Towards a Theory of Visualization by Dynamic Geometry Software
Paradigms, Phenomena, Principles

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Dynamic geometry software (DGS) has become widespread. The extending scope of experiences increases the need for a background theory: *„To understand students' (mathematical) experiences, it is important to characterize DG's behaviour (in a mathematical way). Such an analysis might be treated as a definition of (existing) DG and should naturally precede empirical work with students: It is the preparation after which we can ask which phenomena students notice, and how they perceive and interpret those phenomena."* (Goldenberg & Cuoco 1998). In this paper, we reconsider some common DGS ***paradigms*** in view of several crucial ***phenomena*** and reveal underlying ***principles*** that govern any DGS use, yielding constraints for and inevitable side-effects of what one can expect from a DGS.

Our results may influence teacher trainings also in another way: namely to foster the central demand of Whiteley (2000) that "*students as 'cognitive apprentice' can practice geometry. If such an apprenticeship is the goal, then we face issues that are hidden if the discussion is based on our current curriculum and preconception…. The preparation of teachers in the use of living geometry is key"* – and by taking the results below into account, we can obtain that even departing from our current curricular contents: by elucidating their link to fundamental mathematical notions and modern techniques encorporated when presented dynamically. On-topic activities for teacher education and advanced classroom practice that emerge quite naturally from the following examples are presented elsewhere.

I. Paradigms

In search for axioms that characterize the suitable behaviour of DGS (dynamic geometry software) mathematically, one may naturally come up with a superset of the following ones (see e.g. Laborde 1999, Kortenkamp 1999):

Continuity Paradigm *DGS should move objects continuously in drag mode.*

Determinism Paradigm *For any position of the base points, the position of the constructed elements should be uniquely determined.*

But unfortunately, all current DGS like "Cabri" or "Cinderella" violate one of them:

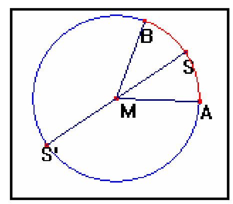
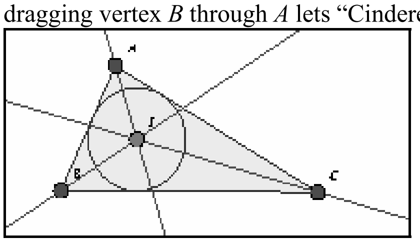


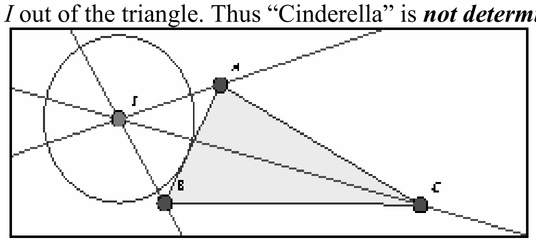
Figure 1

II. Phenomena

Example 1 The bisector w of $\angle AMB$ intersect k in S and S' (fig. 1). Moving A along k with "Cabri", after a full turn S and S' will suddenly "jump" across the diameter to return to their original position – so "Cabri***" behaves discontinuously***!

….**Example 5** By dragging the vertices of the triangle students may discover that its angular bisectors meet inevitably in one point I. However, it may be considered harmful (especially with regard to the interpretation of I as incentre), that dragging vertex B through A lets "Cinderella" move I out of the triangle. Thus "Cinderella" is ***not deterministic***!





Especially the last example, (which actually occurred in classroom!) is apt to put the teacher into real difficulties, so one is driven to demand a DGS that conforms to both paradigms. But unfortunately that is impossible:

III. The first principle of Dynamic Geometry

Though both paradigms above are certainly desirable - they are mutually exclusive!

Exclusion Principle *A continuous DGS cannot be deterministic.*

Before we prove this, some reflections on the didactical implications: Of course it necessitates most careful reflections on *when* to use *what* kind of DGS in order to minimise the unwanted, but unavoidable side effects of violating one of the principles. However, switching between the two worlds without comment may also cause problems for the students, since what is true for one DGS does not hold for the others. So one may even think of deciding once and forever between the principles and confine oneself to the less harmful one. And though at first sight, continuity looks more promising, one may well be tempted to ban it from the classroom, having in mind the possibly disastrous after-effects of Example 5 on students' understanding. Well – but the matter is even more intricate: In V. an example will be presented where one may *want* the incentre to move out of the triangle for good reasons.

IV. Towards a Theory of the Drag-Mode

As usual, a figure F is conceived as set of points in the Euclidean plane E. Dragging a base point along a curve $\mathcal{C}$ amounts mathematically to traversing a parametrization $\gamma : [0, a] \to E$ of $\mathcal{C}$. By virtue of the construction of F, the DGS then produces a family F_t of figures, where $t \in [0, a]$ denotes time. The entirety of the dragged figures is comprised to a new entity, the **drag-figure** $\mathcal{F}$, by the following construction

$$\mathcal{F} = \bigcup_{t \in I} \{t\} \times F_t \subset I \times E.$$

In $\mathcal{F}$, the individual figures F_t are arranged like slides in a projector. Now we can formalise our paradigms:

Definition 1 *A DGS is **deterministic** iff for every figure F and every drag path* γ*:* $[0, a] \to E$

$$\forall s, t \in I : \gamma(s) = \gamma(t) \Rightarrow F_s = F_t .$$

Definition 2 *A DGS is **continuous** iff for every figure F and every drag path* γ *the drag-figure* $\mathcal{F}$ *is defined by equations that depend continuously on t.*

Applying these notions, we can now give a rigorous proof of the exclusion principle that can be readily visualized as follows: the drag-figure of the figure in example 1 is just a Moebius-strip, see fig. 6. It is well-known that this is a non-orientable surface. From this, the theorem is easily derived (in the paper).

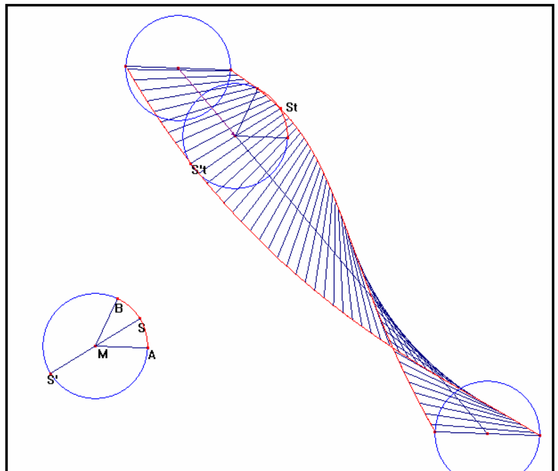


Figure 6

V. Examples that necessitate a new notion of construction

(For lack of space we give here but one, see paper for more.)

Locus problems are a powerful realm for developing strategies and deepening the understanding of fundamental ideas. An important one is surely the *Cartesian Correspondence* between curves and their equations. It can be explored by a profitable interplay of CAS and DGS:

Example *I* The locus of the incentre *I* of an isosceles triangle *ABC*, when *C* moves on a circle through *B* centred at *A*. The straightforward way of constructing the locus $\mathcal{J}$ yields seemingly a curve with a cusp (Figure *I*a). But with elementary trigonometry one obtains a parametric representation for *I* that can be transformed to an equation by some algebraic yoga with Pythagoras' theorem. This equation, however, turns out to describe a strophoid – and the strophoid is known to have a node as singular point! What's going on? Some curve plotting reveals that the geometric locus is only the "inner part" of the curve that is described by either the parametrization or the equation. So one is tempted to conclude that the continuous behaviour of "Cinderella" is didactically favourable in this case as it allows to produce the complete locus without extra effort (Fig. *I*b). But the price one has to pay for this is to accept that then the incentre *I* has to move out of the triangle *ABC* in every second pass of *C* through *k*.

Both drag modes yield thus unsatisfactory results, and because of the exclusion principle we cannot hope to combine them. Thus we are directed towards rethinking the construction as such – which will lead to a surprising remedy…

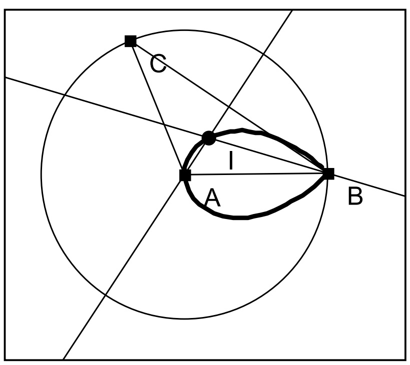


Figure *I*a

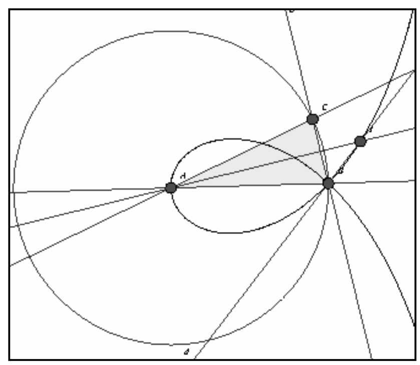


Figure *I*b

VI. The power of the dynamic ruler

Is dynamic geometry "just like with ruler and compass"? To answer this, we take into account *macros* and *loci* as new tools. It will be shown that the rich possibilities of these *dynamic* tools can be concisely described by the properties of a *static* ruler! In particular: the circle can be *dynamically* constructed with the ruler alone!

Beforehand, we show that middle perpendiculars, angle bisectors and altitude can be constructed by ruler as well. This is surprising as they are based on metric properties like "perpendicular" or "halving". These of course cannot be represented by ruler alone – but it is possible to encapsulate them in the set of starting points: From $x_1 = (1,0)$, $x_2 = (2,0)$, $y_1 = (0,1)$ and $y_2 = (0,2)$ one can construct the coordinate axes with their origin *U*. These are thus straight lines on which one has two points and their midpoints. For such a straight line *AB*, however, the parallel through a given point *F* can be drawn, provided one has another point *R* on *AF* at one's disposal whose existence must be ascertained beforehand: If *E* is the given(!) centre of *A* and *B*, $D = ER \cap BF$ and $G = AD \cap BR$, thus $AB \parallel FG$, cf. fig. 14.

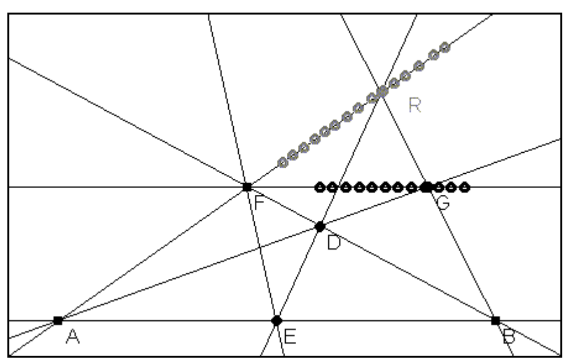


Figure 14

In the same vein, most of the elementary geometric constructions can be reduced to drawing parallels (the centres required as auxiliary points always being easily found). Consequently, a parallel ruler suffices to solve almost all elementary geometric construction problems:

Perpendicular to a straight line g through U: g shall intersect with the x-axis in Q. The parallel to x_1y_1 through P shall intersect with the x-axis in P', the parallel through Q shall intersect with the x-axis in PQ'. $P'Q'$ is the mirror image of $g = PQ$ at the first bisector. Let P'' be the point of intersection with the parallels to the coordinate axes through P' and Q'. Then UP'' is perpendicular to g, see figure 15.

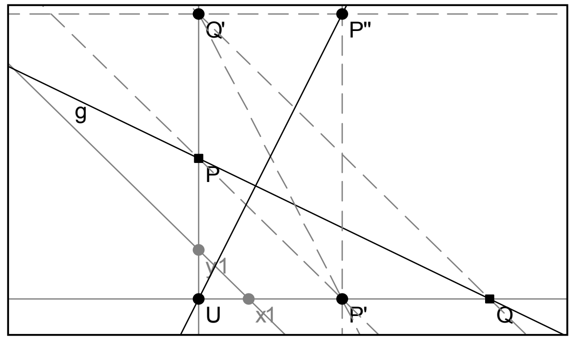


Figure 15

Perpendicular to g in an arbitrary point R: This perpendicular is obtained as parallel to the perpendicular just constructed through the point R.

Midpoint $M = MP\,(A,B)$ of the points A and B: Let C be a point outside of AB, and D be a point on the parallel to AB through C. Let Because of the harmonic properties of the complete quadrilateral, $M = AB \cap EF$, $E = AC \cap BD$ und $F = AD \cap BC$ (see fig. 16 and Bieberbach 1952).

Mid perpendicular of the points A and B: this is of course the perpendicular on AB in M.

The medians and altitudes of the triangle are obtained correspondingly. Consequently, the centre of the triangle's circumcircle, the point of gravity and the orthocentre can also be constructed only by means of the ruler!

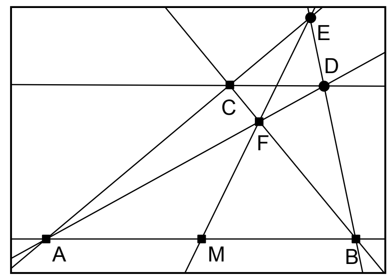


Figure 16

In doing this, we have seen that one cannot only reduce drawing parallels to dropping the perpendicular as usual, but also vice versa. For reasons of simplicity, it is suggestive to use a ruler which commonly (at least in German schools) serves to construct perpendiculars as well: the *angle-hook*. It can thus be concluded:

Elementary geometry is geometry by angle-hook!

NB: The power of the angle-hook does *not* exceed that of the common ruler, it only enhances the practical feasibility of a ruler constructiong, the starting points of which contain the metrical data.

Many ruler constructions are classical (Pappus, Steiner). For practical purposes, however, these constructions were not feasible because of the necessary effort. This is where dynamic geometry software (DGS) provides its *one* decisive contribution: The "rulerized" constructions can be encapsulated in macros and thus for the first time comprehensively carried out and concatenated.

For constructing the circle, however, another contribution of DGS is essential: *dynamics*. Once a "general" right angle above the segment PQ has been constructed utilizing the angle-hook, one produces the entire circle as locus of its vertex S by moving any point R of the angle-hook on a straight line g: The circle thus proves to be a ***dynamic ruler construct***, as depicted in fig. 17.

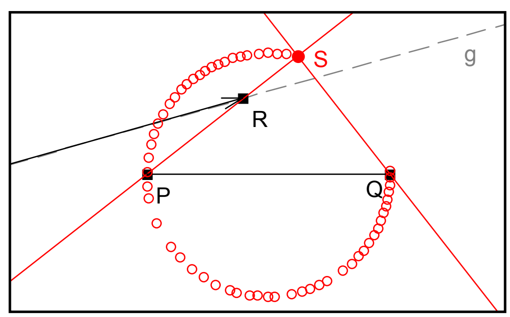


Figure 17

VII. The second principle of Dynamic Geometry

It is well known in static geometry that the ruler-constructible points are just those which have coordinates rationally dependent on the coordinates of the starting points. Dynamizing this we get an analogous description of rational parametrizable curves. By combining two classical results of elementary algebraic geometry (Brieskorn & Knörrer 1986) , namely the Plücker formula and the Lüroth theorem, we derive a handy criterion for ruler constructibility:

The ruler principle *Most elementary geometric constructions can be accomplished dynamically by ruler alone: A locus is a ruler curve iff the following relation between the degree d and the number r of double points and cusps holds:* $r = (d-1)\,(d-2)/2$. *If so, a construction for its general point of the curve will most probably be also rulerizable.*

We apply this line of thought to the example of the strophoid generated by a moving triangle's incentre (the paper has more examples!): Though from the classical theory we could derive readily ruler constructions for altitude, perpendicular bisectors etc., the situation is different with angular bisectors. As a rule, these are not ruler constructible – but nevertheless this applies to the incentre!

With regard to the angles themselves, consider first that they are constructible by ruler iff the angle's tangent is rational. Accordingly, not every constructible angle can be halved by ruler, e.g. the 45^{o} angle cannot. But the locus of I while varying C on k is a strophoid which is a ruler curve by the criterion above – this is strong evidence that one may also find a ruler construction of the incentre itself! To get the right idea, take into account that ruler constructions are necessarily deterministic, so one thereby obtains only that part of the strophoid which lies within *k*. Thus in order to make a deterministic incentre leave the triangle one has to restructure the construction. This can be achieved by the following construction: Let the rotation of the angular bisector *wB* about *B* be guided by $\beta = \angle ABC / 2$ instead by *C* to– say by moving the point *Z* on a circle around *B* (fig.20).

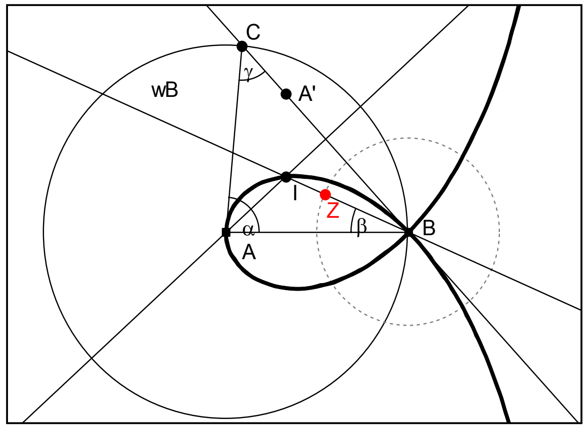


Figure 20

Since the bisector at *A* is ruler constructible as perpendicular bisector of *BC*, we obtain a ruler construction for *I* from one of the point *Z*. But we know from VI how to get this. Consequently, the whole strophoid as well as the extra-triangular incentre can be produced by a deterministic DGS via a ruler construction.

In the paper, we discuss more examples:

- the loci of the orthocentre – these yield all types of quadrics and cubics,
- the conchoids of Nicomedes.

But there are a lot more interesting examples. By way of contrast, it is far more difficult to find “classical “ curves to which the ruler principle does not apply: Perhaps the most elementary example are the ovals of Cassini. For them, the use of a compass is essential.

VIII. Perspectives

Let us close with a theoretical perspective. That most of the points and curves hitherto are constructible by ruler, is good news in the first place: for rational curves, equations can be effectively determined, and intersection points can be exactly calculated. This makes them suitable for any kind of professional geometric data processing. In computer-aided design (CAD), the term “parametrization” has meanwhile even become synonymous with “rational parametrization”, for these obviously represent the only practically usable ones.

Last not least, the exclusion principle applies also in parametric CAD: it serves to explain the “persistent naming problem” (Hoffmann 1996) and makes clear that there can be no easy ad-hoc remedy. But of course it is a real nuisance when parts of a technical drawing jump whilst changing continuously a parameter. But now the ruler principle shows a conceptionally sound way to refine CAD software in order to circumvent the “persistent naming problem”.

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