SIMPLIFICATION OF RATIONAL ALGEBRAIC EXPRESSIONS IN A CAS ENVIRONMENT: A TECHNICAL-THEORETICAL APPROACH

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In this paper we analyze and discuss the influence of CAS technology and an Activity designed with a Technical-Theoretical approach on two 10th grade students' thinking on a Task related to simplifying rational algebraic expressions. The theoretical elements adopted in this study are based on the instrumental approach. Results indicate that CAS and a technical-theoretical-oriented Activity provoked students to theorize on certain aspects of the simplification of rational expressions, thus illustrating the epistemic role of CAS technique and its influence in improving students' learning with respect to specific technical-theoretical components of rational expressions.

INTRODUCTION

Ever since the arrival of Computer Algebra Systems (CAS), many researchers have studied the role of this kind of technology in the learning of algebra (Thomas, Monahan & Pierce, 2004). According to some researchers (e.g., Artigue, 2002; Lagrange 2003) the technical aspect of algebra (i.e., the symbol manipulation) is fundamental in order to promote students' conceptual understanding. Accordingly, Kieran (2004) has pointed out that, due to the fact that conceptual understanding can come with technique, the study of algebraic transformations will be an area of research interest during the years to come. Thus, it is not a coincidence that in the past few years CAS has played a major role, mainly in those studies related to that aspect of algebraic activity that Kieran (2004) has identified as transformational activity.

In this sense, many studies (e.g., Kieran & Damboise, 2007; Kieran & Drijvers, 2006, Hitt & Kieran, 2009) related with the use of CAS and a technical-theoretical approach to algebra, have indicated the potential of this kind of technology in algebra learning. These studies have shown that the use of CAS promotes conceptual understanding if the technical aspect of algebra is taken into account. For instance, Kieran and Damboise (2007) pointed out how weak algebra students can improve both technically and theoretically by means of a CAS experience involving the factoring of algebraic expressions. Kieran and Drijvers (2006) showed that techniques and theory co-emerge in CAS environments where tasks promote the interaction between CAS and paper-and-pencil media.

According to the reported literature, with respect to CAS studies, little or nothing has been said on the role of CAS technology in students' thinking on the simplification of rational expressions – a task embedded in the transformational activity of algebra. Our interest in studying this domain of school algebra in a CAS environment is based on more than three decades of research that has recognized (e.g., Davis, Jockusch & McKnight, 1978; Matz, 1980) that students have difficulty when they try to manipulate (simplify) rational expressions, making well known errors in tasks of this sort. Thus, the aim of this study is to answer the following research question: Which technical and theoretical aspects are promoted [or emerge] in students' thinking by the use of CAS and an activity designed with a technical-theoretical approach to the simplification of rational algebraic expressions?

THEORETICAL FRAMEWORK OF THE STUDY

The instrumental approach to tool use has been recognized as a framework rich in theoretical elements for analyzing the processes of teaching and learning in a CAS context (e.g., Artigue, 2002; Lagrange 2003). This approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). According to Monaghan (2007), one can distinguish two directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics. In the former, the focus is the development of mental schemes within the process of instrumental genesis. Within this approach, an essential point is the distinction between artifact and instrument (for more details see Drijvers & Trouche, 2008).

In line with the anthropological approach, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology. This approach is grounded in Chevallard's anthropological theory. Chevallard (1999) points out that mathematical objects emerge in a system of practices (praxeologies) that are characterized by four components: *task*, in which the object is embedded (and expressed in terms of verbs); *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology.

Artigue (2002) and her colleagues have reduced Chevallard's four components to three: *Task, Technique,* and *Theory*, where the term *Theory* combines Chevallard's *technology* and *theory* components. Within this (Task-Technique-Theory) theoretical framework a *technique* is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values (Artigue, 2002). According to Lagrange (2003), technique is a way of doing a task and it plays a pragmatic role (in the sense of accomplishing the task) and an epistemic role. With regard to the epistemic value of technique, Lagrange (2003) has argued that:

> Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency. (p. 271)

According to Lagrange (2005), the consistency and effectiveness of the technique are discussed in the theoretical level; mathematical concepts and properties and a specific language appear. This epistemic value of techniques is crucial in studying students' conceptual reflections within a CAS environment. In our study, this T-T-T framework was taken into account in all aspects: the designing of the Activity related to the task "simplifying rational expressions", the conducting of the interviewer interventions, and the analyzing of the data that were collected.

THE STUDY AND METHODOLOGICAL CONSIDERATIONS

In the present paper we discuss and report the results of the first section of the designed Activity, which is part of a wider research study on the role of CAS and a Technical-Theoretical approach to algebra on the simplification of rational expressions.

Rationale of the Designed Activity.

It is important to mention that in this study we use the term Task as is defined in the T-T-T framework; it refers to a question embedded within the Activity. That is, as Kieran and Saldanha (2008) state, the Activity is a set of questions related to a central Task – in our case, the simplification of rational expressions. Since we have adopted the T-T-T framework for conducting the research study, the Activity was designed so that Technical and Theoretical questions were central and, hence, that students would have the opportunity to reflect on both Technical and Theoretical aspects in both paper-and-pencil and CAS environments. In the present report, only the following parts of the activity are reported: first, students' paper-and-pencil work (with Technical and Theoretical questions); second, their subsequent CAS work (Technical question); and, finally, Theoretical questions related to their work in both environments.

Population.

The participants were eight 10th grade students (15 years old) in a Mexican public school. The selection of the students was made by their mathematics teacher, who believed that they were strong algebra students. None of the students were accustomed to using CAS calculators; consequently, at the outset of the study, all the students received some basic training from the interviewer on how to use the TI-Voyage 200 calculator for basic symbol manipulation.

Implementation of the Study.

The data collection was carried out by means of interviews conducted by the researcher. Students worked in pairs; each work session lasted between two and three hours. Each team of two students had a set of printed Activity sheets as well as a TI-Voyage 200 calculator. Every interview was audio- and video-recorded so as to register the students' performance during the sessions.

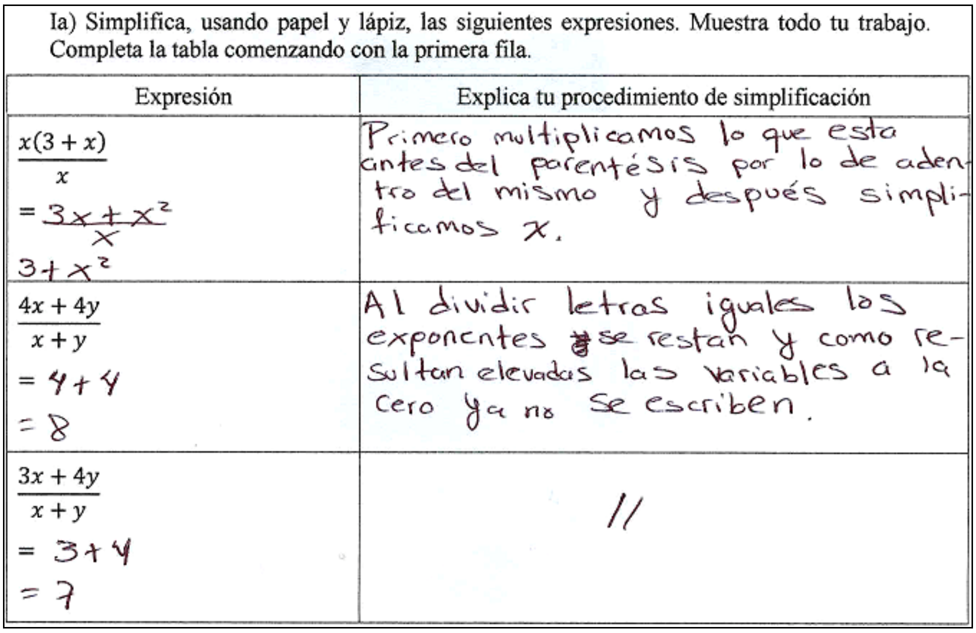
ANALYSIS AND DISCUSSION OF THE DATA

In this report we analyse and discuss only one team's work. This team was chosen for the report because we consider that their work was typical of all participants and represents the role played by both the CAS and the designed Activity (we'll call each member of the team S1 and S2). The analysis, which is qualitative in nature, is based on the team's work sheets, as well as the video-recorded interview. The analysis and discussion of the data is detailed below as follows.

On the paper-and-pencil work related to Technique and Theory.

As per the task design, the first section of the activity helped reveal the students' Technique and Theory related to their paper-and-pencil simplification of rational expressions (see Figure 1). From their work, we confirm that, in this environment, students made the expected errors: they eliminated the 'literal components' that were common to both numerator and denominator, without taking into account whether these 'literal components' were, in fact, a factor of both the numerator and the denominator.

We note too that whenever there were parentheses, the students first tended to expand the expressions of the numerator and denominator (see the first example of Figure 1) before cancelling. This initial expanding, which was not preceded by a first observation in terms of factors, was something that hindered their theoretical reflection and seemed to lead them to make the kinds of errors that are reported in the literature. In their written explanations, they used the terminology of *dividing* (see the second example of Figure 1, where the students wrote, "we divide the same letters").


Ia) Simplifica, usando papel y lápiz, las siguientes expresiones. Muestra todo tu trabajo. Completa la tabla comenzando con la primera fila.

Expresión	Explica tu procedimiento de simplificación
$\frac{x(3+x)}{x} = \frac{3x+x^2}{x}$ $3+x^2$	Primero multiplicamos lo que esta antes del paréntesis por lo de adentro del mismo y después simplificamos x.
$\frac{4x+4y}{x+y}$ $= 4+4$ $= 8$	Al dividir letras iguales los exponentes se restan y como resultan elevadas las variables a la cero ya no se escriben.
$\frac{3x+4y}{x+y}$ $= 3+4$ $= 7$	//

Figure 1. Simplification of expressions: Paper and pencil work.

On the New Technique and Theory, Based on the Use of CAS.

In the context of the designed Activity, the use of CAS led the students to rethink their first techniques and explanations and provoked a theoretical reflection that could explain for them the results given by the CAS. The differences between the two sets of results led them to wonder about their paper-and-pencil techniques and explanations. They began to question the theoretical underpinnings of their work. Figure 2 shows the corresponding students' CAS work.

Ib) Verifica tus respuestas de Ia), para ello, utiliza la calculadora (usa la tecla *enter*). Escribe los resultados dados por la calculadora en la siguiente tabla.

Introduce en la calculadora	Respuesta dada por la calculadora
$\frac{x(3+x)}{x}$	$x+3$
$\frac{4x+4y}{x+y}$	4
$\frac{3x+4y}{x+y}$	$\frac{3x+4y}{x+y}$

Figure 2. Simplification of expressions: CAS work.

For the expressions that involve just one term in the denominator (as in the first example of Figure 2), the students could see that their paper-and-pencil technique was not correct, but could also see how to fix it. As the following extract suggests, they were able to make a quick adjustment to their first technique (adjustment without theoretical justification that called for cancelling each occurrence of the given term in the numerator) so as to eliminate the discrepancy between the results:

1 S1: What is it? [*Asking for the result given by the calculator for the first expression of Figure 2*]

2 S2: *x* plus 3 [*the CAS result for the first expression of Figure 2*]

3 S1: And we wrote 3 plus *x* squared [*She refers to the result which they got by paper and pencil at the time they obtain the CAS result for the first expression of Figure 2*]

4 S2: Yes, We must have taken off only one *x* [*Meaning that they had to eliminate another x*]. No matter. What's next?

However, for the second and third examples of Figure 2, the students could not easily come up with a simple adjustment to their paper-and-pencil technique for simplifying those expressions containing a binomial as the denominator. The following extract illustrates their bewilderment at the CAS results for the last two expressions:

5 S2: Yes, here [*Referring to the first expression of the Figure 2*], it makes sense [*the result given by the calculator*] because the *x's* were taken off, it first multiplied and we missed taking off the two *x's*. [*She states the multiplication procedure that she thinks the calculator did, just as they had expanded the numerator of the first expression of the Figure 2*]. But in here, I'm not quite sure why it's 4, neither the result in here [*Referring*

to the last two results (Figure 2) given by the calculator]. Why it is the same [*referring to the 3rd result of Figure 2*], I don't have any idea.

While they could accommodate the result given by the CAS for the first example, the other two examples remained mysterious. They kept asking themselves if there were other ways to think about these simplifications. How might they justify the results given by the CAS? The following extract underlines their dilemma, but then student S1 suddenly had an idea:

6 S2: It's believed that in this case we should've taken off the *x* and the *y*, we take off both [*The repeated terms in the numerator and the denominator of the 2nd expression in Figure 2*]. But why is it 4? [*The result given by CAS*]

7 S1: Let's see [*Pause*]. This is a division of polynomials!

It is clear that the CAS Technique provoked a conceptual change in one of the students (line 7 of the above transcription). This theoretical reflection induced by the discrepant results moved the students from a Technique involving eliminating literal symbols that are repeated in the numerator and the denominator to a Technique involving division of polynomials (the long division of polynomials algorithm) as can be seen in the next Figure 3.

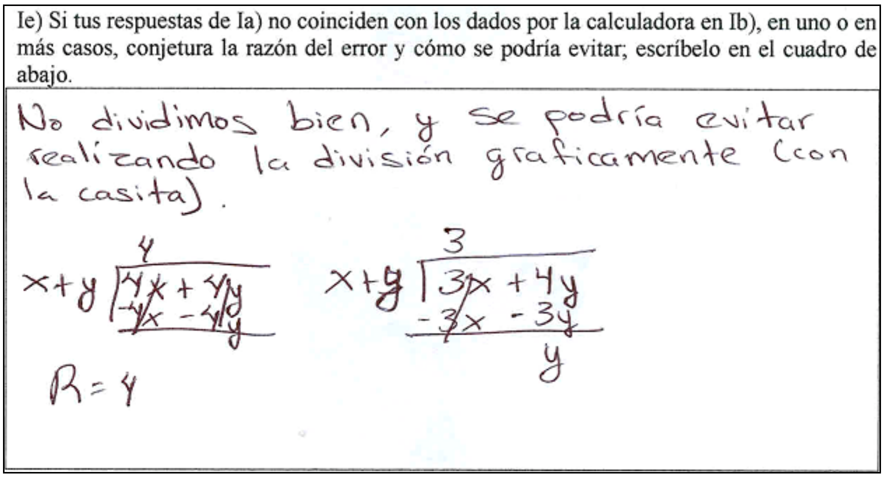

Ie) Si tus respuestas de Ia) no coinciden con los dados por la calculadora en Ib), en uno o en más casos, conjetura la razón del error y cómo se podría evitar; escríbelo en el cuadro de abajo.

Figure 3. New paper and pencil Technique for simplifying rational expressions

It is interesting to see how the students came to adapt their new technique and theory so as to make it also fit the case of rational expressions that could not be simplified. They found, on their own, that if the quotient works out exactly, then the rational expression can be simplified – the quotient of the division being the final simplification. But if the division is not exact, then the rational expression can not be simplified and the CAS calculator will give as the result the same expression. For those cases where the denominator is a monomial, the students continued to believe that the technique of cancelling the monomial of the denominator with all of its occurrences in the numerator is workable.

CONCLUSIONS

In this report we have showed the epistemic role of CAS Technique, in the sense that the use of the CAS provoked in students a spontaneous theoretical reflection that allowed them to think of new techniques for simplifying rational expressions. The use of the CAS provoked a change in the students' technique for simplifying rational expressions whose denominator is a binomial (from canceling 'literal components' that were common to both numerator and denominator to using the polynomial division algorithm as the new Technique). This epistemic role played by the CAS was evident also in terms of the students' language, the students' initial language evolving from "canceling and dividing" terms to using terminology involving the division of polynomials.

However, other technical-theoretical aspects did not emerge, such as noticing the structure of the expressions in terms of factors. Thus new questions arise, such as, *How to promote in students' thinking a focus on seeing the expressions in terms of factors?* CAS technology and appropriate tasks may not be sufficient; teacher intervention may be critical in encouraging technical-theoretical noticing of other aspects of this domain on the part of students.

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