HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inſcripſit, eſt in ordine Decimus.

HAbes in hoc libro, ſtudioſe Lector, Regulas Algebraicas (Itali, de la Coſſa uocant) nouis adinuentionibus, ac demonſtrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam ſeptuaginta euaſerint. Neque ſolum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc autem librum ideo ſeorſim edere placuit, ut hoc abſtruſiſsimo, & planè inexhauſto totius Arithmeticæ theſauro in lucem eruto, & quaſi in theatro quodam omnibus ad ſpectandum expoſito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore faſtidio perdiſcant.

THE GREAT ART
or *The Rules of Algebra*

by GIROLAMO CARDANO

Translated and Edited by
T. RICHARD WITMER
with a Foreword by Oystein Ore

The M.I.T. Press
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS, AND LONDON, ENGLAND

1968

[DEDICATION]

Girolamo Cardano, Physician, to the most erudite *Andreas Osiander*,[1] greetings:

I have considered nothing so deeply, learned Andreas, as the names of those who, by their writings, deserve to be commended to posterity. I have especially asked whether they combine liberal learning with erudition. Hence, since I know that you have a far from mediocre knowledge not only of Hebrew, Greek, and Latin letters but also of mathematics and since, moreover, [you are] the most humane man I have ever encountered, it seemed to me that this, my book, could be better dedicated to no one than you by whom it may be corrected (if my pen has gone further than the power of my mind), read with pleasure, understood, and, indeed, recommended authoritatively. Unless I am mistaken, others will follow this example and dedicate their works [to you], whatever branch of learning they pursue. Accept this, therefore, as a perpetual testimonial of my regard for you, of your assistance to me, and, above all, of your distinguished scholarship. And, although you may well be such that your merits are known to all men, yet just as Alexander and Caesar wished to have their most notable deeds inscribed in the chronicles of others and as Plato, who preserved such marvels in his own writings, still desired to be praised in the writings of others, so I hope that this, my offering to you, will not be displeasing, whatever its merit, because in such matters there is a certain fortune which controls and the better perish while the poorer survive. And whatever your judgment may be in matters of this sort, it nevertheless is certain to me that it is my duty to repay my debts. I hope also that, by this clear example, there may be known my affection for all who have that same candor of spirit for which you are recognized among the scholars of our time. But perhaps a better occasion [for this] will be given; if not, I still do not wish this one, such as it is, to be lost to me.

Farewell. Pavia, the 5th of the Ides of January, 1545.

[1] Andreas Osiander (1498–1552) of Nürnberg was one of the minor leaders of the Reformation. He is generally credited with authorship of the anonymous preface to Copernicus' *De Revolutionibus* (see Preserved Smith, *A History of Modern Culture* (New York, 1930), I, 40 ff; Henry Osborn Taylor, *Thought and Expression in the Sixteenth Century* (2d ed., New York, 1930), II, 336–337; Thomas S. Kuhn, *The Copernican Revolution* (Modern Library ed., New York, 1959), p. 187; A. C. Crombie, *Medieval and Early Modern Science* (New York, 1959), II, 168, 186). James Eckstein in his *Jerome Cardan* (Baltimore, 1946), pp. 7, 23, credits Osiander with having edited *The Great Art*.

Index to the Matters Contained in this Book[1]

Chapter

[1] Some of the chapter titles in this Index vary in minor respects from those that appear at the heads of the chapters themselves.

[2] *De aestimatione generali & equatione, cum media denominatio aequatur extremae & numero.*

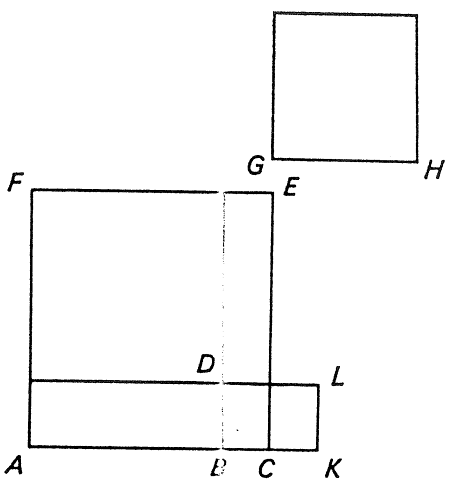
[3] *De regulis maioribus singularibus.*

CHAPTER XI

On the Cube and First Powe· Equal to the Number

Scipio Ferro of Bologna well-nigh thirty years ago discovered this rule and handed it on to Antonio Maria Fior of V:nice, whose contest with Niccolò Tartaglia of Brescia gave Niccolò occasion to discover it. He [Tartaglia] gave it to me in reponse to my entreaties, though withholding the demonstration. Armed with this assistance, I sought out its demonstration in [various] forms. This was very difficult. My version of it follows.

Demonstration



For example, let GH^3 plus six times its side GH (1)equal 20, and let AE and CL be two cubes the difference between which is 20 and such that the product of AC, the side [of one], and CK, the side [of the other], is 2, namely one-third the coefficient of x.(2) Marking off BC equal to CK, I say that, if this is done, the remaining line AB is equal to GH and is, therefore, the value of x, for GH has already been given as [equal to x].

In accordance with the first proposition of the sixth chapter of this book, I complete the bodies DA, DC, DE, and DF; and as DC represents BC^3, so DF represents AB^3, DA represents $3(BC \times AB^2)$ and

DE represents $3(AB \times BC^2)$.[1] (3) Since, therefore, $AC \times CK$ equals 2, $AC \times 3CK$ will equal 6, the coefficient of x; therefore $AB \times 3(AC \times CK)$ makes $6x$ or $6AB$,[2] wherefore three times the product of AB, BC, and AC is $6AB$. Now the difference between AC^3 and CK^3 — manifesting itself as BC^3, which is equal to this by supposition — is 20,(4) and from the first proposition of the sixth chapter is the sum of the bodies DA, DE, and DF. Therefore these three bodies equal 20. (5)

Now assume that BC is negative:

$$AB^3 = AC^3 + 3(AC \times CB^2) + (-BC^3) + 3(-BC \times AC^2), \quad (6)$$

by ~~that~~ the demonstration. The difference between $3(BC \times AC^2)$ and $3(AC \times BC^2)$, however, is [three times] the product of AB, BC, and

[1] 1570 and 1663 vary considerably from here on to the end of the demonstration. They read:

We will have, therefore, four propositions, two of which have already been mentioned — namely, that $AC \times CK$ or CB is 2 and that the difference between AC^3 and CB^3 is 20. The third can be deduced from these and is that, since the product of $3AB \times BC \times AC$ is equal to the sum of [the text has, "the difference between"] DE and DA and that $3AB \times AC \times BC$ is $6AB$ for, from the first proposition, the product of AC and CB is 2, therefore three times this is 6, and this product times AB is $6AB$. This, however, is the sum of [the text has, "the difference between"] DE and DA. The fourth [proposition], which derives from the second and third corollaries in the sixth chapter, [is] that DF [i.e., AB^3] is the difference between $AC^3 + 3(AC \times CB^2)$ and $CB^3 + 3(CB \times AC^2)$. Let, therefore, α be AC^3, β BC^3 [the text has ABC^3], γ $3CB \times AC^2$, δ $3AC \times CB^2$, ϵ the difference between α and β, ζ the difference between γ and δ, and η [1570 has β; 1663's character is illegible] the difference between $\alpha + \delta$ and $\beta + \gamma$. Therefore [there is what appears to be a superfluous *cum* inserted at this point] ϵ is composed of $\zeta + \eta$ [1570 again has a β and again 1663's character is illegible], as can readily be demonstrated numerically and by example as shown in the margin. But ϵ is 20, from the second assumption, ζ is $6AB$, and η [1570's character is illegible and 1663 has θ] is AB^3. Therefore, $AB^3 + 6AB$ — that is, plus $6x$, for AB is the root of its cube — is equal to 20. Therefore, since $GH^3 + 6GH$ [the text has $BH^3 + 6BH$ here and the next place it occurs] is equal to 20, $GH^3 + 6GH$ will be equal to $AB^3 + 6AB$. Hence AB is x and this is the difference between two sides the product of which is 2 and the cubes of which differ by 20, which was to be demonstrated. From this we construct the rule.

	24	1		25	
20			13		7
	4	14		18	

As it is obvious from the preceding the text of 1570 and 1663 is quite corrupt. These items are especially bothersome:

(1) the *differentiae DE et DA* which occurs twice. To make sense, we have either to assume, as I have here, that *differentia* is an error for *aggregatum* or that, in the earlier definitions of DA as *triplus CB in quadratum AB* and of DE as *triplus, AB in quadratum BC*, AB should be replaced by AC. Either is consistent with the later development of the argument.

(2) the *cubus abc* which I read as a typographical error for *cubus bc*.

(3) the confusion between eta, beta, and theta at various points.

(4) the misprinting of BH for GH at four places.

[2] *fiunt 6 res AB, seu sexcuplum AB.*

(7)
AC.[2] Therefore, since this, as was demonstrated, is equal to $6AB$, add $6AB$ to the product of $3(AC \times BC^2)$, making $3(BC \times AC^2)$. ~~But~~ since BC is negative, (8) it is now clear that $3(BC \times AC^2)$ is negative and the remainder which is equal to it is positive. Therefore,

$$3(CB \times AB^2) + 3(AC \times BC^2) + 6AB = 0.^{3} \quad (9)$$

It will be seen, therefore, that as much as is the difference between AC^3 and BC^3, so much is the sum of

$$AC^3 + 3(AC \times CB^2) + 3(-CB \times AC^2) + (-BC^3) + 6AB. \quad (10)$$

This, therefore, is 20 and, since the difference between AC^3 and BC^3 is 20, (11) then, by the second proposition of the sixth chapter, assuming BC to be negative,

$$AB^3 = AC^3 + 3(AC \times BC^2) + (-BC^3) + 3(-BC \times AC^2). \quad (12)$$

Therefore since we now agree that

$$AB^3+6AB^{[4]} = AC^3+3(AC\times BC^2)+3(-BC\times AC^{2[5]})+(-BC^3)+6AB,$$

which equals 20, as has been proved, they [i.e., $AB^3 + 6AB$] will equal 20. Since, therefore,

13
$$AB^3 + 6AB = 20,$$

and since

$$GH^3 + 6GH = 20,$$

it will be seen at once and from what is said in I, 35 and XI, 31 of the *Elements* that GH will equal AB. Therefore GH is the difference between AC and CB. AC and CB, or AC and CK, the coefficients, however, are lines containing a surface equal to one-third the coefficient of x and their cubes differ by the constant of the equation. Whence we have the rule:

Rule

Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate[6] this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. You will then have a *binomium* and its *apotome*. Then,

[3] *faciunt nihil.*

[4] The text has a spare *cum* at this point, as though something more were to be added.

[5] 1545 has AB^2.

[6] 1545 has *seminabis*; 1570 and 1663 have *servabis*. The former, corrected to read *geminabis* in accord with later passages, is followed here.

subtracting the cube root of the *apotome* from the cube root of the *binomium*, the remainder [or] that which is left is the value of x.[7]

For example,

$$x^3 + 6x = 20.$$

Cube 2, one-third of 6, making 8; square 10, one-half the constant; 100 results. Add 100 and 8, making 108, the square root of which is $\sqrt{108}$. This you will duplicate: to one add 10, one-half the constant, and from the other subtract the same. Thus you will obtain the *binomium* $\sqrt{108} + 10$ and its *apotome* $\sqrt{108} - 10$. Take the cube roots of these. Subtract [the cube root of the] *apotome* from that of the *binomium* and you will have the value of x:

$$\sqrt[3]{\sqrt{108} + 10}^{[8]} - \sqrt[3]{\sqrt{108} - 10}$$

Again,

$$x^3 + 3x = 10.$$

Cube 1, one-third of 3, and 1 results; square 5, one-half of 10, and 25 results; add 25 and 1, making 26; add 5 to and subtract it from the square root of this. You will thus form the *binomium* $\sqrt{26} + 5$ and its *apotome* $\sqrt{26} - 5$; whence x equals $\sqrt[3]{\sqrt{26} + 5} - \sqrt[3]{\sqrt{26} - 5}$. Here you have the proof:

	$\sqrt[3]{\sqrt{26} + 5}$	$-\sqrt[3]{\sqrt{26} - 5}$[9]
The cubes of the parts: (As is evident, the sum of these is 10.)	$(\sqrt{26} + 5)$	$-(\sqrt{26} - 5)$
The squares of the parts:	$\sqrt[3]{51 + \sqrt{2600}}$[10]	$\sqrt[3]{51 - \sqrt{2600}}$
Three times the squares of the parts:	$\sqrt[3]{1377 + \sqrt{1{,}895{,}400}}$[11]	$\sqrt[3]{1377 - \sqrt{1{,}895{,}400}}$[12]
The parts themselves:	$-\sqrt[3]{\sqrt{26} - 5}$	$+\sqrt[3]{\sqrt{26} + 5}$
The products of the parts and three times their squares:	$+\sqrt[3]{\sqrt{49{,}299{,}354} + 6885 - \sqrt{47{,}385{,}000} - 7020}$	$-\sqrt[3]{\sqrt{49{,}299{,}354} - 6885 - \sqrt{47{,}385{,}000} + 7020}$

[7] I.e., if $x^3 + ax = N$, $x = \sqrt[3]{\sqrt{(a/3)^3 + (N/2)^2} + N/2} - \sqrt[3]{\sqrt{(a/3)^3 + (N/2)^2} - N/2}$.

[8] 1570 and 1663 have R̸ *b: cub:* R̸ 108 *p:* 10, the *b:* being a misprint for *v:*.

[9] 1570 and 1663 have $\sqrt[3]{\sqrt{27} - 5}$.

[10] 1570 and 1663 have $\sqrt[3]{51 + \sqrt{2900}}$.

[11] 1570 and 1663 have $\sqrt[3]{1277 - \sqrt{1{,}895{,}400}}$.

[12] 1570 and 1663 have $\sqrt[3]{1377 - \sqrt{1{,}865{,}400}}$.

Moreover, the cube roots contain four terms which can be reduced to two, for when 6885 is subtracted from 7020, the remainder is 135, and likewise when $\sqrt{47{,}385{,}000}$ is subtracted from $\sqrt{49{,}299{,}354}$ there is left $\sqrt{18{,}954}$.[13] Therefore these products are

$$\sqrt[3]{\sqrt{18{,}954} - 135} - \sqrt[3]{\sqrt{18{,}954} + 135}.$$

The whole cube, then, from the demonstration in the third book is $10 + \sqrt[3]{\sqrt{18{,}954} - 135} - \sqrt[3]{\sqrt{18{,}954} + 135}$, and three times the root, or $3x$, equals $\sqrt[3]{\sqrt{18{,}954} + 135} - \sqrt[3]{\sqrt{18{,}954} - 135}$. And, finally, having added all together, since the universal cube roots cancel each other, the whole becomes

$$x^3 + 3x = \text{exactly } 10.$$

A third example:

$$x^3 + 6x = 2.$$

Raise 2, one-third the coefficient of x, to the cube and 8 is the result; square 1, half of 2, making 1; add 8 to 1, and 9 is produced, the square root of which is 3. Now duplicate 3 and to one add 1, half the constant, thus making 4, and from the other subtract half the constant, thus making 2. Then subtract the cube root of the less from the cube root of the greater and you have $\sqrt[3]{4} - \sqrt[3]{2}$[14] as the value of x.

Remember what we said in the chapter in the third book on extracting cube roots whenever these universal cube roots are equivalent to a whole number or a fraction. Thus in the first example

$$\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$$

is 2, as is indicated by the rule there given and as is perfectly clear if it is tried out.[15]

[13] 1663 has $\sqrt{18{,}854}$.

[14] 1663 has $\sqrt[3]{1}$.

[15] 1570 and 1663 add:

It is easy to understand both in this and in the following chapters that, having the solution and the coefficient of x, we obtain the constant of the equation by multiplying the solution by the coefficient of x and adding to this product the cube of the same [i.e., the cube of the value of x], for the sum is the constant of the equation. Thus, given $x^3 + 3x$ and the value of x as 2, I say that you should multiply 2 by 3, making 6, and add this to 8, the cube of 2, making 14, the constant of the equation. Similarly, given x^3 plus a certain number of x's equal to 20 and the value of x [the text seems to say "the value of x^3"] as $\sqrt{8} - 2$, for example, we can derive the coefficient of x by raising $\sqrt{8} - 2$ to the cube, making $\sqrt{3200} - 56$, subtracting this from the constant, which is 20, leaving $76 - \sqrt{3200}$, and dividing this by $\sqrt{8} - 2$, the solution, making $\sqrt{648} - 2$, the coefficient of x.

You also know that there may be a solution common to all types of cases as, for instance, of the cube and constant equal to the first power. Thus if

$$x^3 + 12 = 34x,$$

and the value of x is $3 + \sqrt{7}$ or $3 - \sqrt{7}$, and I wish x^3 plus a number of x's to equal 12 with the latter solution, the number of x's will be, according to the preceding rule, $\sqrt{1008} + 2$. By following the procedure of this chapter and taking one-third the coefficient of x, which is $\sqrt{112} + \frac{2}{3}$, and raising it to the cube, making $\sqrt{1{,}438{,}577\frac{7}{9}} + 224\frac{8}{27}$, and adding 36, the square of one-half of 12, the constant of the equation, you will have $\sqrt{1{,}438{,}577\frac{7}{9}} + 260\frac{8}{27}$. Add 6 to or subtract it from [the square root of] this and take the cube root [of the whole] and you will have the value of x, viz., $\sqrt[3]{\sqrt{\sqrt{1{,}438{,}577\frac{7}{9}} + 260\frac{8}{27}} + 6} - \sqrt[3]{\sqrt{\sqrt{1{,}438{,}577\frac{7}{9}} + 260\frac{8}{27}} - 6}$.

Note: In lieu of $\sqrt{1{,}438{,}577\frac{7}{9}}$ at the four places at which it appears above, 1570 and 1663 have these figures:

1570:	$\sqrt{1{,}905{,}552}$	$\sqrt{1{,}905{,}552}$	$\sqrt{190{,}555}$	$\sqrt{190{,}555}$
1663:	$\sqrt{1{,}905{,}552}$	$\sqrt{1{,}905{,}552}$	$\sqrt{1{,}905{,}552}$	$\sqrt{190{,}555}$

1570 also has in the first term *cui* instead of *cub* and in the second *ut* instead of *cub* to indicate that these are universal cube roots. It also has a superfluous ℞ in front of the $260\frac{8}{27}$ the first time it occurs. 1663 omits any indication that the second term is a *cube* root.