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Approaching functions: Cabri tools as instruments of semiotic mediation

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Abstract Assuming that dynamic features of Dynamic Geometry Software may provide a basic representation of both variation and functional dependency, and taking the Vygotskian perspective of semiotic mediation, a teaching experiment was designed with the aim of introducing students to the idea of function. This paper focuses on the use of the Trace tool and its potentialities for constructing the meaning of function. In particular, starting from a dynamic approach aimed at grounding the meaning of function in the experience of covariation, the Trace tool can be used to introduce the twofold meaning of trajectory, at the same time global and pointwise, and leads students to grasp the notion of function.

Key words semiotic mediation · dynamic geometry environment · Cabri-géomètre · trace tool · dragging tool · covariation · trajectory · variable · function

1 Introduction

The notion of function has long been the object of a great number of studies, and the extensive literature reports on numerous difficulties related to different aspects of this notion. This interest is not so surprising since the notion of function is central in mathematics and essential in experimental sciences for modelling real world phenomena. Both curricula and research have mainly focused on numerical functions, especially in an algebraic domain, and on the relationship between various representations. The classic book of collected papers edited by Dubinsky and Harel (1992) provides a good example of such

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studies. It is interesting to note that some studies adopt a perspective that links historic, epistemological and cognitive analyses (Sfard 1991; Sierpinska 1992). Difficulties are also widely described; most of them are related to the interpretation of graphic representation in terms of function (Dreyfus and Eisenberg 1983; Tall 1991, 1996; Vinner and Dreyfus 1989; and, for a thorough review, see Leinhardt et al. 1990). Even if not many authors explicitly address the covariational aspect of function, some empirical data show that students have problems grasping the key idea of function as a relationship between variables (in particular, one depending on the other) (Carlson 1998).

Generally speaking, the notion of variable has been investigated in a number of studies showing its complexity and the influence of particular teaching approaches in algebra. Students show difficulties in grasping the multifaceted idea of variable, and in particular they think of the functional relationship between variables only in terms of discrete values. This difficulty persists over the years, at different school levels, and seems to be related to the fact that students first encounter discrete numerical variables (Trigueros and Ursini 1999, p. 277). Similarly to Carlson's results, Trigueros & Ursini report on the difficulty in identifying the correlated variation of two variables, moreover they claim that this may also constitute an obstacle to symbolizing a relationship (ibid. p. 278).

Primacy of numerical setting (Goldenberg et al. 1992) and the lack of experience of functional relationship in a qualitative way may be considered as one source of the students' difficulties. In a general paper about functions at the undergraduate level, Thompson (1994) likewise claims that in mathematical curricula there was no emphasis on function as covariation and that very little has been investigated in regard to students' concepts of variable magnitude. In order to face some of the difficulties, several teaching approaches were proposed, most of them based on the utilization of computational technology, and sometimes new environments were specifically designed for this purpose (Confrey and Maloney 1996; Schwartz and Yerushalmy 1992).

These approaches focus on the possibility of exploiting multiple representations and in some cases the dynamic control that can be achieved by means of different devices: for instance numerical variation can be achieved through the dragging facilities of a slider, as mentioned in the extensive review by Kieran and Yerushalmy (2004, p. 120). Nevertheless numerical dependency is central and, in spite of its appearance, any dynamic device makes sense only by referring to the variation of the number that it represents.

Taking a different perspective, Hazzan and Goldenberg (1997) propose an interesting alternative to the numerical approach, based on dynamic geometry. The authors focus their attention on geometrical variables and their study investigates students' perception of functional dependency in a dynamic geometry environment. They interview students and observe that students seem to recognize functional dependency between geometrical elements (points, lines...). What the authors observe is the use of the words dependent/independent according to a common sense meaning which sometimes may and sometimes may not clash with mathematical meaning.

In the same vein as the approach of Hazzan & Goldenberg, we present in this paper an alternative approach to function starting in an environment providing a qualitative experience of covariation, and in particular an experience of functional dependency not primarily based on a numerical setting. In fact, as observed by Goldenberg et al. (1992), dynamic geometry incorporates functional dependency between geometrical variables and allows us to think of geometrical links in terms of functional dependency.

According to these assumptions, teaching experiments were set up aimed to introduce students to function as covariation, starting within the geometrical domain, in particular within a dynamic geometry environment (hereafter DGE). In these experiments, the potentialities of a DGE are exploited according to the Vygotskian perspective of semiotic mediation (Bartolini Bussi and Mariotti in press; Mariotti 2002; Vygotsky 1978).

This paper starts with a short epistemological analysis providing a rationale for our experimentation and then reports on some results of the teaching experiment. In particular it aims at illustrating the process of semiotic mediation offered by the Trace tool. This will be done by discussing some written texts produced by students and an episode drawn from a collective discussion.

2 Function as change

Similarly to what happened for other basic mathematical notions, a formal definition of function appeared rather late when, at the beginning of the 20th century, it was formulated within set theory as the correspondence between two sets (Bourbaki 1939).

As pointed out by Malik, a deep gap separates early notions of function, based on an implicit sense of motion, and the modern definition of function, that is "algebraic in spirit. It appeals to the discrete faculty of thinking and lacks a feel for the variable" (Malik 1980, p. 492). The mathematical genesis of the idea of function may be regarded as the longstanding attempt to evacuate the idea of motion; for instance, it was one of the ideas that Lagrange intended to eliminate from the theory of analytical functions (Bottazzini 1990, p. 66), although his previous work, the *Mécanique analytique*, was entirely written in the language of infinitesimals and based on the principle of virtual velocity.

A connection between the formal static definition of function and the basic metaphor of motion can be recognized in the idea of graph. In fact, the curve constituting the spatial representation of a function in a coordinate plane may be interpreted as the trajectory of the moving point P with coordinate (x, f(x)) imagining that point M, whose abscissa represents the independent variable, moves on the axis of abscissas (Laborde 1999, p.170). This complex interpretation requires the reintroduction of time and the consideration of the covariation of P and M as a relation between two interrelated variations depending on time, what we call a *dynamic interpretation of a graph*.

This dynamic interpretation is often neglected in textbooks. In any case, a dynamic interpretation of a graph cannot be externally experienced and remains a sort of mental experiment, impossible to be shared. As a consequence, the expressive potential of graphical representations may remain not completely exploited. Differently from other approaches, in our approach the idea of function is primarily rooted in the experience of functional dependency between motions, so that the idea of trajectory, i.e. a dynamic interpretation of a curve, is expected to be available when the notion of graph is introduced. The following section is devoted to explaining the potential of a DGE with respect to the notion of function and that of trajectory.

3 Variation and covariation in the DGE

Motion is certainly one of the main features of a DGE, and in particular of the Cabri microworld. Using the Dragging tool, activated through the mouse, different objects on the screen can be moved, according to two main kinds of motions: direct and the indirect motion.

- The *direct motion* of a basic element (for instance a point) represents the variation of this element in the plane. In the case of a basic point, this is the way of representing, in

Cabri, a *generic point* in a plane. The motion of a *point on an object* consistently represents the variation of a point within a specific geometrical domain, a line, a segment, a circle, and the like.

➢The *indirect motion* of an element occurs when a construction is accomplished. In this case, dragging the basic points from which the construction originates will determine the motion of the new elements obtained through the construction; this motion will preserve the geometrical properties defined by the construction.

As a consequence, the use of the Dragging tool allows the user not only to experience the combination of two interrelated motions (covariation), but also the dependency between the movement of the basic points and that of the constructed points. In other words, the use of this tool allows one to feel functional dependency in the basic semantic domain of space and time.

The Cabri environment provides an additional tool for representing motion. In fact, the Trace tool displays the trace of a moving point and objectifies (Radford 2002) the motion of both independent and dependent variable points, enabling the progressive display of two correlated trajectories as they are generated point by point. Once they are completely drawn, these trajectories can be globally perceived as two sets of points and both the domain and the range of the corresponding function become visible.

Although the final product of the Trace tool is a static image consisting of a set of points, the use of Trace tool involves time: one can actually feel time passing in the action of dragging, in particular when changing the speed of dragging, but one can also feel time passing in the variation of the dependent point. As a consequence it is possible to grasp simultaneously the global and the pointwise aspect of the product of the Trace tool, which can be related to the global and pointwise aspects of the notion of trajectory.

4 Theoretical framework: Tools, signs and meanings

Artefacts of any kind are central elements in the Vygotskian theoretical frame: products of human activity, they play a fundamental role in cognitive development. The Vygotskian perspective assumes a dialectical dependency between practical tools and symbolic tools, unlike other psychological approaches, that clearly separate them. As Vygotsky (1978) expresses it:

> The invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose, and so on) is analogous to the invention and use of tools in one psychological respect. The signs act as an instrument of psychological activity in a manner analogous to the role of a tool in labour. (p. 52)

Vygotsky stresses both the difference and the commonalities between *tools* and *signs*, taking the case of language as paradigmatic; if “the basic analogy between sign and tool rests on the mediating function that characterizes each of them” (ibid., p. 54), the main difference consists in the “ways that they orient human behaviour” (ibid., p. 55). The difference is clearly expressed in the following passage:

> The tool’s function is to serve as the conductor of human influence on the object of activity; it is externally oriented [...]. The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is internally oriented. (p. 55)

But the link between tools (externally oriented) and signs (internally oriented) goes beyond pure analogy in their functioning and rests on the real tie that can be recognized between particular tools and particular signs. One could say that externally oriented tools may be transformed into internally oriented tools.

The process by which this transformation occurs is based on the central hypothesis concerning the process of *internalization*. According to Vygotsky, in the process of internalization, previous socially lived experience is individually elaborated: interpersonal processes are transformed into intrapersonal ones. Internalization expresses the genetic and evolving relationship between external interpersonal processes and their internal intrapersonal counterpart, as Wertsch and Addison Stone (1985) point out, quoting Leont'ev:

> The process of internalization is not the transferal of an external activity to a pre-existing internal "plane of consciousness": it is the process in which this plane is formed. (p.162)

The process of internalization occurs through semiotic processes, in particular by the use of a semiotic system in social interaction. Although not limited to natural language, the analysis of the internalization process is centred on the functioning of the system of signs involved in the activity, i.e. signs such as words, drawings, gestures and the like accompanying actions (Wertsch and Addison Stone 1985).

According to this interpretation, the two main components of social activity, systems of signs and semiotic processes, become basic components of individual activity as well in the process of internalization. The analysis carried out by Vygotsky is consistent with this hypothesis, which claims that the use of particular signs, and more generally of *speech*, contributes to the emergence of internal processes (mental processes), such as concept formation, ways of thinking, cognitive abilities, and generally speaking, what he calls high mental functions.

In summary, a Vygotskian perspective may explain the contribution of tool mediated action to concept formation: signs generated in relation to the use of a tool, through the complex process of internalization accomplished after social interchange, may shape new meanings. In this respect, a specific tool may function as a *semiotic mediator*. At first, externally oriented, a tool is used in action to accomplish a specific task, then, within semiotic activities under the guidance of an expert (for instance, the teacher), the articulation of new signs, generated by (derived from) actions with the tool, may foster an internalization process producing a new *psychological tool*. This new tool is internally oriented, completely transformed, but still maintains some aspects of its origin.

The complex semiotic process, briefly described above, explains in what sense a specific tool may be considered an *instrument of semiotic mediation*. This expression does not refer to the concrete act of using a tool to accomplish a task, rather to the fact that new meanings, related to the actual use of a tool, may be generated and evolve, under the guidance of an expert (Mariotti and Bartolini Bussi 1998; Mariotti 2001).

As explained above, dynamic geometry offers a powerful environment incorporating the semantic domain of space and time, where the notion of function can be grounded. We call that particular instance of function *dynamic geometrical function*. This general idea can be interpreted in a Vygotskian perspective, according to the notion of semiotic mediation.

The Dragging tool may be considered as a sign referring to the idea of function as covariation between dependent and independent variables. Similarly, the Trace tool may be considered as a *sign* referring to the mathematical notion of trajectory, and as such a potential instrument of semiotic mediation. Personal meanings concerning the idea of variation and covariation as they emerge from students' activities in the Cabri environment, through the combined use of the Dragging tool and the Trace tool, may evolve into the mathematical meaning of function. Moreover, in accordance with a Vygotskian perspective, we assume that an organized teaching intervention may foster the move from personal meanings to a mathematical, socially constructed meaning. This hypothesis deeply affected the design of the teaching experiment described in the following section.

5 The teaching experiment

A sequence of activities was designed and implemented in class, both in France and Italy. This was done in accordance with a Vygotskian perspective and taking into account the main results obtained from the analysis on the correspondence between some Cabri tools and meanings related to the idea of function. The research project involved four 10th grade classes (15–16 year old students), two in France and two in Italy.

5.1 General characteristics of classroom organization

As a consequence of our theoretical assumptions and in order to allow the development of the complex semiotic process described above, the design of the teaching sequence centred on the following types of activities.

- ➢Lab Activities. Students are placed in pairs and given tasks to be carried out in Cabri (DGE), where the use of specific Cabri tools is promoted. This type of activity is generally used at the beginning of a unit of the sequence. As explained in the following, particular situations can be designed within a DGE to foster the emergence of meanings of covariation and dependency in relation to the use of particular tools and the generation of signs. Working in pairs at the computer certainly promotes social exchange, accompanied by words, sketches, gestures and the like.
- ➢Individual writing. Students are involved individually in different semiotic activities, most of which concern written productions. For instance, after Lab Activities, students are asked to write individual reports at home on their own experience and reflections, including doubts and questions which have arisen. They also have to write their own version of the final shared mathematical formulation of the main conclusions coming from a collective discussion. All these activities are centred on semiotic processes, i.e. the production and elaboration of signs, related to the previous activities with tools. Although the social interchange during Lab activities or the following collective discussions also involve semiotic processes, this type of activity differs in that it requires a personal contribution in order to produce written signs, which by their very nature start to be detached from the contingency of the situated action. Because of their nature, written signs (in particular words) can be shared, differently from other signs, like gestures. For this very reason they may be used in collective discussions, where they may evolve.
- ➢Collective Discussion. The whole class is involved in a collective discourse, usually started by the teacher, explicitly formulating the theme of the discussion. Collective discussions play an essential part in teaching and learning and constitute the core of the semiotic process on which teaching/learning is based. The whole class is engaged in a mathematical discourse: for instance, after problem-solving sessions, the various solutions are discussed collectively, but it may also happen that students' written texts or other texts are collectively analysed, commented on, elaborated. Very often, and sometimes explicitly, they are real *mathematical discussions* (Bartolini Bussi 1998), in the sense that their main characteristic is the cognitive dialectic, promoted by the

teacher, between different personal senses and the mathematical meaning related to specific signs (most of the times related to mathematical terms). The role of the teacher is assumed to be crucial; in fact, personal meanings emerging from the activity with tools in a specific DGE are expected to evolve under of the guidance of the teacher. It is quite difficult to fully explain the nature of this *guidance*. It cannot be identified with what is called the *institutionalization* process (Brousseau 1997), although it is not contradictory with it. The main objective of the teacher's action in a mathematical discussion is that of fostering the move towards mathematical meanings, taking into account individual contributions and exploiting the semiotic potentialities coming from the use of particular tools.

5.2 Outline of the teaching sequence

Summarizing the previous discussion, the teaching sequence was based on the following fundamental assumptions.

- ➢One crucial aspect of the notion of function is the idea of variation or more precisely of covariation, that is to say a relation between two variations, one depending on the other one.
- ➢Motion, that is to say the change in space according to the change in time, constitutes a basic metaphor for covariation.
- ➢A DGE, such as Cabri-géomètre, can provide a semantic domain of space and time within which variation can be experienced as motion.
- ➢According to the Vygotskian theoretical perspective of semiotic mediation, particular tools and objects students interact with in a DGE, can be thought of as signs referring to the notion of function as covariation and, as such, they may become instruments of semiotic mediation, specifically exploited by the teacher in class activities.

The sequence was structured into three parts:

1. Variation and covariation, as well as dependency, are introduced through exploring the effect of particular Cabri macro-constructions. Collective discussions referring to the experience with Cabri are carried out with the aim of formulating a tentative definition of function. Such a tentative definition is socially constructed in the classroom, based on the interpretation of students' experiences with Cabri in terms of function as well as notions such as range, pre-range, domain, range, co-domain. The Dragging tool and the Trace tool are the key elements on which the process of semiotic mediation is activated by the teacher.
2. From geometric functions students move to numerical functions and approach the problem of geometrically representing numerical functions. Reinvesting the idea of trajectory, the graph is introduced as a geometrical function associated with a numerical one through a well-defined process. In this part of the sequence a new instrument of semiotic meditation is introduced that is an excerpt drawn from a text of Euler (1743/1945).
3. Finally, the use of the graph of a function is promoted as a means to solve problems.

The whole sequence was carried out in approximately 2 months. In this paper we limit ourselves to reporting some results, concerning the first part of the sequence, the objective of highlighting how the idea of function can be related to the grounding metaphor of motion and in particular to the idea of trajectory.

5.3 The first activities and the notion of trajectory

The first teaching session is carried out in the computer laboratory. Students are grouped in pairs and must produce a common written answer on a worksheet. In the first task, the students are asked to apply an unknown Cabri macro-construction to three given points: *A*, *B* and *P*; they obtain a fourth point *H*. The first question asks them to explore systematically the effect of moving one of these points at a time and to fill in a table explaining what moves and what does not move when dragging each point. The second question suggests using the Trace tool and asks students to observe and describe the movement of the different points, using the current language of geometry.

The given macro-construction provides point *H* as the orthogonal projection of point *P* on to line *AB*. The choice of this construction was motivated by the need of obtaining a significant visual phenomena on the screen, in particular related to the difference between domain and range (see Fig. 1).

Students easily answered the first question, immediately grasping the difference between a direct and an indirect motion. This situation became the reference situation for the meaning of variable (independent and dependent).

According to our assumption the use of the Trace tool contributed to the emergence of the twofold meaning of trajectory. In fact, the conception of trajectory both as a *globally perceived object* and as *an ordered sequence of points* can be found in the students' formulations, as shown in the following examples drawn from students' work sheets (Mariotti et al. 2003; Falcade 2002).

Different meanings emerge from the interpretation of their words and expressions.

> Federica (I) drew the trace of both an independent variable point and the dependent variable point H and wrote: "Dragging B, H forms a circle, passing through P and A. Dragging P, H forms a straight line, passing through B and A, which touches the circle in two points. Dragging A, H forms a circle, passing through P and B."

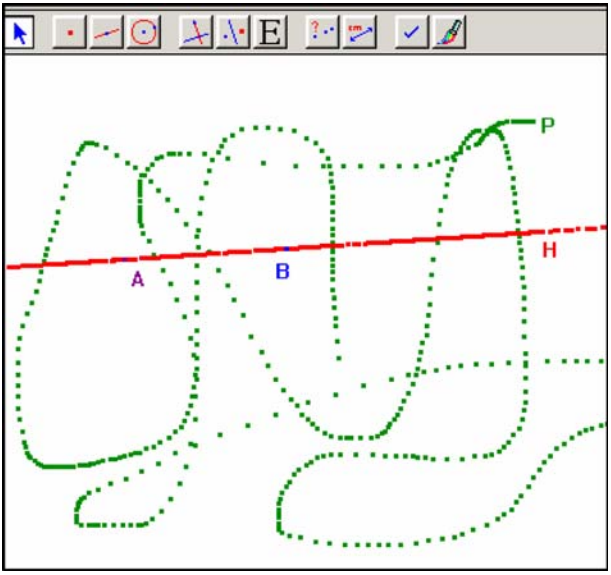

Fig. 1 Traces as they appear on screen: The irregular trace is the trace of the independent point P and the linear trace is the trace of dependent point H

> Laurent (F) wrote: "When one moves P, the point H makes a straight line [AB]. When one moves A, point P forms a circle through B and P. "

The expressions "forms a circle/ a straight line" and "makes a straight line" refer to the global aspect of the trajectory; the circle is the final product of a completed process. But, in addition to this global conception of the trajectory the idea that point H is moving on that object is also expressed.

> Tiziano (I): "If one moves (drags) point P, H moves on the line containing (Italian: che comprende) segment AB."

> Andrea (I): "When the position of P varies, H leaves a trace which stays on the line passing through A and B."

> Catarina (F): "When one uses A, H makes a circle passing through B and P ... when one uses P, H moves on a straight line passing through A and B."

> Sarah–Julia (F): "When A moves, H draws a circle around diameter BP."

> Sonia (F): "If one moves A, A forms a trajectory and H moves on a circle of diameter BP."

Expressions like "moves on a straight line/ a circle" "draws a circle" and "leaves a trace" incorporate both the components of the conception of trajectory (global, as an object, and pointwise, as the sequence of positions taken over time). Sometimes the same student, like Sonia, expresses both the global and the pointwise meaning of the term trajectory.

The students' answers explain the use of the Dragging tool to identify on the one hand the nature of variables and on the other hand the domain and the range of a function as trajectories. In the following task, students were asked to invent and create a function in Cabri. The students' texts show some evidence of the fact that the Dragging tool moved from an external use to an internal one, i.e. the Dragging tool was used to determine the nature (dependent or independent) of a variable. For example, after finishing their construction, Chrystelle and Cécile hesitated to discriminate between the dependent and the independent variable. Finally they evoked the dragging test which led them to write the correct relation between the variables and wrote:

> We associated point P to point Q because when we move P, Q is moving on the trajectory (NN'). Therefore Q depends on P (F(P)=Q).

This example gives some evidence of the internalization of the Dragging tool and its contribution to the emergence of meanings corresponding to the notion of independent and dependent variable.

The following episode likewise gives some evidence of the fact that an internalization process was achieved and that the Trace tool became a psychological tool.

5.4 The case of Sonia and Julie

Faced with the task of conceiving their own function, Sonia and Julie correctly describe a construction relating point *M'* to an independent point *M* and comment:

> M' varies in the plane and it's the reflected point of M with respect to the angle bisector of angle MRM'. M' is dependent on M through a geometrical transformation.

According to our observation notes, they use the Trace tool to check that *M*' depends on *M*: they activate the Trace Tool on both *M* and *M*', then by dragging *M*, verify that *M*' is moving because *M*' is leaving a trace (see Fig. 2). They use the Trace tool as it was initially introduced in relation to dependency between variables. However, evidence of the internalization of this tool is given by the fact that it is now used to solve a new problem. Indeed, in order to determine the nature of the function relating *M* and *M*', the two girls decide to move *M* along a small curve, deliberately regular (a knot), looking for an analogy between the trajectories of *M* and *M*'. They realize that the two traces look identical; after this observation they identify the function as a geometric transformation, a "reflection." This may be interpreted in terms of semiotic mediation.

The range and domain of function as reified by the Trace tool are intentionally utilized in checking a conjecture based on their previous geometrical knowledge about reflection. That means that the Trace tool is used not only in exploration (externally oriented), but also in the reasoning which leads to the solution of the problem: it has become an intellectual tool (internally oriented) used to answer the question concerning the nature of a function.

Besides illustrating the potentialities of the Trace tool as *semiotic mediator*, the episode discussed below shows the contribution of this tool to the emergence of the idea of function as point to point correspondence. In spite of its singularity, this episode shows the potentialities of the Cabri tools in attaining this sophisticated conceptualization. However it also shows how difficult it is for students to accept the correspondent definition of function, relating the independent and the dependent variables but neglecting any procedural idea.

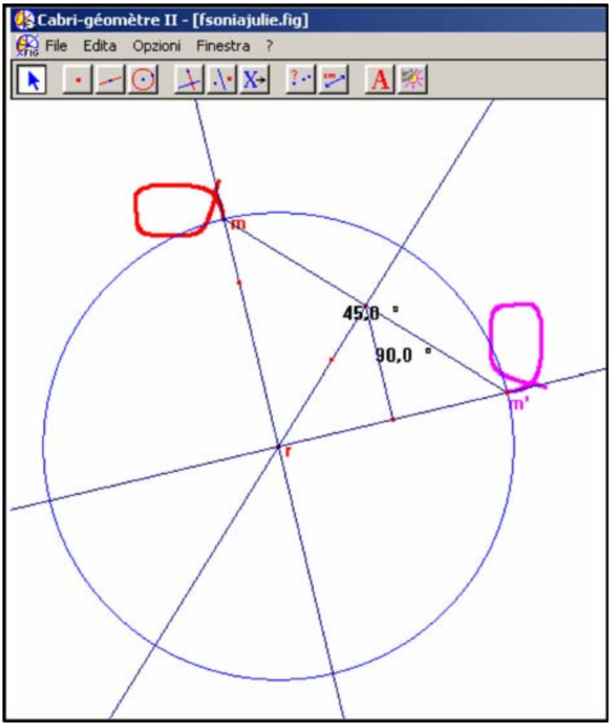

Fig. 2 What appears on the screen of Sonia and Julie

5.5 The episode

5.5.1 From acting to defining

After the first phase of activities, a collective discussion is carried out. It must be stressed that in this class there is a well established *classroom didactic contract* (Brousseau 1997) concerning the functioning of collective discussions: both teacher and students are familiar with this form of interaction. The discussion is articulated into two parts and takes place during three sessions (lasting a total of approximately 3 h). It is organized by the teacher with the explicit and shared objective of collectively elaborating a tentative definition of function. It is planned with a twofold aim corresponding to:

- Clarifying and systematizing some of the ideas which emerged during the previous activities, and
- Expressing these ideas in a *mathematical statement*, i.e. the definition of function.

From the very beginning, students are asked to characterize a function. Both the students and the teacher refer to Cabri tools and Cabri phenomena, as experienced during the first activities. The first part of the teaching sequence has become a reference situation for the students, where a system of signs is established, on the basis of which the meaning of variable is introduced by the teacher referring to the different kinds of motion in the DGE. Different elements in play are highlighted by the students: the two types of variables, the domain and the range. The difficulties arising in entering the mathematical world make the role of the teacher relevant; the teacher has the difficult task of mediating between culture and students' meanings related to their personal experience, between mathematics, as a product of human activity, and students' individual learning.

5.5.2 A definition of equal functions

The difficult task concerning the characterization of a function arrives at a crucial point when students realize that characterizing a function implies determining when two functions can be said to be *equal*. The teacher shifts the focus of the discussion and asks the students to attempt a *definition of equal functions*.

A first attempt of a definition simply states:

Two functions are equal if they have the same domain and the same range.

At this point it is impossible to say whether students, when they speak about domain and range, are thinking globally or pointwise.

The teacher asks the students to go back to Cabri and to look for different examples that can corroborate or invalidate this first conjectured definition. After this moment working in pairs, students are invited to express the new ideas which have arisen from their activity in Cabri. The following definitions about *two equal functions* were proposed; they clearly refer to students' experience with Cabri. The attention is focused on three main elements, the domain, the range and the construction procedure, although not everybody attaches the same importance to these elements.

Andrea–Alessandro: "Two functions are equal if they have the same domain and the same range for all the subsets of the original domain which define the functions".

Gioia–Federica: "Two functions are equal if they have the same number of variables, the same domain, and the same procedure (in the construction of the macro)".

Marco–Gabriele: “Two functions are equal when they have the same range and (when) the same domain is fixed (for both).

Tiziano–Sebastiano: “In our opinion two functions are equal if having the same domain and the same definition procedure they have the same range. If either the domain, or the definition procedure, or the range is not equal, neither are the functions equal.”

Surprisingly, one of the definitions (that of Andrea and Alessandro) presents a characterization in which the domain is thought of in terms of subsets. It’s a static definition that shows no traces of variations and uses a quantifier “for all.”

This way of thinking may appear quite strange, if one does not take into account the very peculiar experience that students had in the previous activities and the relation built up between the idea of trajectory and that of range. In previous work, in order to compare two different functions, students compared two constructions on the same domain, observing that each construction produced a different trace/trajectory. From this observation they concluded that different construction procedures produce different trajectories, that is, different sequences of image-points. Nevertheless, the link between what was done in Cabri and Andrea and Alessandro’s formulation is not immediate at all. We can suppose that the process of internalization of the Trace tool, which transformed the experience of dragging points and producing trajectories into such a static formulation, “for all the domain subsets,” was quite important. However, let us see how this way of thinking was shared in the class and evolved.

5.5.3 A reliable definition, hard to accept

The second part of the discussion starts when students are asked to compare the different definitions they have produced (see students’ definitions above).

Here is an excerpt of this discussion:

1. Teacher: “We must find an agreement on a definition, which can be one of these, or an improvement of one of these, or the fusion of these... We must decide.”
2. Andrea : “According to me, Gabriele and Marco’s definition is wrong.”
3. Teacher: “So, Andrea, according to you, Gabriele and Marco’s definition is wrong. Let’s read it again” (she reads again) “Two functions are equal when they have the same range and (when) the same domain is fixed for both.”
4. Andrea: “Because to get to the same range, someone could pass through... we could have several journeys; in fact, if there were a subset of the domain... we can’t say that the functions are...”
5. Teacher: “Tiziano, could you try to explain it better?”
6. Tiziano: “Yesterday, we saw that we can, by doing the same domain, we can create the same range and this, with different functions (he means construction procedures).”

The teacher redirects the discussion on the comparison between the definition of Andrea and Alessandro and those referring to the procedures.

44. Teacher: “Let’s read the text. You say that if they have the same domain and the same range for each subset of the domain...”
45. Tiziano: “But, here it’s like having the same procedure.”
46. Teacher: “Hum, and why is it like having the same procedure?”
47. Several voices: “...Because...”
48. Gabriele: “...As we go further, the subsets of the domain and vice versa...”
49. Teacher: “Do you agree, Andrea?”

50. Gioia: "The domain is the plane, then you have the straight line, then a segment..."
51. Teacher: "What are these?"
52. Andrea: "The domain can be whatever."
53. Gioia: "They are subsets."
54. Teacher: "And then, the procedure, what does it do? That is to say, I.... Where does it start from?"
55. Andrea: "The domain can be one point too... if we want!"
56. Teacher: "The subset of the domain can be one point too. Oh!"
57. Andrea: "For whatever point, we get the same point of the range."
58. Teacher: "And this gives us the idea to say that..."
59. Gioia: "I'm doing the same procedure."
60. Andrea (together with Gioia): "I'm doing the same procedure."
61. Teacher: "I'm doing the same procedure. Therefore, for whatever point of what?"
62. Andrea: "For each point of the domain we have the same... as the result of the function, the same point of the range."
63. Teacher: "Do you agree?" (referring to Tiziano)
64. The students are perplexed. Silence.
65. The teacher writes at the blackboard and reads: "For each point of the domain, we have as the result of the function, the same point as the range."

This excerpt of discussion allows one to observe the emergence of the idea of a point by point coincidence, as it originates in the trajectory by trajectory coincidence, considering each time a trajectory already included in the previous one, and passing to the limit situation when the subset becomes a point. This is the case for Andrea, in which the process of internalization of the Trace tool turns out to be quite substantial.

The first part gives the opportunity of observing how the teacher organizes the sequence of interventions to reach her objective of coming to a definition of function decontextualized from a DG context. When speaking about procedures, Andrea evokes the nice metaphor of a journey: in order to go from one place to another, one can take different paths. Then the teacher prompts the intervention of Tiziano with the explicit request to reformulate it in mathematical words, to use the terms officially introduced in the collective discussion of the day before: "Tiziano, could you try to explain it better?" (see line 5).

Students seem to accept that "having the same domain and the same range for each subset of the domain" is like having "the same domain and the same procedure" (see lines 44, 45). The teacher invites the students to compare their own definitions with the one of Andrea and Alessandro. Here we can identify two instances of situated abstractions, referring to the term used by Noss and Hoyles (1996, p. 122) "to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting, which, in turn shapes, the way the ideas are expressed." Drawing on the recent experiences in Cabri, the first definition is strongly related to the construction procedure, whilst the formulation of the second one focuses on the correspondence between trajectories, identifying a trajectory with a subset of the domain and extending this identification to a point. After the question of the teacher (line 54) about the role of the procedure in his definition, Andrea does not reply in terms of procedure, rather he goes on in his generalization arriving at the correspondence between two points, so that Gioia (line 59) could recognize the procedure in this correspondence. Avoiding widening the discussion on the delicate link between procedure and point to point correspondence (line 61), the teacher proposes a reformulation of Andrea's definition for collective agreement. This leaves the other students perplexed, and at the end, when the final formulation is written on the blackboard and read by the teacher (line 65: "For each point of the domain, we have as the result of the function, the same point as the range"), for most of the students it becomes difficult to accept it. The difficulty (and the reluctance) of accepting this new definition may correspond to the difficulty of overcoming the conceptual move from a definition, based on real experience and tightly related to Cabri activities, to a purely mathematical definition. Indeed, further discussion was needed to reach the acceptance of the new abstract definition based on the comparison of functions point by point.

Nevertheless, what is interesting is the fact that such a *point by point* comparison emerged and apparently it did so from the internalization of the Trace tool, which bases the interpretation of a function on the idea of dynamic relation between two trajectories, i.e. a point by point relationship.

The analysis of collective interactions shows the role of interpersonal exchanges in the formulation of a definition. Following the prompts of the teacher, Andrea shares with the other students the elaboration of a new definition; in particular, the role of Gioia seems to have been critical. According to a Vygostkian interpretation, the contributions of Gioia and Andrea to the discussion lead to an interpersonal construction of the meaning of domain made of one single point. Gioia gives the impulse by enumerating a sequence of nested domains: plane, line, segment (line 50), that Andrea completes by evoking "point" (line 55). But it should be noticed that Andrea did not mention "point" at the beginning: he interprets the enumeration by expressing that a domain can be any subset of the plane (line 52) and only after this intervention he seems to discover something he had not dared to think: "The domain can be one point too... if we want!" (line 55). The conclusion, "if we want" shows how audacious Andrea feels his idea is and simultaneously reflects the freedom that abstraction provides in this descending process of considering smaller and smaller subsets.

5.5.4 The role of the teacher

As is clearly shown in the previous excerpt, the role of the teacher is crucial both in the evolution of meanings and in helping students to face the move from describing their activities to the elaboration of a definition.

Indeed, the role of the teacher turns out to be determinant all through the discussion. At the beginning (line 1), she states the classroom didactic contract within which the discussion should be developed. On different occasions (lines 44, 46, 51, 54, 58, 61), she resorts to various ways in order to redirect the discussion and focuses on the main objective, as commented on above in the example of Andrea and Gioia about the necessity of taking into account the procedure. She prompts the intervention of a given student, or she asks him/her to explain his/her claims or she repeats a student's intervention. When she grasps the relevance of Andrea's contribution, she repeats (line 56), with emphasis, his statement. She is aware of the important mathematical implications of Andrea's observation and pushes the discussion further in this direction. We wonder whether without this intervention, the discussion would have been completely different, we wonder whether students would even have noticed Andrea's remark. It is important to note the teacher's attention to maintain the link with the Cabri context.

In other moments (lines 5, 49, 61) she tries to involve students who seem not to participate in the discussion. In general, what she tries to do is to orchestrate all the interventions in order to foster the discussion of certain mathematical meanings, emerging from the contribution of particular students, by all the other students.

6 Conclusions

From the combination of observation and action students grasped variability as motion, while the idea of covariation, incorporated in the coordinated movement of points on the screen, was experienced through the coordination between eyes and hands. In most of the cases, students' formulations reflect the asymmetrical nature of the independent and dependent variables and the twofold meaning of trajectory; this is consistent with the results obtained in the different contexts of geometric transformations by Jahn (2002). In that case students obtained the image of a circle through an unusual transformation of the trajectory of the image of a point moving on the circle.

The episode of Sonia and Julie gives some evidence of the complex process of internalization through which Dragging and Trace tools are transformed into psychological tools. This episode is particularly relevant in so far as it is very difficult to get direct access to the internalization process and, in particular, to the change of status of a tool, from externally to internally oriented. According to our hypothesis the internalization of the Dragging and Trace tools may contribute to introducing function as covariation, and the construction of global and pointwise aspects of trajectory may contribute to introducing the notions of domain and range.

Furthermore, the excerpt of the collective discussion shows a particular way in which the Trace tool can function as potential semiotic mediator. In fact, it is possible to recognize that the idea of trajectory, as it emerges from the activities carried out in Cabri, substantially contributes to Andrea's elaboration of a definition of equal functions as a pointwise correspondence. Nevertheless the ultimate perplexity of the other students and their reluctance to accept such a definition of equal functions shows the difficulty of shifting to a formulation that neglects the construction procedure and thus is completely detached from the physical experience in which it originated.

The activities developed in the Cabri environment could even have reinforced a natural procedural tendency. Future teaching sequences should plan situations with equal functions differing in their procedure construction. On the other hand, within Cabri, the available tools (Dragging, Trace, Macro...) and the particular signs (segments, rays, Cabri figures representing either the domain, on which the independent variable varies, or the range, on which the dependent variable varies) offer a common semiotic system that the students and the teacher can elaborate on.

This excerpt also shows the importance of the teacher's role. Firstly she promotes different semiotic activities related to the use of Cabri tools and aimed to produce personal meanings. Afterwards she orchestrates the discussion in order to guide the semiotic process towards the construction, necessarily inter-subjective, of a specific mathematical meaning which may be, sometimes, quite different from the students' personal meanings. For this reason, in some cases, and this is one of them, it may happen that the teacher even forces certain conceptual moves in order to help students to accomplish these processes completely. Actually, the definition of function as a set of pairs was not the main objective at this point of the sequence, and the teacher decided to postpone a further development of this discussion to the future. Nevertheless, the main objectives seem to be achieved (function as covariation and the twofold conception of trajectory) and are ready to be reinvested to introduce the notion of graph of a function as a trajectory, i.e. a set of points representing a covariation. But this is a new story and a new paper will be devoted to it.

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