Interpreting and Assessing the Answers Given by the CAS Expert: A Reaction Paper[1]

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Abstract:

In this reaction to the paper presented by Werner Peschek, I take issue with the notion put forward that, in general mathematics education, we ought to assign the field of competence for operative knowledge and skills primarily to the experts and, consequently, that this competence could be delegated almost completely to the electronic mathematical expert that is the CAS. Citing research evidence from current work, I argue that it is quite unrealistic to expect students to be able to interpret and assess the answers produced by the CAS if their general mathematical education does not include provision for developing operative knowledge and skills. Furthermore, students themselves are not satisfied at not being able to do such interpreting.

Key words: CAS; computer algebra systems; high school algebra with technology; interpreting answers in CAS environments

1. INTRODUCTION

Werner Peschek argues in his plenary paper that he is missing "the perspective of general mathematics education, the anchoring of the use of CAS in an adequate framework of general mathematics education for all" (p. 95). I offer a few remarks in this regard and then go on to question the appropriateness of the Fischer framework, which Peschek proposes as a suitable candidate. With its attendant entailments regarding experts' and laypersons' educational needs, I argue against the application of this framework as a model for thinking about CAS use and draw upon an example from my own research to support my argument.

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2 "GENERAL MATHEMATICS EDUCATION FOR ALL" FRAMEWORKS: PRESENTING THE ISSUE OF OPERATING KNOWLEDGE AND SKILLS

Peschek proposes that, "an adequate framework of general mathematics education for all could give orientations for many considerations and decisions regarding curriculum and CAS-supported teaching and learning" (p. 95). I suggest that such general frameworks do indeed exist in many countries. For example, in Canada and the United States, we have the *Principles and Standards for School Mathematics* (NCTM, 2000) -- intended to be a guide for those who make decisions that affect the mathematics education of <u>all</u> students in prekindergarten through grade 12. The PSSM, with its set of ten standards, the first five of which describe mathematical content goals, and the next five of which address the processes of problem solving, reasoning and proof, connections, communication, and representation, can indeed serve to validate the use of CAS in mathematics classrooms. However, as we are all well aware, this document can be and is interpreted in many different ways.

Peschek then goes on to discuss operating knowledge and skills. One has the sense, initially at least, that he believes that attaining such knowledge is very important within a general mathematics education for all because he states in his paper that "generalized calculating according to rigidly formulated rules describes a very important mathematical activity" (p. 95). He also bemoans the dilettantism that exists in his own country and critiques, as well, the PISA 2003 mathematics assessment for its limited number of items that demand algebraic transformations or even elementary arithmetic transformations. The fact that PISA promotes a view of mathematical literacy for all that emphasizes, "the capacity of students to analyse, reason, and communicate effectively as they pose, solve and interpret mathematical problems in a variety of situations involving quantitative, spatial, probabilistic or other mathematical concepts" (OECD, 2004, p. 37), seems not to agree with Peschek's notion of mathematics for all. Thus, at this particular juncture of Peschek's paper, one begins to think that the problem that Peschek raised initially about there not being an adequate framework of general mathematics education for all that would allow for the anchoring of CAS use, is really one related to there not being a framework that emphasizes operative knowledge and skills – as these are, according to Peschek, "operating knowledge and skills seem critical to the CAS discussion" (p. 95). But having started to build this argument, we suspect this has been a purely rhetorical device, because he proceeds to demolish it by suggesting that we consider the framework developed by Roland Fischer.

3. MATHEMATICS FOR ALL WITHIN THE SOCIO-PHILOSOPHICAL FRAMEWORK OF FISCHER: REDUCING OPERATIVE KNOWLEDGE AND SKILLS

While I am not personally familiar with Fischer's framework, it would seem to rest on a distinction between the knowledge needed by experts and that needed by laypersons. According to Peschek, Fischer identifies three fields of competence as "those which are to be acquired for every subject" (p. 96): (a) basic knowledge (notions, concepts, forms of representation), (b) operative knowledge and skills (operational transformations [drawn from p. 98], solving problems, proofs, in general: generating new knowledge), and (c) reflection (possibilities, limits, and meaning of concepts and methods). Fischer apparently considers the fields of basic knowledge and reflection to be particularly important for the generally educated layperson, emphasizing that operative knowledge and skills are not needed for the generally educated layperson, but are (along with basic knowledge) the domain of the expert. While Fischer is said to consider that "doing operations" cannot be completely removed from the framework of a general mathematics education, he is ready to reduce expectations with regard to such knowledge among students in a general mathematics education.

Peschek clearly sees value in adopting the Fischer framework because it appears to follow our concept of general mathematics education (or the concept of mathematical literacy used in PISA test-items). However, he also sees a problem as far as CAS work is concerned: "Despite the fact that CAS can help a lot in developing basic knowledge and reflected (conceptual) understanding, such mathematics teaching seems not to be an inviting field for using CAS because of the lack of operative work" (p. 98).

Peschek's proposed way out of this impasse is to consider -- in accordance with the work of his colleague Edith Schneider -- the CAS as the expert:

> "According to our concept of general mathematics education, I assign the field of competence for operative knowledge and skills primarily to the experts. This turns out to be exactly the same field of competence that could be delegated most completely to CAS. In this (narrow) sense we can perceive CAS as a simple electronic model of a mathematical expert." (p. 8)

Thus, according to this line of thinking, the student is put into the position of a layperson consulting the expert when she/he uses the CAS. In my opinion, this is a dangerous metaphor for CAS use, precisely because the layperson within the Fischer framework will not be equipped to interpret the products of the operative knowledge and skills of the expert. As I will attempt to illustrate, this framework offers a far-from-adequate model for thinking about intelligent use of CAS.

4. DELEGATING OPERATIVE KNOWLEDGE AND SKILLS TO THE CAS EXPERT

Peschek emphasizes that successful and profitable interaction with human as well as electronic mathematical experts requires the following four components:

- the willingness and ability to ask the "right" questions, to be precise when formulating one's own questions and considerations and to present them in a form that can be interpreted by the expert;
- exact conceptions of the possibilities and limits of the mathematical expert's knowledge and skills, and of the range of validity of mathematical expressions;
- wide basic knowledge of mathematics (especially knowledge about important mathematical forms of representation); and
- verification as well as an appropriate interpretation and assessment of the answers given by the expert. (pp. 8-9)

Is it possible to imagine a layperson being able to do all of the above, in particular, "to interpret and to judge the expertise" (p. 4) of the CAS expert without some basic knowledge of the domain of expertise of the given expert -- in this case, operational knowledge and skills. I think not. However, my dispute with Peschek's position is based on more than opinion.

Peschek makes quite clear his belief that, "in our concept of higher mathematics education, operative abilities and skills are not the crux and can largely be delegated (outsourced) to mathematical experts (as CAS)" (p. 99). While he acknowledges that many of his mathematics colleagues are strictly against his view, he continues to argue on the basis of what he calls an *educational* viewpoint that:

> "Mathematics is relatively secure, socially accepted, codified knowledge, which notably allows for a separation between understanding and doing. It owes its high social

relevance to the fact that, in utilizing outsourcing, it even works when the user has no idea anymore as to why." (p. 100)

In his argument, Peschek fails to take into consideration the findings of decades of research work on, for example, the learning of algebra where an overly rapid reliance on the syntactic aspects (an example of outsourcing in mathematics that is offered by Peschek) left far too many algebra students in the dust (see, e.g., Kieran, 1992). For these students, algebra had become a meaningless activity where they could not fall back onto its numerical or property-based foundations, even if they wanted to. Peschek also fails to take into account the many studies emanating from the French school of *didactique* that have shown, for example, that, when the learning of technique is neglected, CAS does not lighten the technical work of doing mathematics, nor does it enhance students' conceptual reflection (Lagrange, 2003).

To add further support to my case against Peschek's position, the research example that I have chosen to use is one that is drawn from a recent CAS study that my colleagues and I have been conducting (Kieran & Drijvers, 2006) and which relates directly to the issue of "interpreting and assessing the answers given by the CAS expert" (p. 99) – one of the four dimensions outlined by Peschek as being required for successful interaction with the CAS expert.

5. INTERPRETING AND ASSESSING THE ANSWERS GIVEN BY THE CAS EXPERT: SOME EVIDENCE FOR THE POOR FIT OF THE FISCHER FRAMEWORK WITH STUDENT CAS WORK

The task that I shall use as a backdrop for discussing this issue is one that involves deriving general rules for the factoring of $x^n - 1$, a task that has figured in French CAS work (Mounier & Aldon, 1996) and which has been described by Lagrange (2000, 2003). In our own study, we developed a two- to three-lesson activity around this task for 16-year-old algebra students – an activity that was designed to push students beyond their curricular experience and place them in a situation devised to expose limitations in the thinking frames they are using. (See Kieran & Saldanha, 2007, for details on the task design considerations.) Further, in designing the activity, we noted that Drijvers (2003) had emphasized, in his study, that more attention needed to be paid to the roles of both paper-and-pencil work throughout CAS activity and focused classroom discussions, and so we took steps to include such features in our activity.

Students in our study had already learned to factor the difference of squares (e.g., $x^2 - 1$) and the sum and difference of cubes (e.g., $x^3 - 1$). The initial part of the activity was designed to support them in working toward a general factorization of $x^n - 1$, by helping them construe a factoring pattern for n = 2, 3, and 4, which they might not have noticed before when using only 2 and 3 as values of n. Then, activity with the systematic replacement of n by ever-increasing integral values in $x^n - 1$ was intended to provide an arena for making conjectures and working toward (and refining) the generalization taking place. To facilitate both the process of generalization and the identification of the constraints on the form of the complete factorization of $x^n - 1$, we initiated the practice of reconciling the factored form obtained by their paper-and-pencil work with the form of the CAS output.

Based on their observations of the factors $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$, and the resulting products for these three examples, students quickly abstracted the general factored form for the factorization of $x^n - 1$, for integral values of n. Then, they proceeded to fill in the activity table (see Figure 1) for n = 2 to 13 in $x^n - 1$, according to the general rule for factoring that they had just generalized. They were, however, surprised to see the factorizations that the CAS produced. Figure 1 presents a relevant section of the factoring activity.

Toward a generalization (activity with <u>paper & pencil</u> and with <u>calculator</u>)

1. In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down.
If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using <u>paper and pencil</u>	Result produced by the <u>FACTOR command</u>	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		

$x^6 - 1 =$		

2. Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
   - i) contain exactly two factors?
   - ii) contain more than two factors?
   - iii) include $(x+1)$ as a factor?

Please explain:

Classroom discussion

3. As above, each line of the table below must be filled in completely (all three cells), one row at a time before proceeding to the next row. Start from the top row and work your way down.
If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^7 - 1 =$		
$x^8 - 1 =$		
$x^9 - 1 =$		
$x^{10} - 1 =$		
$x^{11} - 1 =$		
$x^{12} - 1 =$		
$x^{13} - 1 =$		

Figure 1. Part of the Activity on Factoring (from Kieran & Saldanha, 2007)

In general, the students' factorization with paper-and-pencil (p/p) for x^4-1 had taken the form $(x-1)(x^3+x^2+x+1)$. When the CAS produced (x-1)(x+1)(x^2+1) as the factored form, some students reconciled the discrepancy between the two responses by multiplying the second and third CAS factors to produce the second factor that they had originally obtained with p/p; others used factoring by grouping of the second p/p factor in order to produce the same three factors as the CAS had displayed; still others refactored the given expression x^4-1 as a difference of squares, which they applied twice.

During the discussion that followed the completion of Questions 1 and 2, several students were curious to know how they might refactor x^6-1 so as to obtain directly the factors that the CAS had produced. Some students proposed treating x^6-1 as a difference of squares, and then using the sum of cubes and difference of cubes techniques. Another suggested the application of the difference of cubes immediately to x^6-1.

At first, we thought that students might simply be wanting to save reconciling time by arriving immediately at the factored form that the CAS would display. But when they reached the expression x^9-1, we realized that there was more to the issue than speed. The CAS had produced a factored form that most of the students were unable to obtain themselves. Even though they had reconciled their factorization obtained by the general rule for x^n-1 with the factors produced by the CAS -- by multiplying all the CAS factors except (x-1) to yield their second p/p factor -- they were not satisfied. They insisted on knowing how to factor x^9-1 themselves, and explicitly requested such help from the teacher: "How do you get those factors?" The teacher suggested that they might try to "see" x^9 as $(x^3)^3$ and thus (x^9-1) as $((x^3)^3$-1), which could then be treated as a difference of cubes. When students arrived at the next expression to be factored, x^{10}-1, which some factored as $(x^5+1)(x^5-1)$, and subsequently factored to $(x^5+1)(x-1)(x^4+x^3+x^2+x+1)$, they were again surprised by the CAS result: $(x-1)(x+1)(x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1)$. This led a few of them to begin conjecturing and testing a new general rule for factoring (x^n+1).

The analysis of classroom discussions and student written work during this part of the activity, as well as students' continued work in the rest of the activity, suggested to the members of the research team that the machine technique of FACTOR, and its output, had disclosed to our students that there were certain factoring techniques that they were missing. As a consequence, they insisted on finding out about these techniques, or at least ways of seeing how the expression $x^n - 1$, for particular integral values of n, could be viewed in such a way as to be factored into the form produced by the CAS. Thus, the CAS provoked in our students the desire to better understand certain aspects of factoring, thereby serving to facilitate their interpretation of such CAS outputs. This need to understand factored CAS outputs and to be able to explain them in terms of a certain structure or by means of p/p techniques that would produce the same results seemed important to our students.

Herein lie the grounds for my disagreement with Peschek's advocacy of the position that we "delegate the field of competence for operative knowledge and skills to CAS" (p. 98). Clearly, students – at least, the secondary level students we observed – have a tendency to insist on understanding what the CAS produces and will not accept working with a tool that produces answers that they personally do not understand or to which they cannot give meaning. Furthermore, "students' ability to interpret and assess the answers given by the CAS expert" (p. 98) would seem to require some support in the form of p/p techniques and/or theorizing that, if not already known by the students, would need to be developed by them within the context of the activity itself. Of course, the nature of the task is a crucial factor in this discussion, as are the teacher and the didactical contract concerning technology and paper/pencil.

Given a non-trivial task, our research findings suggest that it is quite unrealistic to expect students to be able to interpret and assess the answers produced by the CAS if their general mathematical education does not include provision for developing operative knowledge and skills. Furthermore, students themselves are not satisfied at not being able to do such interpreting. Thus, the Fischer framework, along with its entailments, provides a poor fit with students' observed behavior in CAS learning environments and would appear to be rather inadequate as a model for thinking about CAS use by inquiring minds in high school mathematics classes.

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BIOGRAPHICAL NOTE

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