to be able to explain them in terms of a certain structure, or by means of paper-and-pencil techniques that would produce the same results, seemed important to the students (and to us!).

In all, the confrontation of students' paper-and-pencil factors with the CAS factors led to the development of new theoretical ideas. In the process of making sense of the CAS factors, the students extended their view of the range of the difference-of-squares technique. They also came to see that exponents that have several divisors can generally be factored in more than one way. They began to look at expressions in terms of multiple possible structures. Their understanding of the notion of complete factorization evolved. Finally, as will be seen in the next section, some students were even able to detect new patterns, and with the aid of the CAS, developed another general rule.

4.5.3 Proving: Theorizing more deeply on the factorization of $x^n - 1$

As the concluding activity of the sequence, students were presented with the task of proving that $x+1$ is always a factor of $x^n - 1$ for even ns. During its early discussions related to task design, the research team had produced several proofs for this problem, including:

$$
\begin{aligned}
x^n - 1 &= x^{2k} - 1 \qquad \text{(for } n \text{ even)} \\
&= (x^2)^k - 1 \\
&= (x^2 - 1)(x^{2^{k-1}} + x^{2^{k-2}} \ldots + 1) \\
&= (x + 1)(x - 1)(\quad \ldots \quad)
\end{aligned}
$$

However, the students of this study had not had any prior experience with proving. Furthermore, Mounier and Aldon's (1996) report had not included any elements of students' proofs with general factorisations of $x^n - 1$. As well, the existing research literature on proof methods in algebra with this age group of student has tended to focus on number theoretic proofs (e.g., Healy and Hoyles, 2000; see also Mariotti, 2006) rather than on proofs of the kind of problem we were proposing. Thus, we were not sure what the 15-year-old students of our study would be able to do with it.

After having given the class about 15 minutes to work on the task and having circulated during this time in order to see what kinds of proofs the students were attempting, the teacher invited selected students to come to the board, one at a time, to present their work and discuss it with the class at large.

One of the first "proofs" that was proposed was the following by Paul: "When n is an even number greater than or equal to 2, $x^2 - 1$ is always a factor, and so $x+1$ is a factor." However, he could not really show or explain why $x^2 - 1$ is always a factor.

A rather different approach to the task was presented by Janet (Figure 31). The proof, which she and her partner had constructed, was generic in that it embodied the structure of a more general argument. From their earlier work on reconciling CAS factors with their paper-and-pencil factoring, which had been based on the general rule for $x^n - 1$, they had noticed that for even ns, the number of terms in the second factor was always even. Janet argued as she presented her proof at the board, using $x^8 - 1$ as an example, that it would work for any even n. She explained how the terms of the second factor could be grouped pair-wise, yielding a common factor of $(x+1)$:

$$
\begin{aligned}
x^8 - 1 &= (x-1)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \\
&= (x-1)(x^6(x+1) + x^4(x+1) + x^2(x+1) + 1(x+1)) \\
&= (x-1)(x+1)(x^6 + x^4 + x^2 + 1)
\end{aligned}
$$

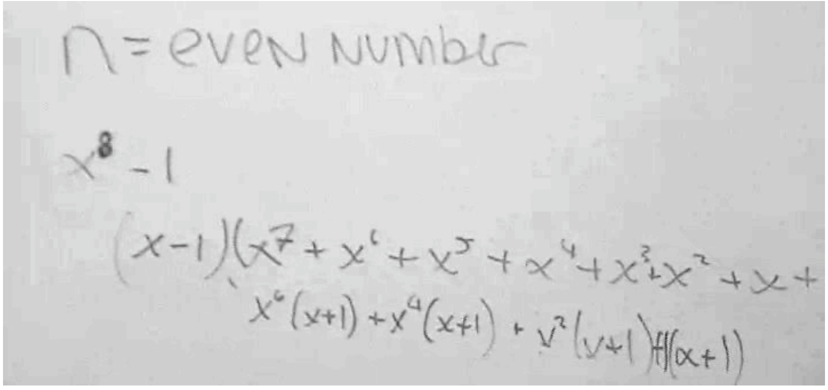


Figure 31. Janet's proof by grouping

In yet another approach, one that was related to Paul's difference-of-squares "proof", Bryan (who had earlier experienced difficulty in making sense of general symbolic notation) came to the board and wrote $x^n - 1 = (x^{n/2} + 1)(x^{n/2} - 1)$, while stating that, if the exponent n is even, then it can be divided by 2. He then tried to start a proof involving the factor $(x^{n/2} + 1)$, but could not make progress. His team-partner, Andrew, came forward to continue the proof and share the conjecture that their small group had come upon in the earlier task involving the reconciling of paper-and-pencil factors with CAS factors. It seems that when these students had been working on the expression $x^{10} - 1$, they had factored it as $(x^5 + 1)(x^5 - 1)$, the latter part of which they refactored according to their general rule and subsequently wrote on their sheets: $x^{10} - 1 = (x^5 + 1)(x - 1)(x^4 + x^3 + x^2 + x + 1)$. But the CAS produced as its factored form $(x - 1)(x + 1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1)$. They then looked at the factors that the CAS had output for $x^5 + 1$: $(x + 1)(x^4 - x^3 + x^2 - x + 1)$. Andrew noticed something important: "Isn't that how it works for the sum of cubes?" So, they began to conjecture and test a general rule for the following factorization:

$$x^n + 1 = (x + 1)(x^{n-1} - x^{n-2} + \ldots - x + 1).$$

Andrew, in presenting this conjecture to the class, insisted that, even though "it does not seem to work for even ns, it is true for all odd numbers n, and $x + 1$ would always be a factor of it." While Andrew and Bryan never actually proved that $x + 1$ is a factor of $x^n + 1$ for all odd ns, they had succeeded in providing the missing link for Paul's difference-of-squares proof. A proof of Andrew's and Bryan's 'rule' might have been approached by applying some of the grouping ideas that had been used by Janet, along with the addition of zero-based pairs for the missing terms (i.e., a generic proof where 5 in $x^5 + 1$ represents any odd positive integer):

$$\begin{aligned} x^5 + 1 &= x^5 + x^4 - x^4 - x^3 + x^3 + x^2 - x^2 - x + x + 1 \\ &= x^4(x + 1) - x^3(x + 1) + x^2(x + 1) - x(x + 1) + 1(x + 1) \\ &= (x + 1)(x^4 - x^3 + x^2 - x + 1) \end{aligned}$$

Although the grouping proof by Janet was the only one to have clearly established that $x + 1$ is always a factor of $x^n - 1$ for all even values of n, the proving activity provided for the development of several other key mathematical notions related to the factoring of $x^n - 1$. The difference-of-squares "proof", for example, with its accompanying treatment of the case $x^{n/2} + 1$, for odd values of $n/2$, served to extend the thinking of students in the class. The $x^n + 1$ conjecture, which had issued from the earlier work of a group of students with the factoring of $x^{10} - 1$, helped others to integrate their ideas about odd, even, and prime exponents – theoretical ideas that had been generated in interaction with the various CAS and paper-and-pencil techniques that had evolved throughout the entire factoring activity, but especially during that part of the activity on reconciling CAS and paper-and-pencil factors.

4.6 Synthesis on the theme of generalizing and proving within factoring

While theoretical elements have been the organizing principle of this presentation of student activity, their emergence among the students would not have been possible without the accompanying technical demands raised by the tasks. In fact, the development of CAS techniques, simple as they were in most cases, was vital to theoretical advances in at least four different respects.

One involved the confrontation of the CAS factored forms with those that students produced by paper and pencil – based on their existing techniques, including the newly generalized rule for factoring $x^n - 1$. This confrontation was found to be very productive for most students, but especially so for those who, upon realizing that they could not generate the same factors as had the CAS, insisted on finding out how to do so either from the teacher or from individual peers or during the follow-up classroom discussions. These CAS encounters resulted in the evolution of not only students' paper-and-pencil factoring techniques but also their theoretical perception of the structure of expressions (e.g., seeing that $x^6 - 1$ could be viewed either as a difference of squares, or as a difference of cubes, or as an example of the general rule that they had earlier generated). These same encounters also led to insights that provided students with the technical and theoretical tools needed for the proving part of the activity.

A second way in which CAS techniques were constitutive of theoretical advances for the students involved their noticing in a CAS output a certain structure that they had not noticed in prior examples. As we have seen, this led to the serendipitous discovery by a group of students of a rule for the factoring of $x^n + 1$, for all odd ns. A third way was directly related to the nature of the reflection questions such as, for example, "Conjecture, in general, for what numbers n will the factorization of $x^n - 1$ contain exactly two factors?" To answer this question, students first formed a tentative conjecture based on the examples they had already generated, and then tested their conjecture by means of their CAS techniques, Factor and Expand. The 'to-ing and fro-ing' between conjecturing and testing was illustrated in the search for the elusive prime-number response to the question. In such activity, the CAS commands seemed to be more a part of the background than a center-stage player. The issue was not the relation between the form of the output and the command itself – as was the case in the confrontational role played by the CAS output – but rather the use of CAS techniques as a servant in obtaining results to questions of a quite different sort.

Finally, CAS techniques, in accompaniment with paper-and-pencil techniques, were found to play a role in the deepening of theoretical thinking. For example, after students had explored empirically the question of those values of n for which the factorization of $x^n - 1$ includes $x + 1$ as a factor, they later went on to attempt to prove that this is always the case for even values of n. Such a task involved mobilizing and coordinating several pieces of theory that had emerged throughout the activity. This coordination included the further refining of partial conjectures with the aid of the CAS, as was seen for instance in the efforts of the group that had developed the $x^n + 1$ factorization rule, and which they applied to the proof of $x^n - 1 = (x^{n/2} + 1)(x^{n/2} - 1)$ for even values of n.

The need to make sense of the CAS outputs, and the ability to coordinate these with existing theoretical notions and paper-and-pencil techniques, was fundamental to the students' theoretical and paper-and-pencil-technical progress. Evidence that this progress did indeed occur in an ongoing way throughout the unfolding of the activity set has, we believe, been sufficiently provided. However, the students did not manage all of this on their own, even with the aid of the CAS. The teacher was essential to the process. He encouraged students to grapple with the complicated notions of the tasks and gave them adequate time in which to carry out their own explorations. The students clearly experienced difficulties at times, but they were not abandoned in these situations. At regular intervals, the teacher stimulated classroom discussion of the important mathematical ideas that were at stake. He supported students in presenting their work and in justifying their thinking. Without the teacher orchestrating the theoretical and technical development of the task situation, and asking key questions at the right moment, the advances made by the students would likely have been less dramatic. While the study did not intend to focus explicitly on the teacher, his role throughout was of the utmost importance.

5. Concluding discussion

In this concluding section we come back to the initial research question and summarize our findings on this issue. These findings concern both themes and will be related to the theoretical framework, which is the instrumental approach to tool use, and in particular the anthropological view that was summarized by the three T's: Task-Technique-Theory (TTT). We also briefly discuss the findings pertaining to CAS techniques and paper-and-pencil techniques, as well as issues of language and discourse, and relate these two aspects to the TTT framework.

5.1 The co-emergence of technique and theory

The research question we phrased earlier in this article is:

> *In which ways does the interaction between technique and theory foster students' algebraic thinking when working in a combined CAS/paper-and-pencil environment?*

Of course, this is in a sense not a neutral question; it implicitly refers to the TTT framework. Indeed, the notions of task, technique and theory, which are closely intertwined in learning, guided the study in its different phases. In the preliminary design phase of the study, we identified possible ways in which the tasks would invite both technical and theoretical development, which was helpful for structuring the design process. During the teaching experiments, these TTT relations framed our data collection, particularly for the mini-interviews in the 2005 experiment. In the retrospective phase, the framework guided the data analysis towards the identification of students' going back-and-forth between theoretical thinking and developing techniques, both with the CAS and with paper and pencil, an intertwined process that characterizes instrumental genesis.

The main finding of the study is that we clearly found evidence for the relation *theory – technique* within the setting of the designed tasks, which confirms the importance and productiveness of the TTT approach. Technique and theory emerge in mutual interaction. The observations in both themes show how techniques gave rise to theoretical thinking, and, the other way around, how theoretical reflections led students to develop and use techniques. This interaction proved to be very productive in cases of confrontation, or even that of conflict, between the techniques – particularly the CAS techniques – and the students' theoretical thinking. A tendency to reconcile CAS work and theory was observed; students seemed to strive for consistency, and used the CAS on several occasions as a means of checking their theoretical thinking. A second aspect of reconciliation concerns CAS techniques and paper-and-pencil techniques, an issue that will be addressed below.

Even if our findings in both themes underpin the importance of the co-emergence and intertwining of theory and technique in a task setting, the two themes also illustrate that the actual relation between task, technique and theory depends on the situation. In the approach and task design of theme 1, CAS techniques played an important role. In fact, the CAS peculiarities such as the 'equality test' for equivalence, automatic simplification, and neglect of restrictions were the technical issues that most provoked theoretical reflection. In theme 2, however, paper-and-pencil techniques were important, too, and it was probably the coordination and the reconciliation of paper and pencil with CAS that evoked theoretical progress. The last part of theme 2 on proving clearly shows this interplay between the two media and the impact of the students' prior theoretical reflections. The fact that, in spite of the differences between the two themes, they both provided support for the relevance of the TTT approach suggests a wide applicability of this theoretical framework.

In spite of this, we encountered some phenomena in our teaching experiments that turned out to be difficult to coordinate with the current TTT framework, and which suggest elaboration or further adaptation. The two issues we will address now are the relation between CAS techniques and paper-and-pencil techniques, and the issue of language and discourse.

5.2 CAS techniques and paper-and-pencil techniques

Earlier we quoted Lagrange (2003, p. 271) as saying that, "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration." This potential epistemic role of technique was at the core of our desire to better understand the interaction between the technical and the theoretical in students' developing algebraic thinking. However, CAS environments *combine two types of techniques: paper-and-pencil techniques* and CAS techniques. Artigue (2002) has emphasized that, while it is certainly easy to recognize the pragmatic value of CAS techniques, it may be less easy to grasp their epistemic value. She has suggested that the epistemic value of CAS techniques can be found, for example, in the greater diversity of representations of mathematical objects than is usual in classroom work with paper and pencil. We too found this to be the case in the set of task situations involving equivalence and equality, where the CAS techniques for determining equivalence of expressions were somewhat different from those involving paper and pencil. The variety of representations produced by the CAS provoked students to reflect in ways that would have been considerably more difficult to achieve with paper and pencil only. However, we did not find this to be the case in the set of tasks involving factoring. In fact, the epistemic value of the CAS techniques was to be found less in themselves and more in the way in which the output from the CAS techniques elicited a need for the epistemic value that could be derived from other techniques, namely paper-and-pencil.

The set of tasks involving factoring could be said to be quite different from the set of tasks involving equivalence, equality, and equation. These factoring tasks involved an area of algebra that is typically considered primarily manipulative. The CAS techniques used in the tasks were quite simple in that they included just two commands: Factor and Expand. However, the fact that the output of the Factor command was quite often a form that was unexpected evoked in students the action of trying to produce the same form with their pencil-and-paper techniques.

Even when they could reconcile by other means, including CAS – such as multiplying some of the factors from the CAS factorization to obtain the form of their incomplete paper-and-pencil factorization – this was found not to be satisfying to them. More than half the students in the class we followed in theme 2 wanted to be able to produce themselves the factored form that was output by the CAS. If the CAS factored form could not be explained by students' existing knowledge of factoring, they wanted to learn more in this regard. This is but one example of the way in which CAS and paper-and-pencil techniques were found to be interrelated epistemically, that is, co-constitutive of students' theoretical development.

Thus, our research findings lead us to suggest that the epistemic value of CAS techniques by themselves may depend both on the nature of the task and on the limits of students' existing learning. When students cannot explain, in terms of their current theoretical and technical knowledge, that which a CAS technique produces, reliance on additional CAS techniques may not suffice. In such cases, the epistemic value of paper-and-pencil techniques would seem to play a complementary, but essential, role. Recent research that has used the TTT theoretical framework for analyzing the learning of mathematics in technological environments has tended to pay less attention to the role of paper-and-pencil in interaction with CAS techniques in promoting theoretical growth. Our results point to this as a fruitful area for research involving, in particular, young high school algebra learners.

5.3 Language and discourse: A matter for further research

Several examples were presented that illustrate the difficulties that the students experienced with both: i) interpreting the language used in the task questions that dealt with reflection issues, and in responding to them, and ii) finding suitable language with which to talk about algebraic objects and processes, as well as their own algebraic thinking, within the classroom discussions. It will be recalled that, while the development of a theoretical discourse is one of the four main components of Chevallard's anthropological theory, which he refers to as *technology*, this component has been folded into the theoretical component of the task-technique-theory framework by Artigue, Lagrange, and others of the various French teams working in this area because of the ambiguity of the word *technology* in this context. In view of Artigue's (2002) position that, "building a theoretical discourse relevant to some given instrumented techniques and well adapted to the students' cognitive state is not a trivial task" (p. 262), it seems somewhat surprising that the discursive component has not been given more attention by these latter researchers. Perhaps, a first step would be to reconsider the decision to collapse the two – technology and theory – and to emphasize the discursive component in the same ways as they have done so for task, technique, and theory.

As the TTT framework is derived from the anthropological didactical theory of Chevallard and is thus situated within the ensemble of human activity and social institutions, this framework cannot sidestep the issues associated with language and discourse. While mathematics education researchers (e.g., Noss and Hoyles, 1996; Lerman, 1998; Bartolini Bussi and Mariotti, 1999) have over the past decade or so focused on the role of language and other mediational tools in the teaching and learning of mathematics, Radford's (2000) emphasis on the fact that language in use evolves theoretically throughout mathematical activity and comes to have different meanings over time seems especially pertinent here. It will be recalled that the students of our study struggled with sorting out the meaning of words such as *common form, solution, equivalent, equal, expression, equation,* and so on. Their initial meanings for these terms both shaped their interpretations of the task situations and were shaped by them. Meanings evolved as students grappled with reflection questions and attempted to communicate their thinking to classmates and teacher. However, as researchers, we found that the task-technique-theory framework seemed to provide little in the way of tools for analyzing the role of language and discourse in our study. We point to the role played by language and discourse in interaction with technique and theory as an area for further research and theoretical development.

Acknowledgments

The research presented in this article was made possible by a grant from the Social Sciences and Humanities Research Council of Canada – INE Grant # 501-2002-0132. We wish to thank our project consultant, Michèle Artigue, for her invaluable contributions during the task-design process. We express our appreciation as well to the two classes of students and their mathematics teacher who participated in the research. In addition, we are grateful to Texas Instruments for providing the TI-92 Plus calculators and view screens used in the study. The affiliations of authors and collaborators are as follows: A. Boileau, F. Hitt, C. Kieran, D. Tanguay: Université du Québec à Montréal, Canada; P. Drijvers: Freudenthal Institute, the Netherlands; J. Guzmán: CINVESTAV, Mexico; L. Saldanha: Portland State University, USA.

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics* 14(3): 24-35.

Artigue, M. (1997). Le Logiciel 'Derive' comme révélateur de phénomènes didactiques liés à l'utilisation d'environnements informatiques pour l'apprentissage. *Educational Studies in Mathematics* 33: 133–169.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning* 7: 245–274.

Bartolini Bussi, M.G. and Mariotti, M.A. (1999). Semiotic mediation: From history to the mathematics classroom. *For the Learning of Mathematics* 19(2): 27-35.

Boon, P. and Drijvers, P. (2005). Chaining operations to get insight in expressions and functions. In M. Bosch (Ed), *Proceedings CERME4 conference, February 2005*. Retrieved on December 20, 2005 from http://cerme4.crm.es/ .

Cerulli, M. and Mariotti, M.A. (2002). L'Algebrista: un micromonde pour l'enseignement et l'apprentissage de l'algèbre. *Sciences et techniques éducatives* 9(1-2): 149-170.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques* 19: 221-266.

Drijvers, P. (2002). Learning mathematics in a computer algebra environment: Obstacles are opportunities. *Zentralblatt für Didaktik der Mathematik* 34(5): 221-228.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment. Design research on the understanding of the concept of parameter*, Doctoral Dissertation, Freudenthal Institute, Utrecht University, the Netherlands. Also available through www.fi.uu.nl/~pauld/dissertation .

Drijvers, P. and Trouche, L. (in press). From artifacts to instruments, a theoretical framework behind the orchestra metaphor. In M.K. Heid and G.W. Blume (Eds), *Research on Technology and the Teaching and Learning of Mathematics: Syntheses, Cases, and Perspectives*. Greenwich, CT: Information Age Publishing.

Falcade, R. (2003). Instruments de médiation sémiotique dans Cabri pour la notion de fonction. Paper presented at the ITEM conference, June 2003, Reims, France.

Guin, D., Ruthven, K. and Trouche, L. (Eds). (2004). *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument*. Kluwer Academic Publishers, Dordrecht, the Netherlands.

Haspekian, M. (2005). An “Instrumental Approach” to study the integration of a computer tool into mathematics teaching: The case of spreadsheets. *International Journal of Computers for Mathematical Learning* 10: 109–141.
Healy, L. and Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education* 31: 396-428.
Heid, M. K. (1996). A technology-intensive functional approach to the emergence of mathematical thinking. In N. Bednarz, C. Kieran and L. Lee (Eds), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 239-256). Dordrecht, the Netherlands: Kluwer.
Hoyles, C. (2001). From describing to designing mathematical activity: The next step in developing a social approach to research in mathematics education? *Educational Studies in Mathematics* 46: 273-286.
Hoyles, C. and Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A.J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick and F. Leung (Eds), *Second International Handbook of Mathematics Education* (Vol 1, pp. 323-349). Dordrecht, the Netherlands: Kluwer Academic.
Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde and A. Pérez (Eds), *8th International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Sevilla, Spain: S.A.E.M. Thales.
Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey, H. Chick and M. Kendal (Eds), *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study* (pp. 21-33). Dordrecht, the Netherlands: Kluwer Academic.
Kieran, C. (2006). Research on the learning and teaching of algebra: A broadening of sources of meaning. In A. Gutiérrez and P. Boero (Eds), *Handbook of Research on the Psychology of Mathematics Education* (pp. 11-49). Rotterdam, the Netherlands: Sense Publishers.
Kieran, C. and Saldanha, L. (in press). Designing tasks for the co-development of conceptual and technical knowledge in CAS activity: An example from factoring. In K. Heid and G. Blume (Eds), *Research on Technology and the Teaching and Learning of Mathematics: Syntheses, Cases, and Perspectives*. Greenwich, CT: Information Age Publishing.
Kieran, C. and Yerushalmy, M. (2004). Research on the role of technological environments in algebra learning and teaching. In K. Stacey, H. Chick, and M. Kendal (Eds), *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study* (pp. 95-152). Dordrecht, the Netherlands: Kluwer Academic.
Lagrange, J.-b. (2000). L’intégration d’instruments informatiques dans l’enseignement: une approche par les techniques. *Educational Studies in Mathematics* 43: 1–30.
Lagrange, J.-b. (2002). Étudier les mathématiques avec les calculatrices symboliques. Quelle place pour les techniques? In D. Guin and L. Trouche (Eds), *Calculatrices symboliques. Transformer un outil en un instrument du travail mathématique: un problème didactique* (pp. 151-185). Grenoble, France: La Pensée Sauvage.
Lagrange, J.-b. (2003). Learning techniques and concepts using CAS: A practical and theoretical reflection. In J.T. Fey (Ed), *Computer Algebra Systems in Secondary School Mathematics Education* (pp. 269-283). Reston, VA: National Council of Teachers of Mathematics.
Lagrange, J.-b. (2005). Curriculum, classroom practices, and tool design in the learning of functions through technology-aided experimental approaches. *International Journal of Computers for Mathematical Learning* 10: 143-189.
Lerman, S. (1998). A moment in the zoom of a lens: Toward a discursive psychology of mathematics teaching and learning. In A. Olivier and K. Newstead (Eds), *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education (*Vol. 1, pp. 66-81). Stellenbosch, South Africa: PME Program Committee.

Mariotti, M.A. (2006). Proof and proving in mathematics education. In A. Gutiérrez and P. Boero (Eds), *Handbook of Research on the Psychology of Mathematics Education* (pp. 173-204). Rotterdam, the Netherlands: Sense Publishers.
Monaghan, J. (2005). *Computer algebra, instrumentation and the anthropological approach.* Paper presented at the 4th CAME conference, October 2005. Retrieved on December 20, 2005 from http://www.lonklab.ac.uk/came/events/CAME4/index.html
Mounier, G. and Aldon, G. (1996). A problem story: factorisations of x^n–1. *International DERIVE Journal* 3: 51-61.
Nicaud, J.-F., Bouhineau, D. and Chaachoua, H. (2004). Mixing microworld and CAS features in building computer systems that help students learn algebra. *International Journal of Computers for Mathematical Learning* 9: 169–211.
Noss, R. and Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht, the Netherlands: Kluwer Academic.
Rabardel, P. (2002). *People and technology - a cognitive approach to contemporary instruments*. Retrieved on December 20, 2005 from http://ergoserv.psy.univ-paris8.fr/ .
Rabardel, P. and Samurçay, R. (2001). From artifact to instrument mediated learning. *New Challenges to Research on Learning* 22. Helsinki, Finland: University of Helsinki.
Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics* 42: 237-268.
Trouche, L. (2000). La parabole du gaucher et de la casserole à bec verseur: étude des processus d'apprentissage dans un environnement de calculatrices symboliques. *Educational Studies in Mathematics* 41: 239–264.
Trouche, L. (2004a). Environnements informatisés et mathématiques: quels usages pour quels apprentissages? *Educational Studies in Mathematics* 55: 181–197.
Trouche, L. (2004b). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning* 9: 281-307.
Vérillon, P. and Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education* 10: 77-103.
Vygotsky, L.S. (1930/1985). La méthode instrumentale en psychologie. In B. Schneuwly and J.P. Bronckart (Eds), *Vygotsky aujourd'hui* (pp. 39-47). Neufchâtel: Delachaux et Niestlé.

CAROLYN KIERAN
Département de Mathématiques
Université du Québec à Montréal
C.P. 8888, succ. Centre-Ville
Montréal, QC H3C 3P8
Canada
Email: kieran.carolyn@uqam.ca

PAUL DRIJVERS
Freudenthal Institute
Utrecht, the Netherlands
Email: p.drijvers@fi.uu.nl