CHAPTER 15

DESIGNING TASKS FOR THE CODEVELOPMENT OF CONCEPTUAL AND TECHNICAL KNOWLEDGE IN CAS ACTIVITY

An Example From Factoring

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In this chapter we describe a classroom instructional activity that integrates the use of Computer Algebra Systems (CAS) in a manner aimed at engaging high school students in substantive mathematical reasoning. The activity draws on students' prior knowledge of factoring and pushes them to use and develop this knowledge beyond what is usual for high-school-level algebra. In particular, we discuss the attendant design and mathematical principles of the activity at some length, explicating the intended role of the CAS. We then highlight aspects of engagement and student reasoning that emerged with the unfolding of the activity in a

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high school mathematics classroom. Three features of the design of the activity are considered to have played important roles in the further development of students' conceptual and technical knowledge of factoring: reconciling the forms generated by the CAS in relation to the forms they had generated with paper and pencil, reflecting on the objects they had reconciled, and proving the relations they had generalized.

THEORETICAL FRAMEWORK

One of the contributions of the French didactique research community (e.g., Artigue, 2002; Guin & Trouche, 1999; Lagrange, 2000) has been the notion that both conceptual and technical knowledge can co-develop within mathematical activity involving technological tools, as long as the technical aspects of that activity are not neglected. According to Vérillon and Rabardel (1995), tools exist as both artifacts (physical objects) and instruments (psychological constructions). These researchers suggest that, when students appropriate tools for themselves, and thereby transform artifacts into instruments, they are not simply learning tool-techniques in response to given tasks. They are actually developing conceptually while they are perfecting their techniques with the tool.

Lagrange (2003) has described research in which French teachers initially used Derive (1996) in their algebra classrooms with the intention that the symbol manipulation software would lighten the technical load, and so they emphasized conceptual activity. The result, however, was that neither the conceptual nor the technical developed. The research team began to look upon techniques in a different light and to view them as a link between tasks and conceptual reflection. Further, Lagrange (2003) points out the following:

> A technique is generally a mixture of routine and reflection. It plays a pragmatic role when the important thing is to complete the task or when the task is a routine part of another task. Technique plays an epistemic role by contributing to an understanding of the objects it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques. (p. 271)

But, for technique to be a link between tasks and conceptual reflection, an immediate question arises: What is the nature of the tasks that support this potential conceptualizing role of technique?

THE NATURE OF TASK-SITUATIONS[1] THAT PROVIDE FOR THE CODEVELOPMENT OF CONCEPTUAL AND TECHNICAL KNOWLEDGE

CAS-Related Literature

In the Derive study mentioned previously, Lagrange noted that the easier calculation afforded by the tool did not automatically enhance students' reflection and understanding. Teachers had

> encouraged students to jump directly from tasks to conceptual reflection ... not seeing that techniques specific to the use of symbolic computation are useful, that the techniques are not obvious to students, and that these techniques may be a topic for reflection. (Lagrange, 2003, p. 271)

In contrast, two experienced teachers, who assigned to their class of 16- to 18-year-old college students the long-term problem of working on the factorization of $x^n - 1$, did recognize the epistemic value of developing techniques of factorization with Derive (Mounier & Aldon, 1996). According to the teachers, as reported by Lagrange, the students learned a great deal at a conceptual level about finding a general factored form that is true for every integer n.

But how can one make "techniques a topic for reflection"? What kinds of situations, in which a task such as the previous one is embedded, could potentially lead to the codevelopment of both conceptual and technical aspects of mathematical activity? This was a central question we faced as we began our CAS research project in 2004 and were confronted with the need to design a set of CAS task-situations for use by tenth-grade students within their algebra course of study. Of equal import to us as researchers was our need to ensure that these tasks would also function as productive vehicles for generating rich data on the codevelopment of conceptual and technical knowledge.

We combed the CAS research literature regarding the nature of tasks susceptible to playing this dual role, looking specifically for design frameworks, or basic principles, for constructing such tasks. Hoyles (2002) had emphasized that a focus on the design of task-situations is at the heart of the "transformative potential of [technological] tools in activities" and that, with this focus, "knowledge and epistemology are brought back to center stage" (p. 284). But the search for design frameworks proved elusive. In fact, Zbiek (2003) found that the "absence of detail in research reports causes difficulty in using the reports to develop deeper insights into the kind of CAS-related mathematical experiences that best support student learning. Writers who wish to have a greater impact on classroom

practice should consider ways to include in their research reports more information about the mathematical tasks and CAS activities used in the classroom" (p. 212).

However, we were able to glean relevant information from various research reports that provided a basis for getting started on our own task construction. For example, Drijvers (2003) emphasized that,

> While designing instructional activities, the key question is what meaningful problems may foster the students' cognitive development according to the goals of the hypothesized learning trajectory, ... three design principles guided the design process: guided reinvention, didactical phenomenology, and mediating models. (p. 26)

Especially useful was Drijvers' suggestion that more attention needed to be paid to the roles of both paper-and-pencil work throughout CAS activity and focused classroom discussions. Artigue, Defouad, Duperier, Juge, and Lagrange (1998) provided several examples of novel CAS tasks, pointing out that they had used three types of situations in their study: guided manipulation, problem solving, and teaching of a specific notion—each with different degrees of teacher intervention. Artigue (2002) also pointed out that some of the levers on which one can capitalize with respect to CAS activity playing an epistemic role include "the surprise effect that can occur when one obtains results that do not conform to expectations and that can destabilize erroneous conceptions, and the multiplicity of results that can be obtained in a short space of time when exploring and trying to understand a certain phenomenon" (p. 344) (translated by Carolyn Kieran and Luis Saldanha). Guin and Trouche (1999) added that situations should aim at fostering investigation and anticipation.

Zehavi and Mann (2003) described how they set out to develop tasks that, in their view, could not be handled practically and effectively without CAS—tasks that had the potential to intertwine student work, CAS performance, and student reflection. The tasks were to highlight the fundamental ideas within a mathematical topic, and were to extend the topic at hand by connecting it with previously learned topics or with topics that would be learned at a later stage. Often they were based on the identification of persistent learning difficulties in current non-CAS practice, and sometimes involved replicating and completing examples. Ball and Stacey (2003), who raised the question as to what students should record when working in CAS environments, suggested that, "Our goal is to shift the focus of students' written records from principally providing the detail to principally providing the overview and emphasizing the reasoning" (p. 290). Finally, Lagrange (1999) proposed that task-situations ought to be created in such a way as to "bring about a better comprehension of mathematical content" (p. 63) via the progressive acquisition of techniques in the achievement of a solution to the task.

Relevant Non-CAS-Related Literature

In addition to the research cited above, which deals specifically with the design and implementation of CAS-based tasks, another perspective on instructional design, relevant to the present discussion, is that developed by Thompson (2002). His is a more general perspective that involves conceptual analyses of mathematical ideas and their contribution to the design of mathematical learning experiences that aim to support learners' conceptual advances. In Thompson's view, doing conceptual analyses entails imagining students having *something* in mind in the context of *discussing* that something.

Toward this end, according to Thompson, instructional activities should be designed with two central aims in mind: (a) to create opportunities for students and teacher to discuss particular things, objects, or ideas that need to be understood and to discuss how to imagine such things; and (b) to create opportunities for the instructor to ensure that specific conceptual issues will arise for students as they engage in discussions with her or him. When goals a and b are realized with regard to a particular idea, they can produce *instructional conversations* (interactions) around that idea.

At the core of this design perspective, then, is a vision of students purposefully participating in conversations that foster reflection on some mathematical *thing*—an object, an idea, or a way of thinking. Thompson (2002) employs the term *didactic object* to refer to "a thing to talk about that is designed to support reflective mathematical discourse involving specific mathematical ideas or ways of thinking" (p. 210). In our view, a didactic object is akin to a tool for a designer or a teacher who conceives using it in ways that enable student engagement and conversations that foster productive reflection on specific mathematical things.

Drawing on the notion of didactic object, we view the CAS as a *didactical tool* in that we intended that it be used in our task-situations as a means for leveraging the kinds of instructional interactions envisioned by Thompson. More specifically, the *things* generated by the CAS in the context of the activity within which it is integrated—for example, the expressions it produces and displays as a result of students' interaction with it in the context of some goal-directed sequence of activities—could be taken as objects of explicit and substantive reflection and classroom discussions.

DESIGNING CAS TASK-SITUATIONS: AN EXAMPLE FROM FACTORING

With the preceding prior experiences of other CAS researchers as a pool from which to draw, and with Thompson's perspective of didactic objects in mind as a background instructional design heuristic, the members of our research team began to think about the nature of the task-situations that might be integrated into our own study of tenth-grade algebra classes. It was clear to the team from the outset that CAS technology was not intended to replace paper-and-pencil work as a technical tool. Nor would it be simply a means for checking paper-and-pencil work. It was, however, seen as a didactical tool for pushing students to come to grips with underlying theoretical ideas in algebra.

Some of the components of our activities that were intended to make CAS a didactical tool were questions that drew on the machine to occasion discussions that do not normally happen in mathematics classes. Tasks that asked students to write about how they were interpreting their work and the related CAS displays aimed to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom, and clarifying ideas and distinctions, in ways that simply "doing algebra" may not require. The design team decided that, if CAS is to be effective at the high school level, it is precisely this kind of usage that needs to be considered—usage that involves a mix of paper-and-pencil work, CAS activity, reflection questions, and classroom discussions of a substantive nature in which the teacher draws out from the students the ideas upon which they are reflecting.

One of the several task-situations that we developed was an elaboration of the $x^n - 1$ factoring task[2] previously referenced in the Lagrange (2003) report.[3] We shall use it as the vehicle for presenting the principles on which we drew as we designed the various questions of the task-situations, a process that involved several refinement loops. (See the Appendix for the task-situation in its entirety—Activity 6: Factoring.[4]) After presenting the aims and design principles of each of the main parts of the factoring task-situation, we briefly describe some of the highlights of its unfolding within one of the classes that we observed and conclude by pointing toward those design principles and features of CAS that supported students' engagement in substantive mathematical reasoning and the codevelopment of their conceptual and technical knowledge.

Overview of the Task-Situation and Its Design Principles

The particular task-situation discussed here as an illustrative example is one that involves seeing patterns in the factorization of an expression. We designed the task-situation with the intention that students would engage in the following meta-level processes of mathematical activity: anticipating, verifying, conjecturing, generalizing, and proving (Kieran, 1996). We structured the situation so as to unfold in phases of student work that would correspond more or less with these processes.

Students had already learned to factor the difference of squares (e.g., $x^2 - 1$) and the sum and difference of cubes (e.g., $x^3 - 1$). As a first design principle, we wished for them to *link their past work* involving these identities with the generalization toward which they would be working regarding the factoring of $x^n - 1$. We aimed to support students' construing a factoring pattern that they might not have noticed before when using only 2 and 3 as values of n.

An additional design principle involved students' *predicting* a factorization for $x^n - 1$, using the symbol n for the exponent, rather than specific numerical values. This principle, which reflected our intention that students would move toward an algebraic formalization of the patterns they noticed, would be invoked within the context of yet another principle—that of *focused substantive classroom discussion* in which the teacher attempts to elicit student thinking rather than give them answers a little too quickly.

Then, activity with the systematic replacement of n by ever-increasing integral values in $x^n - 1$ was intended to provide an arena for making conjectures and working toward and refining the generalization taking place. To facilitate both the process of generalization and the identification of the constraints on the form of the complete factorization of $x^n - 1$, we elaborated as a design principle the student practice of *reconciling* the factored form obtained by paper and pencil with the unexpected form of the CAS output. This design principle is described more fully in Part II of the Task-Situation: Conjecturing and Moving Toward a More Refined Generalization.

The last design principle to be called on was one that we adapted for use with CAS task-situations that aim at supporting the development of conceptual knowledge within technical activity. It involves the *mobilization of the emerging conceptual knowledge* within the multifaceted activity of proving—an activity that has only rarely figured explicitly in CAS research at the secondary school level.

We now present some details of the three main parts of the factoring task situation. We also include brief reference to the suggestions offered for teacher intervention so as to give some indication of the classroom context for which we attempted to provide.

Part I of the Task-Situation: Anticipating and Verifying

The activity begins with a brief review of paper-and-pencil factoring of differences of squares and cubes, accompanied by CAS verification (see Item 1a of the Appendix). The activity entails prompts to suggest that this preliminary work be followed by the teacher asking students to look at the factored forms of $x^2 - 1$ and $x^3 - 1$ to see whether they detect some signs of a pattern. Students might remark that $(x - 1)$ is a factor of both. The teacher could then write the factored forms on the board, while posing the question, "So what do we obtain when we multiply out these factored forms?" In this way the teacher could be helping students to relate explicitly both the factored and expanded forms of the two examples:

$$(x - 1)(x + 1) = \underline{\qquad\qquad} \text{ and } (x - 1)(x^2 + x + 1) = \underline{\qquad\qquad}.$$

Students then return to the activity sheets (Items 1b, 2a–2d) where they are asked to write the result they anticipate for $(x - 1)(x^3 + x^2 + x + 1)$, without doing any paper-and-pencil or CAS manipulation. They then verify their anticipated result by means of CAS. They are asked subsequently to note what the three expressions $[(x - 1)(x + 1), (x - 1)(x^2 + x + 1), \text{and } (x - 1)(x^3 + x^2 + x + 1)]$ have in common, as well as to explain why it is that a binomial results as the product in all three cases. The next section of Part I (Items 2e–2g) presents an item requiring prediction: "On the basis of the expressions we have found so far, predict a factorization of the expression $x^5 - 1$," followed by a few more items of an anticipatory or explanatory nature.

The classroom discussion that brings this first part of the activity to a close focuses on whether students have been able to anticipate a general factorization of the expression $x^n - 1$, for integral values of n. We considered it important that students try to generate the general formulation themselves, tackling issues such as representing decreasing powers of the variable with a general notation. At this point, we suggested that the discussion could address the importance of the factor $x - 1$ and the fact that in multiplying this binomial by any polynomial of the form $x^{n-1} + x^{n-2} + \ldots x^2 + x + 1$, the result is the sum of the first and last terms of the product. This result is obtained by distributing the factor x, then the factor -1, throughout all the terms of the second polynomial, thereby leading to the pair-wise cancellation of all "inner" terms. In sum, for integral values of n we have $(x - 1)(x^{n-1} + x^{n-2} + \ldots x^2 + x + 1) = x^n - 1$.

Part II: Toward a generalization (activity with paper & pencil and with calculator).

II(A) In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		

Figure 15.1. First line of table requesting the systematic replacement of n in $x^n - 1$ by 2, 3, 4, 5, and 6.

Part II of the Task-Situation: Conjecturing and Moving Toward a More Refined Generalization

The second part of the activity focuses, first, on the systematic replacement of n by the integers from 2 through 6 in $x^n - 1$. Students are asked to fill in a table, of which only the first line is shown in Figure 15.1, according to the given instructions.

We anticipated that students might factor $x^4 - 1$ as $(x - 1)(x^3 + x^2 + x + 1)$, according to the pattern they had generalized up to this point, not realizing that the second factor could be further factored. In fact, several students might be initially surprised by the factored form produced by the CAS for $x^4 - 1$, until perhaps noting that they were dealing with a difference of squares or that the second factor could be refactored by grouping. Thus, a reconciliation of the versions they had produced with paper-and-pencil and with CAS would be in order. We also considered it important that the reconciling be done after each example of $x^n - 1$, which explains why students are asked to fill in all three cells of a given row before moving down to the next row of the table.

We conjectured that the reconciling part of the activity would be a crucial factor in students' evolving conceptual development. Not only was the completely-factored CAS output intended to provoke them into thinking more deeply about what it meant to be "fully factored," but also how they might be able to tell, or test, whether a given expression could be factored differently or whether the "factors" they had produced could be further factored. Indeed, the prompt to reconcile is that part of the activity that aims to engage students in the act of reflecting on the "things" (forms, in

this case) generated by the CAS in relation to the forms generated by them.

After filling in the table for $x^2 - 1$ through $x^6 - 1$, students are asked to do some conjecturing (Item II.A.2): "Conjecture, in general, for what numbers n will the factorization of $x^n - 1$: (i) contain exactly two factors? (ii) contain more than two factors? (iii) include $x + 1$ as a factor? Please explain."

The discussion following this first section of Part II was intended to touch upon the different ways of factoring $x^6 - 1$, that is, an expression in which n is a multiple of both 3 and 2, before asking students about the conjectures they had made regarding the factorization of $x^n - 1$. We anticipated that students would first distinguish the cases of n even and n odd. We also thought that many of them would (incorrectly) conjecture that there are more than two factors for even numbers $n > 2$ and only two factors when n is odd. At this point, it was suggested that the teacher might want to offer $x^9 - 1$ as a counterexample to this conjecture, for which the factored form contains three factors. Then students would be urged to work either individually or in groups, in order to revise their conjectures. They would then be prompted to go on to the second section, which involves completing a table as before, but this time for $x^7 - 1$ through $x^{13} - 1$.

The same conjecturing question is repeated after students have completed the table with the extended integral values for n, in order to give them the opportunity to revise and elaborate their initial conjectures (Item II.B.2). The final item of Part II involves applying the generalizations they have formed to a few numerical examples (Items II.C.1–II.C.3): "Does $x^{2004} - 1$ (followed by $x^{3003} - 1$, and then $x^{853} - 1$) (i) contain more than two factors? (ii) include $x + 1$ as a factor? Please explain."

The aim of Part II is to stimulate students to discover the general relationship: $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots x^2 + x + 1)$, for integral values of n. More specifically, we intended to encourage the realization that for n prime, the factorization will be of this form; while for n even, the form of the complete factorization, which involves a refactorization of the second factor, will include both $(x - 1)$ and $(x + 1)$ as factors. In the discussion following the first section of Part II, the counterexample

$$\begin{aligned} x^9 - 1 &= (x-1)(x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \\ &= (x-1)(x^6 + x^3 + 1)(x^2 + x + 1) \end{aligned}$$

aims at promoting distinctions among the cases of n prime, even $n > 2$, and odd n not prime (see Figure 15.2 for examples of the cases of n prime and even $n > 2$).

If n is even and $n > 2$, then $(x-1)$ and $(x+1)$ are two factors of the factored polynomial.

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x^2+1)$$

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$$

$$x^8 - 1 = (x^4 - 1)(x^4 + 1) = (x^2 - 1)(x^2 + 1)(x^4 + 1) = (x-1)(x+1)(x^2+1)(x^4+1)$$

$$\begin{aligned} x^{10} - 1 &= (x^5 - 1)(x^5 + 1) = (x-1)(x^4+x^3+x^2+x+1)(x^5+1) \\ &= (x-1)(x+1)(x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1) \end{aligned}$$

If n is a prime number, the factored expression contains only two factors:

$$x^2 - 1 = (x-1)(x+1)$$

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$$

$$x^7 - 1 = (x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$$

$$x^{11} - 1 = (x-1)(x^{10}+x^9+x^8+\ldots+x^3+x^2+x+1)$$

Figure 15.2. Factoring $x^n - 1$ for cases of n prime and even $n > 2$.

Part III of the Task-Situation: Proving

Part III contains a single item: "Explain why $(x + 1)$ is always a factor of $x^n - 1$ for even values of $n \geq 2$." When we included this item, we were not sure what tenth-grade students would be able to do with it. They had not had any prior experience with proofs such as the following:

$$\begin{aligned} x^n - 1 &= x^{2k} - 1 \qquad \text{(for } n \text{ even)} \\ &= (x^2)^k - 1 \\ &= (x^2 - 1)(x^{2^{k-1}} + x^{2^{k-2}} + \ldots + 1) \\ &= (x+1)(x-1)(\ \ldots\ ) \end{aligned}$$

What we observed was a surprise, both to ourselves and to the classroom teacher.

SOME HIGHLIGHTS OF THE UNFOLDING OF THE FACTORING ACTIVITY WITHIN ONE OF THE CLASSES[5]

The activity unfolded over two 1-hour class periods occurring on consecutive days. Students engaged in the activity, following the intended structure that we elaborated in the previous section, each part concluding with a whole-class discussion of ideas addressed within that part. Our aim in this section is not to provide a detailed research report of what transpired within this unfolding. Rather, we focus on two highlights that illustrate how engagement with the activity brought issues of substantive mathematical reasoning to the fore and pushed students to reason in seemingly new directions. The first highlight, emerging in the classroom discussion at the end of Part I, concerns issues related to the symbolic formalization of the general relationship that we intended for students to broach. The second highlight concerns a creative and elegant proof of the general relationship, advanced by a particular pair of students at the activity's conclusion.

The emergence of these highlights was driven by students' attending to the subtle but important requirement, built into particular parts of the activity, that they reconcile the forms produced by the CAS with their own paper-and-pencil factorizations of expressions of the form $x^n - 1$ for increasing integral values of n. By helping to orient students toward constructing complete factorizations of these expressions, the reconciling requirement thus played a central role in pushing them to conjecture, generalize, and formalize a proof.

Dealing With *ns* in the General Formulation

After the students had shared their thinking with respect to the three given examples of Item 2c in Part I [$(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$, along with their expanded forms], the teacher asked them to predict the factorization of $x^n - 1$. It seemed clear to the researcher-observers that this was the first time that the students had been asked to deal with a general formulation of a factored form.

One student started with the suggestion, "$(x - 1)(x^{n-1} + x^{n-2}$" but then stopped and said, "but I don't know how far to go." This was echoed by several others in the class. The teacher then offered: "down to $x^2 + x + 1$, as with the others." On examining the proposed factored form for $x^n - 1$, another student wondered whether $(x - 1)(x^{n-1} + 1)$ would be the same as $(x - 1)(x^{n-1} + x^{n-2} + \ldots x^2 + x + 1)$. We noted that even the knowledge of how to multiply x by x^{n-1} seemed precarious at this moment. One student suggested adding the exponents, 1 and $n - 1$. In the background, however, remained the question of "how far to go" in the second factor; it had not yet been satisfactorily addressed. The student asked again, "If it's decreasing, how far down do you go?" to which the teacher responded, "You go all the way to x^0, which is 1." However, the link between x^0 and x^{n-n} was not made, and the student in question seemed unsatisfied.

Even though the issue was still not resolved, the teacher asked the students to move on to Part II. So, midway through Part II, when the rest of the class had begun to work on the factoring of $x^7 - 1$ and its successor expressions, the same student went up to the board to repeat his question to the teacher: "When you go like this, going down even to $n - 1$, $n - 2$, $n - 3$, how far do you go with the ns; when will this (circling the −3) be zero?" "It depends on n," offered the teacher. Not yet convinced, the student added that he did not like the dots either (referring to the second general factor where the middle terms were represented by three dots).

Had the students not been asked to predict a general factorization for x^{n-1} and to express it in general notation, we would not have witnessed the difficulties that are inherent in such polynomial notations involving non-numerical exponents for students of this age and particular algebraic experience. It suggested an interesting area for further study.

Proving That $(x + 1)$ Is Always a Factor of $x^n - 1$ for Even Values of $n \geq 2$

After having given the class about 15 minutes to try to generate a proof of this phenomenon—that $(x + 1)$ is always a factor of x^{n-1} for even values of $n \geq 2$—and having circulated around the classroom during this time in order to see what kind of proofs the students were attempting, the teacher invited selected students to come to the board, one at a time, to present their proofs.

One of the first "proofs" that was proposed was as follows: "When n is an even number greater than or equal to 2, $(x^2 - 1)$ is always a factor, and so $(x + 1)$ is a factor." However, the student could not really show why $(x^2 - 1)$ is always a factor. Then another student came forward with what he proposed as a counterexample, $(x^{12} - 1)$, which he manipulated so as to have a sum of cubes (i.e., the first factor of the expression $(x^3 + 1)(x^3 - 1)(x^6 + 1)$), which yielded $(x + 1)$ as a factor. So, the presence of $(x^2 - 1)$, he argued, was perhaps not so crucial after all.

But the quite remarkable attempt appeared when Jane was invited to come forward to present the proof that she and her partner had constructed (see Figure 15.3). Theirs was a generic proof (Tall, 1979), in that the argument was presented as an example that embodied the structure of a more general argument. Their proof was based on noticing that for

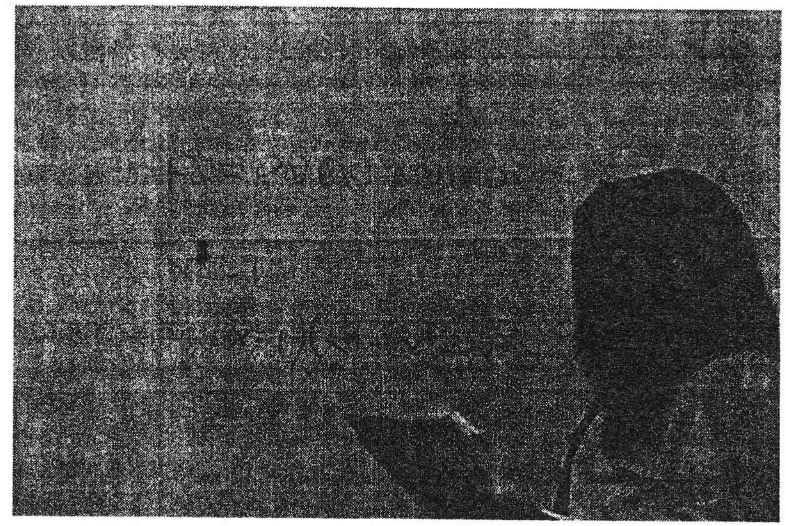
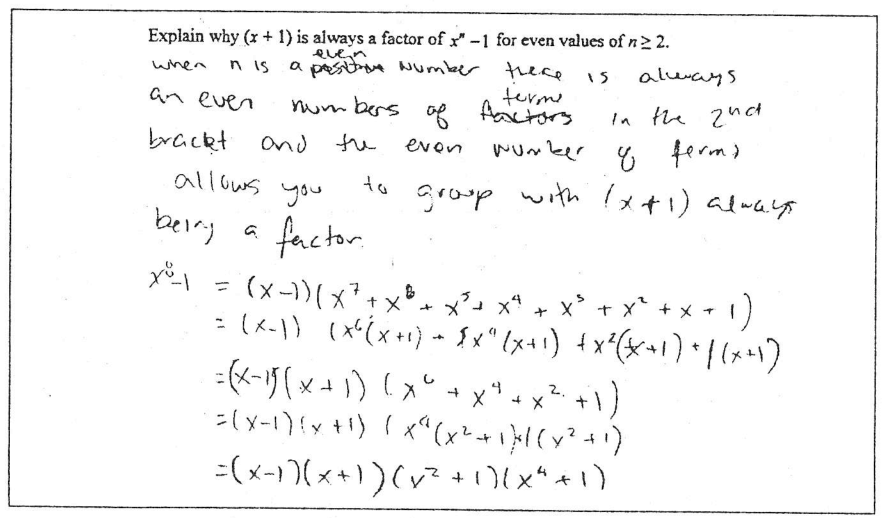


Figure 15.3. The beginning of the proof based on the idea of grouping the even number of terms in the second factor.



Explain why $(x + 1)$ is always a factor of $x^n - 1$ for even values of $n \geq 2$.

when n is a even number there is always an even numbers of terms in the 2nd bracket and the even number of terms allows you to group with (x+1) always being a factor

$$x^8 - 1 = (x-1)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$
$$= (x-1)(x^6(x+1) + x^4(x+1) + x^2(x+1) + 1(x+1))$$
$$= (x-1)(x+1)(x^6 + x^4 + x^2 + 1)$$
$$= (x-1)(x+1)(x^4(x^2+1) + 1(x^2+1))$$
$$= (x-1)(x+1)(x^2+1)(x^4+1)$$

Figure 15.4. Written work illustrating the proof that $(x + 1)$ is always a factor of $x^n - 1$ for even values of $n \geq 2$.

even ns in x^{n-1}, the number of terms in the second factor was even. Thus the terms of this second factor could be grouped pairwise, yielding a common factor of $(x + 1)$. See Figure 15.4 for the written work underpinning,

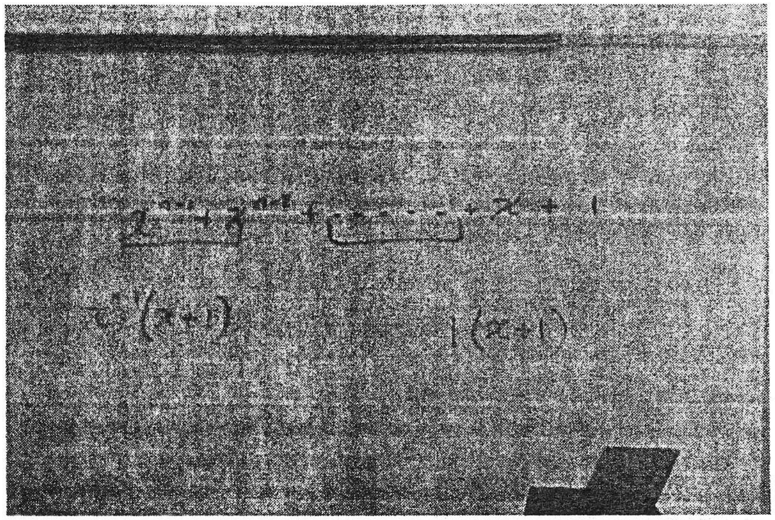


Figure 15.5. Part of a more general notation for the "grouping" proof.

and extending, the presentation at the board. Part of a more general notation for the proof was added by the teacher (see Figure 15.5), but the elegant simplicity of the proof had already been appreciated by many in the class—as well as by the teacher and the researcher-observers.

CONCLUDING REMARKS

We offer the activity described here as an illustration of a productive way to integrate CAS into mathematical activity that at once draws upon students' prior knowledge and skills *and* pushes them to reflect on and use these in a manner that goes beyond the usual for the high school context. From our perspective, in this activity the CAS functioned as a generator of factorizations of expressions of the form x^{n-1}, for different values of n.

This function, together with the important requirement that students reconcile their own paper-and-pencil work with the results produced by the CAS, provided an impetus for their generating conjectures about the form of the complete factorization of $x^n - 1$. This, in turn, oriented students to elaborate a general relationship and to formalize that relationship. It also led several of them to find a way of proving that $(x + 1)$ is always a factor of $x^n - 1$ for even values of $n \geq 2$, with novel methods that intertwined both the conceptual and technical knowledge that had been developed throughout the activity.

The coupling of students' attention to the forms produced by the CAS in comparison to their own productions, we argue, helped constitute these forms and their structure as objects of explicit reflection and classroom discussion. We hasten to add that this function was not inherent to the CAS, but rather was imposed on it within the context of the total intended activity. In other words, that the CAS served this function was very much an intended *design* feature of the activity. It is in this way that we view the CAS as a didactical tool.

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NOTES

1. We use the term *task* to refer to a single question (with possibly one or two subquestions) and the term *task-situation* or *activity* to refer to an extended set of questions related to a given central task or mathematical idea.
2. In these activities, the assumption was made that factoring would be over the set of integers.
3. We wish to acknowledge in particular the contribution of Fernando Hitt to the design of the elaborated version of the basic task reported by Lagrange (2003).
4. Every activity that was developed had a student version, a teacher version that included ideas for classroom discussions, and an answer key that could be given to students after we had collected their activity sheets for use as one of our data sources.
5. We express our appreciation to the students and teacher of the class reported herein. They gracefully accommodated all of our videotaping, observing, and interviewing over a period of several months. We are grateful to Texas Instruments for providing the calculators used in our study.

APPENDIX

Activity 6: Factoring

Part I (Paper & Pencil, and CAS): Seeing Patterns in Factors

1. (a) Before using your calculator, try to recall the factorization of each algebraic expression listed in the left column of this table:

Factorization using paper and pencil	*Verification using FACTOR (show result displayed by the CAS)*
$a^2 - b^2$	
$a^3 - b^3$	
$x^2 - 1$	
$x^3 - 1$	

Classroom discussion of Part I, 1a

1. (b) Perform the indicated operations (using paper and pencil)

$(x - 1)(x + 1)$

$(x - 1)(x^2 + x + 1)$

2. (a) Without doing any algebraic manipulation, anticipate the result of the following product:

$(x - 1)(x^3 + x^2 + x + 1)$

2. (b) Verify the anticipated result above using paper and pencil (in the box below), and then using the calculator.

2. (c) What do the following three expressions have in common? And, also, how do they differ?

$(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$.

2. (d) How do you explain the fact that the following products result in a binomial: two binomials, a binomial with a trinomial, and a binomial with a quadrinomial?

Classroom discussion following Question 2d

2. (e) On the basis of the expressions we have found so far, predict a factorization of the expression $x^5 - 1$.

2. (f) Explain why the product $(x - 1)(x^{15} + x^{14} + x^{13} + \ldots + x^2 + x + 1)$ gives the result $x^{16} - 1$?

2. (g) Is your explanation (in (f), above) also valid for the following equality: $(x - 1)(x^{134} + x^{133} + x^{132} + \ldots + x^2 + x + 1) = x^{135} - 1$?

Explain:

Classroom discussion of Part I, #1 #2

Part II: Toward a Generalization (Activity With *Paper & Pencil* and With *Calculator*)

II(A) 1. In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	*Result produced by FACTOR command*	*Calculation to reconcile the two, if necessary*
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

II.(A).2. Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x + 1)$ as a factor?

Please explain:

Classroom discussion of Part II A

Part II Continued (With *Paper and Pencil*, and With *Calculator*)

II(B) 1. As with Part A above, each line of the table below must be filled in completely (all three cells), one row at a time before proceeding to the next row. Start from the top row and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	*Result produced by FACTOR command*	*Calculation to reconcile the two, if necessary*
$x^7 - 1 =$		
$x^8 - 1 =$		
$x^9 - 1 =$		
$x^{10} - 1 =$		
$x^{11} - 1 =$		
$x^{12} - 1 =$		
$x^{13} - 1 =$		

II.(B).2. On the basis of patterns you observe in the table II.B above, revise (if necessary) your conjecture from Part A. That is, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x + 1)$ as a factor?

Please explain:

II.(C). *Without using your calculator*, answer the following questions:

1. Does $x^{2004} - 1$

i) contain more than two factors?
ii) include $(x + 1)$ as a factor?

Please explain:

2. Does $x^{3003} - 1$
   i) contain more than two factors?
   ii) include $(x + 1)$ as a factor?

Please explain:

3. Does $x^{853} - 1$
   i) contain more than two factors?
   ii) include $(x + 1)$ as a factor?

Please explain:

Classroom discussion of Part II B and C

Part III: Challenge

Explain why $(x + 1)$ is always a factor of $x^n - 1$ for even values of $n \geq 2$.

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