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# SOLVING STRUCTURED GEOMETRIC TASKS ON THE COMPUTER: THE ROLE OF FEEDBACK IN GENERATING STRATEGIES

ABSTRACT. This paper describes a year-long study of the way a group of six 12-year-old children went about solving a special class of geometric problems, using a computer and a limited set of Logo programming tools. We examined children's solution strategies from the perspective of whether the strategies led to insights about the underlying mathematical relations. It was found that the "feedback" aspect of the computer graphics screen encouraged solution strategies that were qualitative and driven by visual data, rather than being based on explicit or implicit relations. These visually based strategies did not lead to the kind of mathematization of problem-situations that we as mathematics educators would like to see and, in fact, seemed to be a barrier to the development of higher levels of geometric thinking.

## INTRODUCTION

In this paper we look at the way children go about solving a very special class of geometric problems, using a computer and a limited set of programming 'tools'. In particular, we examine the way the output on the screen seems to influence the actions taken by the solvers. The research originates from our initial study of children's mathematics while using Logo (see Kieran, Hillel and Erlwanger, 1986) and out of our concern that children rely almost exclusively on visual cues from the output on the screen. Their Logo productions of geometrical figures often fail to account for some of the important underlying mathematical relations. Such a visual approach, which is initially very useful for gaining familiarity and ease with some difficult concepts (e.g. angle of rotation and degree measure), seems to be, later on, a barrier towards gaining a higher level (in the van Hiele sense) of geometrical thinking.

One may wonder to what extent such behaviour is an artifact of a particular computer-based learning environment or whether it is more generally provoked simply by the existence of a graphics screen and a screen output. Originally, we felt that the 'visual schema' was a result of the kind of Logo environment which emphasizes children-generated projects and in which the 'exact' aspects of a geometric figure are not, generally speaking, made very explicit. Hence, the criterion for accepting the correctness of a program is a "more or less" correct screen figure and a visual solution schema is completely adequate. Consequently, we set out to see

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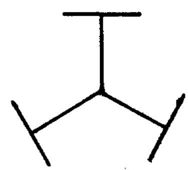
whether changing the nature of the Logo environment by giving specific tasks, which implicitly contained certain geometric relations that had to be satisfied, would provoke a more 'analytical' solution schema. This did not turn out to be the case over the twenty sessions we conducted with twelve-year olds (see Hillel and Kieran, 1987). However, one aspect that emerged out of the research was the mismatch between our and the children's perception of what are the important underlying features of a given task. Often, the attributes of a given geometric figure that led us to its choice were ignored and other features were emphasized. For example, when we gave the children the figure



chosen because of its 3-fold symmetry, they concentrated only on the reflection symmetry about the vertical axis. Consequently their solutions did not involve 120° rotations nor any relation to 360°. The lack of an analytical solution schema could also be related to the fact that the tasks we gave involved mainly angle relations which are still problematic for children of that age.

In the research reported here we have taken another look at the way twelve-year olds solve highly structured geometric tasks. We differed in our approach from the previous study by being specific about the geometric relations that the tasks had to satisfy and by giving, during the first 12 of 24 sessions, problems involving only *length* relations. Our rationale here was that children of that age would have no difficulty in dealing with exact length relations (e.g. sums and differences of lengths of several line segments) and so they could attempt an analytical rather than a visually-based solution. Furthermore, the children were given a very limited repertoire of Logo commands. This meant that they could not solve the tasks, by resorting to the use of special programming 'tricks' which often have the effect of subverting the mathematics of the activity.

Because of the specificity of the experimental situation, we begin the paper by first describing the setting and the tasks (section A). We follow with a detailed description of some solutions of one task from each of the three different types of tasks (section B). In section C, we discuss the results of the research and relate these to other works.

## A. THE EXPERIMENTAL SETTING

### A.1. *The Children*

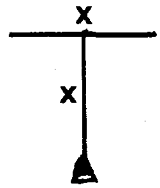
The six children (four boys and two girls) were 12–12 1/2 years old. They were in two different grade six classes of an elementary school close to the university where the computer lab was situated. They were randomly picked from respondents to a letter requesting volunteers to participate in a 'computer project'. They were also asked, in the letter, to come in pairs since our intention was that they would work as partners.

It should be said at the outset that most of the children (particularly the boys) first came to the lab hoping to be able to play computer games. We had to negotiate the 'experimental contract' and make it clear that we wanted them to work, for most of the time, on problems that we had chosen. We solicited a commitment on their part to come to the sessions regularly and we did promise at least one 'computer games' session during each term. Nevertheless, some of the boys remained somewhat reluctant participants and we often had to prod them to get started on a new task or to look back on their work from a previous session. Three of the boys showed up more sporadically after the Christmas break and stopped coming altogether after session #20.

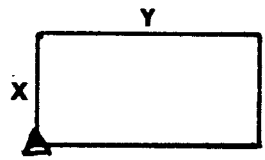
Despite this seemingly negative attitude on their part we should emphasize that, once started on a problem, the children generally got very involved in trying to solve it. Particularly, we noticed a dramatic switch in the level of commitment when we split the pairs and let the children work on their own computers (starting at session #3). For example, Ben, who was up to then the least involved, worked on a task for a good part of an hour and was then heard saying "I am going to get it right no matter how long it takes."

### A.2. *The Available Logo Commands*

The children's Logo vocabulary consisted of six commands. The commands TEE, RECT and BASELINE produced the following geometric objects:
– TEE :$X$ a letter $T$ with both its stem and bar having length $X$ ($X > 0$)



– RECT :$X$ :$Y$ a rectangle with dimensions $X$ and $Y$ ($X > 0$, $Y > 0$)

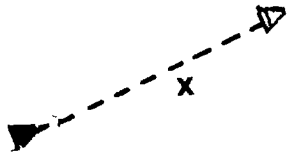



– BASELINE :$X$ a horizontal line of length $X$ ($X > 0$)


In all of the above cases the underlying programs (hidden from the children) were state-transparent, i.e. the turtle returned to its initial position and orientation.

The commands MOVE, TRT and TLT allowed for changes in position and orientation of the turtle, i.e.

– MOVE :$X$ moved the turtle $X$ steps (without changing its heading and without leaving a trace). The turtle moved forward if $X > 0$ and backwards if $X < 0$.




– TRT :$X$ rotated the turtle slowly to its right by $X$ degrees ($X > 0$).
– TLT :$X$ turned the turtle to its left.

The children were also shown how to use the Editor and, during session # 8, the REPEAT command. We should emphasize that the geometric figures Tee, Rectangle and Baseline are very simple and are not, on their own, interesting geometric objects. Rather, they were chosen because they could be used as components of other figures in such a way that the Logo productions of those figures would require a high level of coordination of inputs. This would not have been the case if the children had had access to the usual FD and BK commands. With such commands at their disposal, a visually-based estimate for inputs followed by adjustments usually would produce a reasonable solution. This point should be clearer when we discuss the tasks in the following section.

## A.3. *The Tasks*

Most of the tasks given to the children were highly structured in the sense that dimensions of different components of the figures had to be related in specific ways. These relations were either explicitly described when the tasks were given or had to be inferred from other specifically given constraints. Furthermore, some tasks such as the Baseline tasks (see Figure 1a below) also stipulated a specific starting point.

Tasks with similar conditions were given at different times so as to enable us to observe changes in the approach to their solutions. They fell under the following broad categories:

(i) BASELINE-TASKS includes those tasks which required a figure to be constructed on a fixed baseline (i.e. their productions had to be started with the procedure BASELINE). For example, the Tee-Tower task consisted of a tower of five congruent Tees sitting on a baseline (see Figure 1a).

(ii) CENTRING-TASKS included tasks which contained, as one of their conditions, that one rectangle be centred relative to another (see Figure 1b).

(iii) ROTATION-TASKS included tasks involving $n$-fold rotational symmetry of a figure about a central point, e.g. the 3-Tee Rotation task (see Figure 1c).

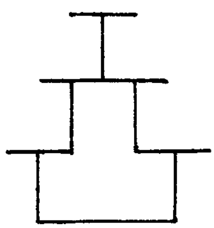


Fig. 1a

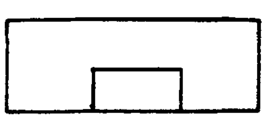


Fig. 1b

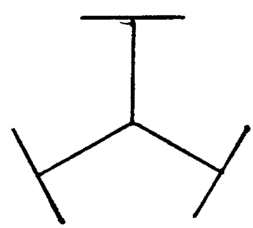


Fig. 1c

We will analyze the characteristics of the tasks using, in slightly modified form, a previously developed framework for task analysis. Thus we view a task as a specific set of *demands* made on a solver which include mathematical/programming demands, structural demands and psychological demands. The mathematical/programming demands refer to specialized, task-specific knowledge required to solve the task (e.g. solving equations or using recursion). The structural demands include the embedded relations, both overt and covert, that need to be coordinated, the untangling of the allowable operations, the sequencing of these operations, as well as the type of 'goal state' and the criterion for deciding that the 'goal state' has been reached. The psychological demands relate to the tendency of a problem to create an inhibiting "set" (see Hillel and Wheeler, 1981, for further elaboration).

*Mathematical/programming demands*: Neither the Baseline nor the Centring tasks seem to make any specific mathematical demands. The Rotation tasks require the notion of 360° as a complete rotation.

Since all the tasks involve simple geometric figures their solutions, from a programming perspective, are also extremely easy. The difficulties do not lie in constructing the defining programs nor in the use of variables, but in the *choices of inputs* to the commands BASELINE, RECTANGLE, MOVE and TRT. (We did not expect productions to fail because of programming errors and when these did occur, they were ignored in the subsequent analysis. In most cases, the children's initial productions resulted in outputs which were not far from the goal figure.)

*Structural demands*: The "weight of difficulty" for these tasks resides in this category, particularly in the *embedded relations*. Their solutions require that inputs to the different commands satisfy explicitly-given conditions ("all the Tees have same size") as well as relations which are the results of other, *geometrically described*, constraints ("the Tee-tower has to sit properly on the Baseline").

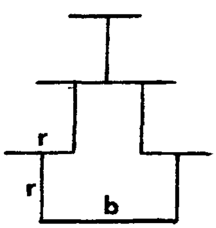



Fig. 2a

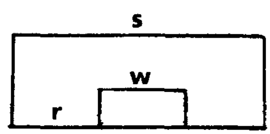



Fig. 2b

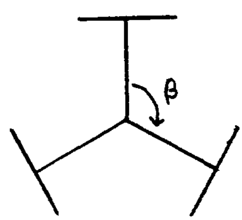



Fig. 2c

For example, in Figure 2a, the input $r$ to the procedure TEE has to be related to the input $b$ to BASELINE (in fact, $r = b/2$), while in Figure 2b, the geometrical requirement of centring the smaller rectangle needs to be reformulated in terms of an input to MOVE which satisfies the relation $r = (s - w)/2$. Similarly, the condition that the three rotations are the same in Figure 2c means that the input to TRT is 360/3.

The *allowable operations* (or transformations) are essentially of two kinds; one involves changing the actual commands and the other, changing the inputs to the commands. Both types of operations are completely constrained by the available Logo commands. The latter type of operation results in either modifying the size of the objects Tee, Baseline and Rectangle, or in modifying the position of the objects (MOVE) or their orientation (TRT, TLT). Furthermore, the *sequencing of the operations* is also partially constrained in those cases when one of the task requirements is that the figure has to be constructed from a particular starting point (equivalently, the program has to begin with a specific command and input).

The *goal state* is given in terms of a computer printout and explicit description of certain geometric conditions that have to hold. The *criterion for achieving the goal* is a bit ambiguous in such a computer-graphics environment. It could either be a screen output which resembles the printout or the underlying program. (The question of which criterion the children use to justify the correctness of their solution will be considered in the paper.)

*Psychological demands*: The way in which the tasks are structured leads to *part-to-a-whole* relations which involve subtraction/division rather than addition/multiplication (cf. $r = b/2$, $r = (s - w)/2$, $\beta = 360/3$). This means that *the choice of inputs to the different commands within a program cannot proceed sequentially as the choice of the commands themselves*. Hence, setting the inputs to some of the commands calls for *anticipating* several steps ahead rather than basing it strictly on the part of the program which is already in place. For example, in the Tee-Tower task the program begins with fixing the input to BASELINE as, say, 70. The next command is TEE, but the exact input $r$ to TEE has to be based on 'working backwards' from the completed figure (see Figure 3a). Modifying a visually based estimate for $r$ within a program necessitates wholesale changes in the program in order to maintain the structure of the figure. This can be contrasted with writing a production for the Tee-Tower with BASELINE as the last command. It would then be easier to establish the additive/multiplicative relation $b = 2r$ since it would be clearer how the input $r$ can be used as a 'counter' to determine $b$, the input to BASELINE (see Figure 3b). Even if the input $b$ is initially estimated and then successively adjusted till the Baseline seems to fit, such modifications of the program would not disrupt any other part of the figure.

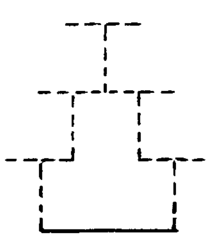


Fig. 3a

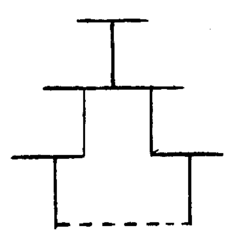


Fig. 3b

### A.4. *Collection of Data*

Each researcher observed a pair of children. Aside from notes taken by the researchers, the main source of data was in the form of 'dribble files' which contained all the on-computer work, including the intermediate screen outputs and the content of and changes in the Editor. All paper-and-pencil work was saved as well.

## B. ANALYSIS OF THE CHILDREN'S PRODUCTIONS

In this section we examine in detail some of the children's solutions to one task from each of the three task-categories. For each such task we begin with a task analysis which highlights both the overt and covert relations that are embedded in the task. We then examine the solutions in terms of the relations that are being attended to, those that are ignored or forgotten, and the kinds of strategies used to satisfy the relations.

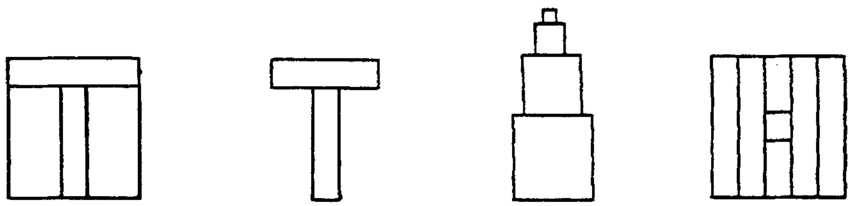
Each child's production is broken up into a sequence of episodes beginning with their initial program which, unless stated otherwise, was planned off-computer or entered directly into the Editor. A new episode is started when either a different kind of output appears (i.e. an output which points to a new kind of mismatch from the goal figure) or a different solution strategy is employed. Because we are particularly interested in the role of the screen output in influencing the solver's subsequent actions, we begin each episode by exhibiting the output on the screen which is the result of the *previous* actions, since it is the 'trigger' for the new set of actions.

We have mentioned in section A.3 that the initial solutions to a task tended to have, barring some elementary programming errors, reasonable resemblance to the goal figure. Thus, changes of the commands making up a program were much less frequent than changes of inputs. The latter resulted in changes of the *placement* of the objects (i.e. modifying the input to MOVE or TRT/TLT) or the *size* of the objects (i.e. modifying the input to TEE, RECT or BASELINE). We are particularly interested whether such changes were qualitative in nature ("this has got to move a little bit more") or exact, at least in a local sense of satisfying one of the problem's conditions ("this has to move another 15 to make those two parts the same"). When the strategies used seem to be qualitative, we label the change in placement **push-pull** and the change in size **stretch-shrink**.

These terms correspond to the way the children, on occasion, described their actions.

## *I. The Centring Tasks*

1. *The Tasks*



(a) Tee-in-the-Square (session # 6)

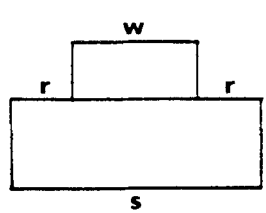
(b) Solid-Tee (session # 6)

(c) Birthday Cake (session # 15)

(d) H (session # 22)

The Tee-in-the-Square task will be described in detail in the next section. All the tasks involved an explicitly stated condition that a rectangle is centred relative to another. The tasks varied to the extent to which other dimensionality conditions were given as part of the problem. For example, (a) and (d) were globally constrained by the dimension of the *initially* chosen outside square, (a) further by the requirement that the height of the bar equal the base of the stem and (d) by requiring the central bar in H be a square. Task (b) had the bar and the stem as congruent rectangles and (c) had explicit dimensions for the squares at each layer. Furthermore, three children worked also on a Castle task (sessions # 17–# 18) which included, in part, a centred rectangle.

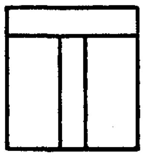
The solution of the centring subproblem requires expressing the geometric condition as a relation involving the *centring-move* $r$ and the bases of the two rectangles, $s$ and $w$, i.e.



2. *The Tee-in-the-Square Task*

(a) *Task analysis*

*Instruction*: The task was given to the children as a computer printout

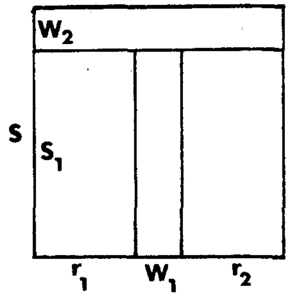


and presented as consisting of a letter $T$ inside a square. The conditions, given verbally, were:

- the outside square had to be constructed first
- in the letter $T$, the height of the bar rectangle was the same as the base of the stem rectangle
- the stem rectangle was centred.

*Analysis of the relations*: The task has a dimensionality and a centring component. Dimensionality requires the coordination of the sizes of the rectangles making up the letter $T$ (the *stem* and the *bar*) relative to the global constraint imposed by the size of the square.

Labelling the different components of the figure as follows




then the pertinent relations are:

$w_1 = w_2$ (explicit in the verbal instruction), and

$s_1 + w_2 = s.$

Furthermore, centring the stem relative to either the bar or the enclosing square involves the *centring-move* $r_1$ (or $r_2$) and the relation

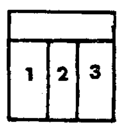
$$r_1 = r_2,$$

subject to the global constraint given by the relation

$$r_1 + w_1 + r_2 = s. \quad \textit{centring} \text{ condition}$$

Note: Moving the turtle from one position to the next often requires a 90° turn prior to, and after, the MOVE command. For this and subsequent tasks, we ignore these 90° turns in the discussion of the solutions.

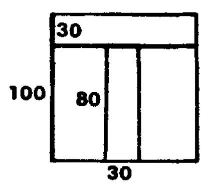
(b) *Children's solutions.* Four of the children proceeded by trying to place a bar and a stem within a specific square. The two other children perceived the task as that of placing three rectangles within a square, i.e.




**Kay**

*Initial Episode*

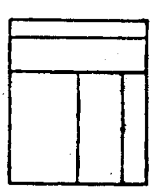
Kay began with a 100 × 100 square, a 80 × 30 stem and a 30 × 100 bar ($s = 100$, $s_1 = 80$, $w_1 = w_2 = 30$).




These choices failed to satisfy the relation $s_1 + w_2 = s$.
Her centring-move was based on a *mid-point* strategy of choosing $r_1 = 1/2s = 50$.

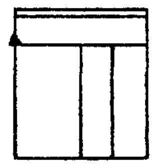
*Episode 1*

Output:



Kay was alerted to the fact that the bar was too high and consequently she lowered the bar using a stretch-shrink strategy. She decreased $w_2$ ($w_2$: 30 → 25) and maintained the condition $w_1 = w_2$ by changing $w_1$ accordingly.

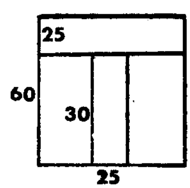
*Episode 2*
Output:



Kay repeated the previous stretch-shrink action of lowering the bar and narrowing the stem by 5 units ($w_1$, $w_2$: 25 → 20). She also noticed that the bar was not centred and used a push-pull strategy of pulling the bar 5 units to the left ($r_1$: 50 → 45). The subsequent output suggested that the stem was still not centred and led her to another push-pull ($r_1$: 45 → 40). The final output convinced Kay that she had solved the task.

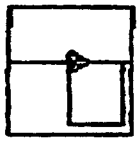
**Rosa**

*Initial Episode*
Rosa started with a 60 × 60 square, a 25 × 60 bar and a 30 × 25 stem ($s = 60$, $w_1 = w_2 = 25$, $s_1 = 30$).



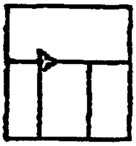

As with Kay, the relation $s_1 + w_2 = s$ was not satisfied. Rosa also used a mid-point strategy for centring the stem ($r_1 = 1/2s = 30$).

*Episode 1*
Since Rosa completed the bar before the stem, the output was



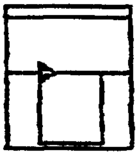
Rosa attended to the centring aspect, first using a push-pull strategy to pull the stem to the left ($r_1$: $30 \rightarrow 25$). The next output led to another push-pull ($r_1$: $25 \rightarrow 15$) and the extension of the length of the stem by 5 ($s_1$: $30 \rightarrow 35$). It is not clear whether the latter choice was analytically based on the relation $s_1 + w_2 = s$ or just a qualitative 'make it longer', but it resulted in all the dimensionality relations being satisfied.

*Episode 2*
Output:



This led Rosa to conclude that the stem was still a bit off-centre. She switched to a stretch-shrink centring strategy, increasing the width of the stem twice ($w_1$: $25 \rightarrow 26 \rightarrow 27$).

*Episode 3*
It was pointed out to Rosa that she no longer had $w_1 = w_2$. She reacted by simply re-establishing the equality by setting $w_1 = w_2 = 30$. However, since she left all other dimensions unchanged, she no longer had $s_1 + w_2 = s$. After several programming errors which threw the production completely off, Rosa ended up with the output:

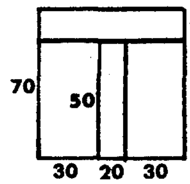


In order to fix the problem with the upper bar, she tried at first to shrink the bar and then modify the global constraint given by $s$ by stretching the square ($s$: $60 \rightarrow 80$). Coupled with some more programming errors, her strategies led her further and further away from the goal. She ended up ignoring the relations and just visually patching up each production. This finally resulted with a figure still with an off-centred stem and she gave up.

**Ben**

*Initial Episode*

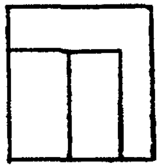
Ben tried placing three rectangles within a square of a fixed size 70. The rectangles were chosen as 30 × 50, 20 × 50 and 30 × 50 respectively, thus satisfying the conditions $w_1 = w_2$ ( = 20) and $r_1 = r_2$ ( = 30).




The relation $s_1 + w_2 = s$ was satisfied by default, but the centring condition $r_1 + w_1 + r_2 = s$ did not hold.

*Episode 1*

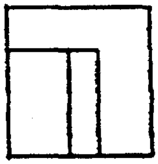
Direct-mode implementation was stopped after two rectangles were constructed since the output showed:



Ben concluded that the stem was not centred. He then used a stretch-shrink strategy to try to centre the stem ($w_1$: 20 → 10 and then 10 → 15).

*Episode 2*

Output:



Ben was convinced that he had solved the centring problem so he tried to place the third rectangle. He began with his initial choice of $r_2 = 30$ then shrunk it ($r_2$: 30 → 20) then stretched it ($r_2$: 20 → 25). Even though neither $w_1 = w_2$ nor $r_1 = r_2$ held any longer, the final output was acceptable to Ben as a solution.

(c) *Discussion.* With the exception of Mark, none of the children seemed to have an analytical centring strategy, i.e. one which takes into account the values for $s$ and $w$ in order to figure out the centring-move $r$. The only analytical attempt was the mid-point strategy of setting $r = s/2$ which was part of the initial solution of three children. It seems that an obstacle to a solution was caused by the children's tendency to focus only on those segments of the figure which the turtle traces or traverses, i.e. segments which have an obvious correspondence with the commands that appear in the program. Thus, they found values for either $r_1$ or $r_2$ (depending on which side of the square the turtle was placed), while centring requires the coordination of both $r_1$ and $r_2$.

For the Tee-in-the-Square task, the screen output provided useful feedback in directing the children's actions towards the goal. They attempted to place the stem at the centre using either a push-pull or a stretch-shrink strategy. While both strategies result in a stem which is more centred, they are not equivalent in terms of the other conditions of the problem. The push-pull strategy leaves all the dimensionality relations intact and hence results in a production which is closer to the goal. The stretch-shrink strategy for the stem tampers with the equality $w_1 = w_2$ and hence improves (or solves) one of the conditions at the expense of another.

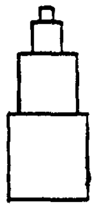
Furthermore, there was no evidence of any analytical solution strategies – for example, none of the children who started with a mid-point as the centring-move, tried to follow it by displacing the stem by a distance corresponding to half of its base. The predominant tendency was to make qualitative changes to the inputs, usually by multiples of 5.

The criterion for accepting a production as a solution was visually based for all children – the proof of the solution resided in the visual verification of the output rather than in the program and none of the children went back to their programs to verify the exactness of the solutions.

### 3. *Other Centring Tasks*

The Solid-Tee task which was given immediately after the Tee-in-the-Square task did not lead to any new centring strategies. For example, Kay's solution to two different-sized Solid-Tees was identical to her previous approach; a mid-point strategy followed by a qualitative push-pull.

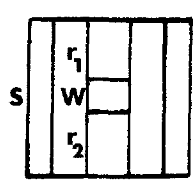
The Birthday Cake task (session # 15)



led to some significant changes in strategy. Because the bases of pairs of adjacent squares in the figure did not differ much in size, the children did not use a mid-point strategy for the centring-move $r$, since it was visually obvious that $r$ was not $s/2$.

Bill abandoned his previous qualitative approach and calculated the centring-move for each layer as $(s - w)/2$. Kay, who prior to working on this task completed a project of her own choosing (a ROBOT) which had a similar centring problem, also switched to an analytic approach. She had become aware that $r$ had to satisfy the centring condition $r + w + r = s$, but she could only solve for $r$ in terms of $w$ and $s$ by forward substitution, i.e. by picking a value for $r$ and checking whether it satisfied the relation. After a few tries, her search for $r$ was systematic. All her calculations were done prior to running her program, pointing to her new awareness that the exactness of the solution lies in the program rather than the screen output.

Rosa didn't change strategies till the H-task (session # 22).

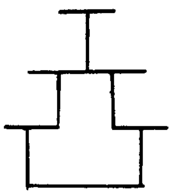



She started centring the middle bar with a mid-point strategy and when the bar appeared too high, she tried rescaling the whole figure, but kept her mid-point strategy. The result triggered another push-pull but this time she calculated the centring-move by coordinating the relations $r_1 = r_2$ and $r_1 + w + r_2 = s$. This was one of the few unprompted shifts to analytical work by Rosa.

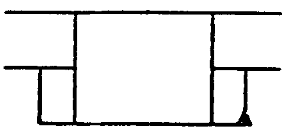
## II. *The Baseline Tasks*

### 1. *The Tasks*

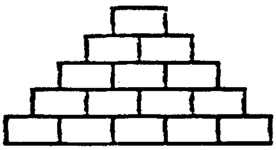
The Baseline tasks consisted of the following:



(a)
Tee-Tower
(session # 2)



(b)
4-Tee
(session # 4 and # 5)



(c)
Bricks
(session # 8)

All the tasks involved an explicitly stated condition that the figures had to constructed *starting* with the Baseline. The fact that the Tees in task (a) were congruent was also made explicit, but the placement of the Tees was assumed to be self evident from the printout. The 4-Tee task was further constrained by the requirement that the Tees be aligned on both sides and that the large Tees just touch. No other explicit conditions were given for task (c), and the number of bricks was not the same for all the children. Depending on their responses, some children were asked to generalise to $N$ bricks on a fixed Baseline and further to a varying Baseline.

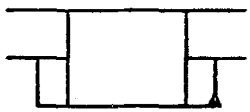
An exact solution of any of the above tasks requires the awareness that the initial choice for the length of Baseline completely fixes all the subsequent inputs. Thus the solutions entail expressing the dimensions of the Tees and Rectangles as fractional multiples of the input $b$ to Baseline. While the Bricks task calls for a fairly straightforward relation, i.e. $N$ bricks of base $b/N$ the other two tasks require a more elaborate analysis in order to come up with required relations. In the Tee-Tower task one needs to coordinate the size $s$ of the Tee (i.e. the height and width of the Tee) with certain horizontal displacements which correspond to $s/2$ so as to arrive at the relation $s = b/2$ (unless one happens to notice that at the second level of the tower, the two bars of the Tees have total length equal to $b$).

Only solutions to the 4-Tee task will be described in this section. We note that this task is further complicated by the fact that there are two different-sized Tees to contend with. Thus, to arrive at the relation $s = b/3$ demands a high level of coordination of multiple relations.

2. *The 4-Tee Task*

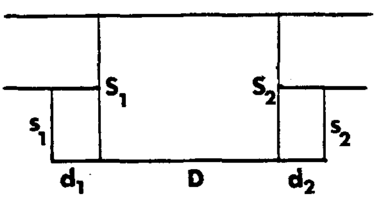
(a) *Task analysis*

*Instruction*: The task was presented as a printout



with the following conditions:
– the task had to be started with the Baseline
– the small and large Tees were to be aligned
– the large Tees were contiguous ("no overlap and no gap")

*Analysis of the relations*: The task has both a symmetry and a dimensionality aspect. The symmetry relations, while not given an explicit mention, seem obvious from the printout. They include:

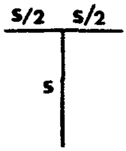



$$\left.\begin{aligned} s_1 &= s_2 \\ S_1 &= S_2 \\ d_1 &= d_2 \end{aligned}\right\} \quad \textit{symmetry} \text{ condition}$$

The placement of the Tees on the Baseline, again obvious from the printout implies the relation:

$$d_1 + D + d_2 = b \qquad \textit{baseline} \text{ condition}$$

The procedure Tee takes a single input which fixes both the height and the width of the figure Tee. Since the moves between adjacent Tees are determined by the width of the Tee figure, it requires viewing the size of the Tee as



This leads to the relation

$$d_1 = s_1/2 \quad \text{and} \quad d_2 = s_2/2 \qquad \textit{placement} \text{ condition.}$$

The two explicit geometric conditions given with the task lead to the following relations:

$$S_1 = 2s_1 \quad \text{and} \quad S_2 = 2s_2 \qquad \textit{alignment} \text{ condition}$$

and

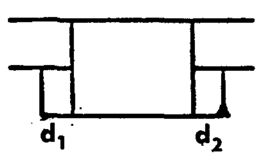
$$D = S_1/2 + S_2/2 = S \qquad \textit{contiguity} \text{ condition}$$

(The placement and alignment condition will generally be expressed as $d = s/2$ and $S = 2s$.)

These five conditions imply that all the inputs are covariant with the length of the baseline $b$. The actual relation between the first two inputs in the procedure, $b$ and $s$, turns out to be $s = b/3$. (This is a rather unexpected relation given the strong "binary" flavour of the task; there are two small Tees, two large Tees and a 2:1 ratio between their size. Thus, the "psychological" demand of the task is to break out of a binary "set". This resembles our previous analysis of the classical "nine-coin weighing" problem (see Hillel and Wheeler, 1981).)

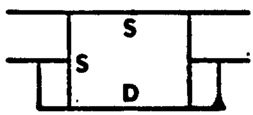
(b) *Children's solutions.* All the six children fixed a baseline and then chose some reasonable value for $s$ and began their construction at the left hand side (which is where the turtle is after executing BASELINE).

Three of the children's initial programs involved a **symmetrizing** approach which relies on the relation $d_1 = d_2$ in




i.e. after completing the big Tee on the left they ended up at the right end of the Baseline by either moving the distance $b - d_1$ and constructing the small Tee or by moving $b - 2d_1$ across and constructing the large Tee on the right. This approach automatically coordinates all but the contiguity condition (unless $s$ is chosen exactly as $b/3$).

Two other children used a **central-interval** approach which relied on the contiguity relation $D = S$ for placing the right hand large Tee, i.e.



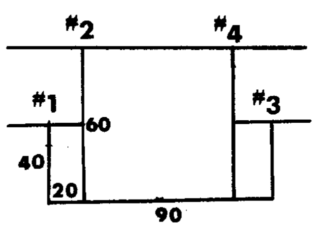

This approach resolves the contiguity condition at the expense of the Baseline condition.

One child, working in direct mode, alternated between the two approaches.

**Ben**

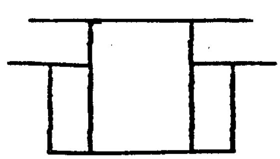
*Initial Episode*

Ben's symmetrizing approach had $b = 90$, $s = 40$, $d = 20$ and $S = 60$ hence his Tees were non-aligned and non-contiguous. He constructed the Tees in the sequence



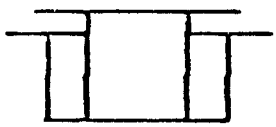

*Episode 1*

Output:



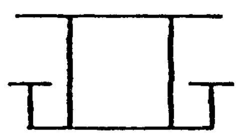
Ben was content that he had solved the problem till he was questioned about the contiguity condition. This led him to a stretch-shrink strategy on the large Tees ($S$: $60 \rightarrow 55 \rightarrow 50$) which was stopped when he visually estimated that the contiguity condition ($S = D$) was satisfied. His new solution with $s = 40$, $S = 50$ still did not satisfy the alignment condition.

*Episode 2*
Output:



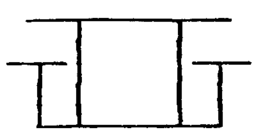
Ben's attention turned to the small Tees which he felt were "too high" (rather than not aligned – i.e. Ben focused on the height rather than the width of the Tees). He repeated his stretch-shrink strategy on the small Tees ($s$: 40 → 20), keeping all the placement moves untouched.

*Episode 3*
Output:



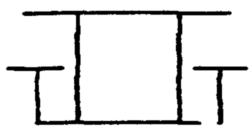
The output now focused Ben's attention on the gap (i.e. on the width of the small Tee's) and he began to reverse his previous action by stretching the small Tees ($s$: 20 → 25 → 30 → 35).

*Episode 4*
Output:



Ben noticed that a gap remained but also that the small Tees were getting "too tall" relative to the large Tees, i.e. he was now coordinating visually both height and width components of the Tees. He tried to deal with the two problems simultaneously by shrinking both the Baseline ($b$: 90 → 80) and the small Tees ($s$: 35 → 25).

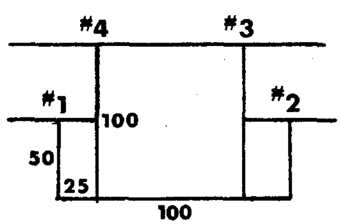
*Episode 5*
Output:



Ben tried to patch up the various gaps by reversing his previous actions. He started to simultaneously stretch the Baseline ($b$: $80 \rightarrow 85 \rightarrow 87 \rightarrow 88$) and the small Tees ($s$: $25 \rightarrow 35 \rightarrow 37$). His program ended up satisfying only some of the symmetry conditions ($s_1 = s_2$, $S_1 = S_2$) and the contiguity condition ($D = S$). (Further changes, during the subsequent session, resulted in a solution which was almost identical to the very initial one. Ben was not motivated to change it any further.)

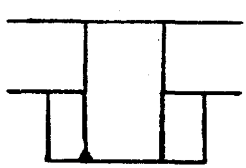
**Ed**

*Initial Episode*
Ed worked in direct-mode. His symmetrizing strategy placed the Tees in the following order:




He chose $b = 100$, $s = 50$, $d = 25$ and $S = 100$.

*Episode 1*
Output:



As was the case with the previous solvers, Ed was content that he had solved the task till the overlap of the large Tees was pointed out to him.

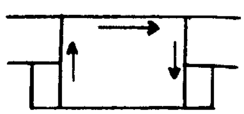
Unlike the other solvers he appropriately used a stretch-shrink on the *small* Tees, halving $s$ ($s$: $50 \rightarrow 25$) and essentially reapplying the symmetrizing approach with the new input. (Actually, Ed proposed first to construct the Baseline *after* the Tees, stating that it would be a simpler program.)

Ed picked up the task again during the next session. Starting with his modified inputs $b = 100$, $s = 25$ and $S = 50$, he choose $d = 15$ (rather than 12.5, if he were consistent with his halving of inputs). His symmetrizing approach now resulted with a gap between the large Tees. Unlike his action of the previous session, this time he applied a stretch-shrink only on the large Tees ($S$: $50 \rightarrow 60$). The resulting output, with unaligned Tees, led him to abandon his solution.

**Kay**

*Initial Episode*

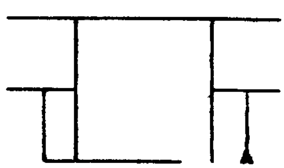
Kay worked directly in the Editor and chose $b = 100$, $s = 50$, $S = 100$ and $d = 25$. Her central-interval approach was achieved by a 'moving along the Tee' strategy:



Her initial solution met all but the baseline condition.

*Episode 1*

Output:

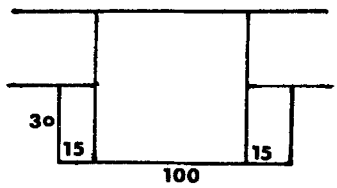


Kay figured analytically that the Baseline is 50 steps short and fixed the baseline condition by modifying $b$ from 100 to 150, i.e. she ended essentially computing $b$ in terms of $s$ and $S$ thus solving an easier additive/multiplicative type problem.

**Mark**

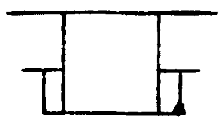
*Initial Episode*

Mark started by symmetrizing, setting $b = 100$, $s = 30$, $S = 60$ and $d = 15$ thus coordinating the alignment and the placement conditions.



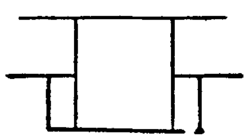

However, in writing the program in the Editor he reverted to the central-interval approach ("since I have 15 and 15 [for $d$], I'll have to move 70 [for $D$]") thus coordinating the baseline condition $D = b - 2d = 70$ with the contiguity condition $D = S$ and, consequently, modifying $S$ ($S$: $60 \rightarrow 70$). This highly analytical approach, which was based on combining relations resulted in a solution which no longer satisfied the alignment condition.

*Episode 1*
Output:



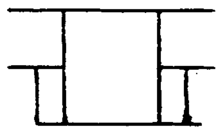
Mark realized that Tees were not aligned and decided to give the small Tees a qualitative stretch ($s$: $30 \rightarrow 40$). He also coordinated the placement condition ($d$: $15 \rightarrow 20$). Since he kept the Baseline fixed, his program now violated the baseline condition, as well as the alignment condition.

*Episode 2*
Output:



Mark attended to the alignment condition by setting $S = 2s = 80$ and extending the Baseline. Keeping the central interval fixed at 70, he first tried $b$: $100 \rightarrow 110$ (using the baseline condition $b = d_1 + D + d_2$). Realizing that the Tees were no longer contiguous, he extended $b$ once more, reasoning analytically that "the Tees are 40, 40, 80 . . . should be 120".

*Episode 3*
Output:



Mark understood immediately the reason for the incorrectly placed Tees ("I forgot to *push* this guy [the larger Tee on the right] over") and modified the central interval $D$ to 80, ending with a program which he knew was the correct solution even prior to executing it.

(c) *Discussion.* Five of the children ended up having to deal with non-contiguous large Tees due to an inappropriate central interval. They dealt with it by modifying either the size of the Tees or the size of the Baseline. The actual placement of the Tees on the Baseline was not tampered with.

The modification of the size of the Tees was generally a qualitative stretch-shrink action. All but one of the children applied a stretch-shrink strategy to the large Tees only (stretching to close a gap and shrinking in the case of an overlap) hence inadvertently abandoning the alignment condition. Ed, on the other hand, anticipated after his initial try that he had to act on the small Tees first and modify the other inputs accordingly. This approach could have led at least to an effective trial-and-adjustment strategy had he persisted with it.

Five of the solvers also operated on the Baseline to deal with non-contiguous Tees (thus altering the nature of the task since the size of the Baseline was supposed to be fixed). Such action was generally followed by a cascade of other changes including the re-placement of the Tees on the new Baseline.

Bill tried to narrow the gap between the large Tees using a rescaling operation where all the inputs were halved. This resulted with a gap half as small as the previous one but proportionally the same relative to the new figure. The rescaling operation seemed to have been based on a certain confusion between absolute and relative measures.

The three children who solved the task on their own did so by modifying the Baseline to deal with the incorrect placement of the Tees on it (i.e. in cases where only the baseline condition failed to hold). As we have remarked earlier such strategy was tantamount to solving a simpler additive/multiplicative problem whereby $b$ is expressed in terms of $s$ and $S$.

As with the centring tasks, the screen output contained information about both the matches and mismatches between a current production and the target figure. However, unlike the centring tasks, the output in this case seemed to have led the children to generate strategies which were not fruitful, given the high interdependence among the different components of the figure.

Because the set of available Logo commands was limited, each particular component of a figure corresponded, in an obvious way, to a program line. When a screen output pointed to a particular mismatch (e.g. a gap, an overlap, a misplaced Tee, etc.), children tended to attribute it to an input(s), of the corresponding command(s). In other words, their reaction to a particular output was to *associate effect with cause*. In general, for 'sectors' of the output figure which matched the goal figure, the corresponding program lines were left intact (e.g. the placement of the Tees on the Baseline was not tampered with when the Tees were non-contiguous). The overall approach was the invocation of what could best be described as 'patch-up' strategies. Many of the above descriptions of the children's solution episodes are stories about a de-structuralization of a highly analytical task, caused by local and qualitative patching-up strategies.

Both the *initial* symmetrizing and central-interval approaches involved a fair degree of analytical work. The relative sizes of the Tees and their placement on the Baseline (i.e. the inputs to the various MOVE commands within the program) were based on a conscious attempt to satisfy as many of the overt relations as possible. However, subsequent solution strategies were dominated by a qualitative push-pull or stretch-shrink, typically modifying sizes of the objects and their placement by multiples-of-5, doubling or halving of inputs. In some instances, the strategies were analytical but only in a local sense, i.e. modifications were based on satisfying a single condition and not taking into account the effects on the other relations.

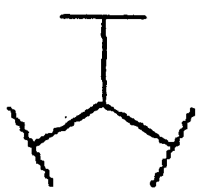
None of the children emerged with an awareness that there is a relation between the input $s$ and the input $b$. The first choices for $s$ were visual estimates (and reasonable ones at that, as they tended to fall between $b/4$ and $b/2$) which were modified in the course of the solution in various ways. On the other hand, three of the children were able to, eventually, figure out analytically the additive relation $b = 1/2s + 1/2S + 1/2S + 1/2s$ but not explicitly the multiplicative relation $b = 3s$.

Initially, all the children judged success by the output. Since the output did not alert the children to the fact that the Tees overlapped, four children concluded erroneously that they had solved the task. However, when three of the children felt, later in the session, that they had solved the task, they based their claim on their exact computation of $b$ in terms of $s$ and $S$. This points to a move away from a strictly visual assessment of the correctness of a solution.

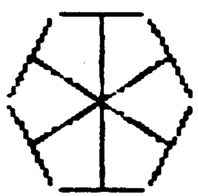
## III. *The Rotation Tasks*

### 1. *The Tasks*

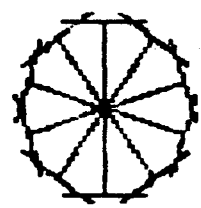
The Rotation Tasks consisted of a sequence of eight tasks including:



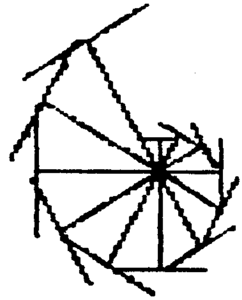
(a)
3-Tee
(session #12)



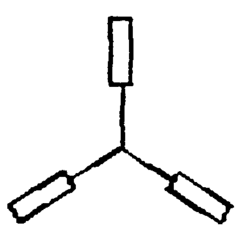
(b)
6-Tee
(session #12)



(c)
10-Tee
(session #12)



(d)
12-Shell
(session #13)



(e)
3-Paddle
(session #15)

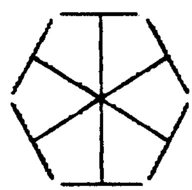
All the tasks involved an explicitly stated condition that all of the angles (or "turns" if the children did not understand the term "angles") were the same size. In addition, the children were told that the Tees were the same size in tasks (a), (b) and (c); that the Tees increased by the same amount for task (d); and that the three rectangles and stems in task (e) were the same size.

From our perspective, the solution of the rotation subproblem of these tasks requires expressing the geometric condition of $n$ equal angles as a relation involving 360, i.e., $360/n$. Children's attempts to solve the 6-Tee Rotation task are typical of their work with the other rotation tasks, and are thus presented in detail.

2. *The 6-Tee Rotation Task*

(a) *Task analysis*

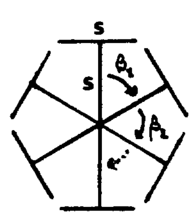
*Instruction*: This task was presented as a printout,



after the children had already worked on these two tasks

during the same session. They were told that all the angles (i.e., turns) were the same size.

*Analysis of the relations*:



The fundamental relations in this problem are:

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6$$

and

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 = 360$$

(or equivalently, $\beta_1 + \beta_2 + \beta_3 = 180$). Thus, the input to each turn is 360/6 or 60. The task demands knowing not only that a complete turn is 360 (or that half a complete turn is 180) but also how this is relevant to the task. In particular, the final turn $\beta_6$ is not a necessary part of the production of the figure, yet it must be taken into account for an analytical calculation of the angle. In this respect, the rotation tasks have something in common both with the calculation of the centring-move in the centring tasks and the calculation of, say, the central-interval in the 4-Tee task.

(b) *Children's solutions.* Three of the children initially proceeded by choosing 45 as input to the turn; the other three used 60 though only in one case was it based on the notion of 180 or 360.

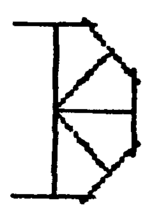
**Ben**

*Initial Episode*

Ben began with $\beta = 45$. He continued in direct mode until five Tees had been produced.

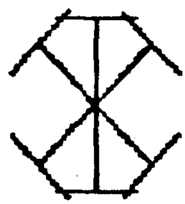
*Episode 1*

Output:



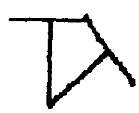
This output suggested to Ben that he was getting too many Tees. He decided to eliminate the horizontal Tee. He began his six Tees again and kept $\beta_1$, $\beta_3$, and $\beta_4$ at 45, but changed $\beta_2$ and $\beta_5$ to 90.

*Episode 2*
Output:



This output pleased him, though it was not clear whether he thought that he had solved the given task. When he was asked if his screen picture was the same as the printout of the task, Ben replied, "No, I'll just make the turn larger". This time he used a push-pull and increased $\beta_1$ ($\beta_1$: 45 → 50).

*Episode 3*
Output:

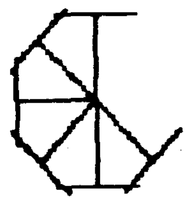


Since the screen output indicated two overlapping Tee's but the Tees in the printout did not overlap, Ben used two more push-pull's ($\beta$: 50 → 55 → 60). He was satisfied with the final output.

**Bill**
*Initial Episode*
The previous task

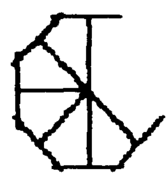
had concluded with a discussion of the role of 360 degrees in the total turn of the turtle. Thus, the interviewer initiated this task with the question, "What do you think would be the size of each angle?" Bill replied that it would probably be 60 because 360 divided by 6 is 60. However, he did not incorporate this idea into his procedure for the figure. He used TEE 40 and TLT 45 in a REPEAT 6 construction.

*Episode 1*
Output:



The output alerted Bill to the overlapping Tee's which he tried to remedy by shrinking the bar of the Tees. He decreased the input to the TEE procedure ($s$: 40 → 35), but kept the turn input the same as before.

*Episode 2*
Output:

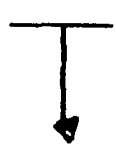


The output was similar to the previous one. Bill again tried to manipulate the size of the Tee's, this time to "stretch the stems", presumably to distance the top pieces of the *T*'s from each other. He increased the input to the TEE ($s$: 35 → 60). The interviewer then drew Bill's attention to the angles of the figure. This hint resulted in changes to the inputs for both the Tee figure and the turn ($s$: 60 → 40) ($\beta$: 45 → 60). It is not clear whether the latter change was just a "pull" or was analytically based. The output satisfied Bill.

**Kay**

*Initial Episode*
Kay drew, in direct mode, the first Tee ($s = 50$) followed by a turn of 45.

*Episode 1*
Output:



The screen output indicated to her that she should increase the size of the turn. She used a push-pull ($\beta$: 45 $\rightarrow$ 50) but the output still did not look right to her ("It looks as if the tops of the Tee's might touch"). The input to the turn was further increased ($\beta$: 50 $\rightarrow$ 60). She then proceeded to "draw" the rest of the Tees and with the same inputs. The final output satisfied her.

(c) *Discussion.* In this task, only two children chose their initial input to the turn command based on analytical approaches. One used 360 – 180 relationships; and the other used the relation between the angles of a 6-Tee Rotation and the angles of a 3-Tee Rotation and halved the input to the turn of the 3-Tee figure. (We note that while the latter approach was effective for this task it is not a generalizable one.) The remaining children chose an initial input of 45 for the turn, because it "looked like half of a 90 turn" – evidence of the 45–90 schema, a schema that is described elsewhere (Kieran, Hillel and Erlwanger, 1986; Hillel and Kieran, 1987). To try to eliminate the discrepancy between the screen output and the printout, the children used either a push-pull or a stretch-shrink strategy.

The most commonly used stretegy was to "push" the Tee's farther apart about the central point. The input to the turn was increased until it looked as if the Tee's would be properly distanced from each other. The children used the overlap/non-overlap of the Tees as cues that they were getting closer to the goal.

The stretch-shrink strategy was used by Bill when he saw that the bars of the Tees were overlapping. He seemed unaware that decreasing the input to Tee would simply produce a smaller but similar figure, even though he had been working with the TEE procedure throughout the previous four months. In his next attempt he began to focus on the stem of the Tee and he tried stretching the stems, in order to eliminate the overlap of the top pieces which resulted, again, in similar Tees. His strategy here was consistent with his attempt to use rescaling to get rid of the gap between the Tees in the 4-Tee task (see section B.II).

The children who began with a visually-based estimate for the turning angle continued by making qualitative adjustments to their productions, by comparing the output with the printout. The output in this case provided useful feedback in directing these qualitative strategies towards the goal. It did not, however, lead the children to focus on the structural features of the figure and to attempt a more analytical approach.

The criterion for success remained the visual verification of the output. The three children who used a qualitative approach did not try to verify the 6-fold symmetry of the figure by checking that the remaining turn ($\beta_6$) was the same as the first five turns.

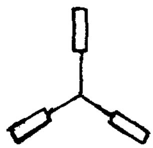
3. *Other Rotation Tasks*

The two main solution strategies that were used with the 6-Tee rotation task were also used with the other Rotation Tasks.

With the various rotation tasks, it became clear to us that the children were not taking into account the $n$th angle of an $n$-Tee rotation task. Rather, they persisted with the strategy of "drawing one object and then turning" and continuing until the last object was drawn. Thus, the last angle of the figure was formed without their having to assign an input. For example, with the 3-Tee task, Kay eventually ended up with 110 and 130 as inputs for $\beta_1$ and $\beta_2$. When she was reminded that the three angles had to be the same, she then changed both inputs to 125. With this same reminder, Rosa also changed her two inputs from 105 and 120 to 130, i.e. the children continued to consider only the explicit turns for which they had to provide inputs. They did not take account of the last angle that was created by default.

During session # 14, we discussed with the children, on an individual basis, the relationship between 360° and the $n$-angles of the previous tasks. These discussions had varying effects on different children. For example, with the 10-Tee task given to Ben in the previous session (session # 13), he began again with a turn of 45. After viewing the output, he decided to make two adjustments: shrink the Tee's in order to cut down the overlap of the top pieces and to pull the Tee's closer by "not moving so much space". Though Ben knew that "360 makes a full circle", he did not use this knowledge. In session # 14, Ben started a 6-Rectangle rotation task by choosing, once more, an input of 45 for the turn. The resulting screen output had led him to increase this input to 55, producing a figure which looked good enough to him. When the interviewer started discussing with Ben 360 degrees in relation to the total number of turns, Ben changed his 55 to 60. He then began to look for a pattern in subsequent tasks, but not one involving 360. "Since 6 turns require TRT 60, a 10-turn figure would require TRT 100". Though he had stated that the turn for a 10-Rectangle figure would be smaller than for a 6-Rectangular one, he insisted on using his REPEAT 10 [RECT 120 10 TRT 100]. For a subsequent 3-Rectangle figure, he again used the same pattern REPEAT 3 . . . TRT 30.

After discussion on the role of 360°, Kay attempted to solve the 3-Paddle task (session # 15),



by calculating on paper the following products: 96 × 3, 105 × 3, and finally 120 × 3, i.e., she solved for the appropriate input using a forward substitution in the (correct) relation, just as she had done for calculating the centring-move for some Centring tasks (see section B.I).

It is probably fair to say that within the time frame allocated to these rotations tasks, the role of 360 degrees was not appreciated by most of the children. We did not observe a shift from a qualitative to an analytical solution strategy that took place with, for example, the Centring tasks.

## C. CONCLUSION

There are several perspectives from which one can examine the problem solving behaviour of solvers. One may want to come up with a general description (or a model) of the solution process which is independent of the specific problems being solved. Alternatively, one may examine the efficacity of the process in terms of its success in resolving the problem(s) given. We, on the other hand, have examined the children's solution behaviour from another perspective; namely, whether the solution strategies led to insights about the actual nature of the problem and about some of its underlying mathematical relations. This perspective is of particular interest when a solver is presented, over time, with many tasks which embody the same or very similar characteristics.

A computer environment has been touted as a very good environment for problem solving work, particularly because the screen output provides an instant feedback as to the validity of a particular attempted solution. Our experience has been that the computer environment is certainly an engaging one for children, and creates a good motivational level to persist with the solution of a problem. Some of the attempts to solve a particular problem took nearly two sessions, a level of engagement rarely seen with children of that age in a more traditional paper-and-pencil solving activity. We also do not underestimate the role of the Turtle as an 'object to think with' in our more specific Logo-based environment, though we have not discussed this aspect in our description of the children's work. For example, Mark, who was by far the most sophisticated solver in the group (and hence seemed very likely to think a problem out, independently of the Turtle) was often overheard saying to himself things like "o.k., I am the turtle and I am here and I have got to move this much across . . . ".

The issue taken by our study relates to the 'feedback' aspect of the computer environment. It stems from our observations (and of others in the field) that this feature of the computer encourages solution strategies which are qualitative and driven by visual data, rather than being based on explicit or implicit relations. Such solution behaviour often results in successful solutions but, we argue, does not lead to the kind of mathematization of the problem that we, as mathematics educators, would like to see. The question we posed to ourselves was whether changing the nature of the tasks typically given to children in a Logo environment would lead children to adopt a more analytical solution approach. To this end, we were very deliberate in separating 'programming' issues from those relating to 'properties of figures', by choosing tasks with minimal programming demands. We were also very explicit about the geometric conditions that needed to be satisfied by a given geometric figure; we constrained the possible operations that could be performed and we insisted on exact solutions. Some of the tasks involved multiple interrelated conditions, thus rendering their solutions difficult with a visually-based approach. The children's task was to write the programs which would produce the given figures with the available set of actions (commands). This required the *quantification* of the given geometric conditions by specifying inputs to the commands appearing in the program.

We have described in detail the children's solutions to three representative tasks, one from each set of tasks sharing certain common structural features. We might start by asking how the behaviours of these novice solvers working on non-routine problems differ from those observed in a more traditional paper-and-pencil setup, that is, we can try to delineate the general features of the solution process from those features that are specific to the computer environment. To answer this question, we take up the description given by Bell (1981), one which was found to give a reasonably good 'fit' for the case of novices solving 'non-standard' problems in a traditional setting:

The [problem solving] process . . . consists primarily of selecting a subset of the data small enough to be processed, leading to a production of a new piece of the knowledge integrating that subset; the end of such a phase is often marked by an act of verification, a check. Attention

> then moves to a new subset of data, possibly including the new item, and this new subset is similarly processed. In this way the number of items to be co-ordinated is progressively reduced until they can be processed simultaneously to resolve the problem. Blockages occur when one of these attempts at co-ordination either fails or produces an incompatibility. (pp. 110–111)

There are, obviously, features of the children's solutions that resonate very well with the above description. The solutions of the problems requiring the coordination of multiple data (e.g. Centring and Baseline tasks) showed the solvers attending to different problem conditions after each verification stage of the solution. However, there are two striking differences. While Bell's description suggests a progressive coordination of more items of data as the solution advances, this was not the observed behaviour of our solvers. Often, their *initial* solution attempts were the ones in which they coordinated the largest set of data. Subsequent attempts involved coordinating one or several items of data independently of the other conditions of the problem. In the cases where a successful solution was reached, it was more often via 'patching-up' rather than due to a simultaneous processing of the multiple conditions. This was particularly so in the case of the 4-Tee task, where we have described the solution process of some of the solvers as a 'destructuralization' of an initial, well-coordinated plan. We believe that this 'anomaly' in the progression of the solution is very revealing. *The initial planning, in the absence of any output on the screen, was based on some of the given problem conditions. Subsequent attempts, in the presence of a screen output, focused on eliminating the discrepancies between the output and the given figure.* They were visually-based attempts, resulting mostly in qualitative strategies. On occasion, the solution attempts were analytical in a 'local' sense (i.e. trying to satisfy one condition based on the chosen inputs but forgetting its relation to the other conditions). Only on rare occasions were they analytical in the 'global' sense referred to in Bell's description, that is, the simultaneous processing of all the conditions.

The other aspect which differs from Bell's description is the *total absence of 'blockages', so commonly observed with paper-and-pencil tasks.* Though we did not indicate the time span between successive episodes, at no time were the children stuck after getting an output. An output always triggered a fairly immediate action. It is hard to say whether this is a positive aspect in a problem-solving situation. It certainly helps to keep the solution process going, while 'blockages' may result in solvers' becoming completely stuck. On the other hand, this spontaneous reaction to the output does not lead to a reevaluation of strategies, while 'blockages' are often the catalysts to more fruitful solution strategies.

Another aspect of the children's solution behaviour was their criterion for having successfully completed a task. As we have pointed out in the description of their work, it was based on the appearance of the output. In a way, this is also not specific to a computer environment but is a more general phenomenon related to the presence of geometric figures. For example, Zykova (1969) and Schoenfeld (1986) have written about student's inferring properties of figures by using "it looks like" arguments. What is different in our situation is the availability of the children's programs as an alternate 'proof' of a solution. Yet, whenever they were pressed to justify the correctness of their solution, the children generally referred to the output on the screen rather than check that their inputs to the commands were consistent with the conditions of the problem. This was particularly so if their solution was arrived at using qualitative strategies. This suggests that these children did not view procedures as "a summary of relationships and sub-procedures which are the logical analogs of geometric action/objects" (see Olson, Kieren and Ludwig, 1987). Rather, procedures were viewed simply as lists of actions which drive the Turtle around.

Finally, we address the question of whether the particular computer environment (including the set of tasks) was conducive in promoting a change of awareness about the nature of the problems and about the way to solve them. The problems that we gave differed quite a bit from each other in their degree of complexity. However, the children's solution strategies were fairly uniform across the tasks – each output usually led to either a push-pull or a stretch-shrink action. The effectiveness of these actions varied substantially with the task. The push-pull strategy was certainly an effective trial-and-adjustment strategy for both the Centring and Rotation tasks. In the case of the Centring tasks, this qualitative strategy was followed in time by an analytical strategy based on the centring-relation, at least for some of the tasks. It is reasonable to assume that the use of the push-pull strategy led to an eventual understanding of the essence of the centring problem. This was not the case with the rotation tasks as the role of 360 remained hidden by the use of the push-pull strategy, one which persisted even after several discussions about 360. By contrast, these qualitative strategies were not effective even as trial-and-adjustment strategies in the 4-Tee task, as they often resulted in movement away from the goal. Even when this (admittedly very difficult) task was solved, its actual structure was not very well understood. It is evident that the children were not discriminating in their use of these strategies. Rather, they were evoked spontaneously as a reaction to an output, without reference to the nature of the problem at hand.

Balacheff (1986) reminds us that a pupil acts as a practical man and not as a theoretician and that "the problem of the practical man is to be *efficient* not *rigorous*; it is to produce a *solution*, not to produce *knowledge*". This comment puts into focus the gap between our concern (knowledge) and the children's (solution), and our contention that the computer can exacerbate this difference. Since efficiency for the children was measured in terms of producing a reasonable screen figure, the cycle of outputs followed by patching-up strategies was very effective even if it took a long time to complete. It leads us to conclude that the interaction child-machine alone does not lead to the mathematization of problem-situations for problems involving multiple conditions; that the computer can serve as an effective problem solving tool only if accompanied by more traditional forms of discourse between pupils and teacher.

## ACKNOWLEDGEMENTS

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