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INTRODUCTION TO PROOF: THE MEDIATION OF A DYNAMIC SOFTWARE ENVIRONMENT

ABSTRACT. This paper, which reports on a long-term teaching experiment carried out in the 9th and 10th grades of a scientific high school, is part of a larger co-ordinated research project. The work constitutes a 'research for innovation', in which action in the classroom is both a means and result of a study aimed at introducing pupils to theoretical thinking and at studying the ways in which this process is realised. The purpose of the study is to clarify the role of a particular software, Cabri-géomètre, in the teaching/learning process. Assuming a Vygotskian perspective, attention is focussed on the social construction of knowledge and on the semiotic mediation accomplished through cultural artefacts; the functioning of specific elements of the software will be described and analysed as instruments of semiotic mediation used by the teacher in classroom activities.

1. INTRODUCTION

This paper analyses a long-term teaching experiment carried out in the 9th and 10th grades (15–16 years) of a scientific high school and aimed at introducing pupils to theoretical thinking. The experiment is part of a co-ordinated research work concerning the introduction to theoretical thinking at different age levels (Mariotti et al., 1997).

The research project I am presenting is rooted in the theoretical frame of a Vygotskian perspective, with particular regard to the social construction of knowledge and semiotic mediation accomplished through cultural artefacts. As explained in the following sections, in this experiment, besides the spontaneous forms of social interaction, there are specific forms of controlled and planned classroom verbal interaction realised by means of 'Mathematical Discussion' (Bartolini Bussi, 1999, 1996).

In fact, the analysis of the teaching experiment is aimed at discussing the specific role played by a dynamic geometry software, with particular regard to the characteristics which make it possible to introduce pupils to theoretical thinking. In particular, the paper will be devoted to discussing on the process of semiotic mediation related to the emergence of the meaning of proof, strictly related to the meaning of Theory. (i.e what we have called Mathematical Theorem (Mariotti et al., 1997)).

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We shall limit ourselves to the description of some elements of the software, in order to present an analysis of the process of semiotic mediation (Vygotskij, 1978) that can be realised in classroom activities.

The following sections (§2-§3) are devoted to clarifying the theoretical frame; in particular, the notion of 'Field of Experience', and 'Mathematical Discussion' will be discussed in relation to their utilisation both in the design and analysis of the experimental project.

The results will be presented in terms of educational goals and analysis of the teaching-learning process, since the research project is to be considered a 'research for innovation', in which action in the classroom is both a means and a result of the evolution of research analysis (Bartolini Bussi, 1994, p. 1) On the one hand pupils achieved a theoretical perspective in the solution of construction problems; the theoretical meaning of geometrical construction provided the key to accessing the general meaning of Theory. On the other hand, the study made it possible to clarify the role of a particular software in the teaching/learning process; the functioning of the specific elements of the software will be described and discussed according to the theoretical reference frame. The software utilised is Cabri-géomètre (Baulac et al., 1988).

2. THE THEORETICAL FRAMEWORK: AN OVERVIEW

Geometrical Constructions constitute the field of experience in which classroom activities are organised. According to Boero et al. (1995, p. 153), the term 'field of experience' is used to intend

> the system of three evolutive components (external context; student internal context; teacher internal context), referred to a sector of human culture which the teacher and students can recognise and consider as unitary and homogeneous.

The development of the field of experience is realised through the social activities of the class; in particular, verbal interaction is realised in collective activities aimed at a social construction of knowledge: i.e. 'Mathematical Discussions', that is

> polyphony of articulated voices on a mathematical object, that is one of the objects – motives of the teaching – learning activity (Bartolini Bussi, 1996, p. 16).

Polyphony occurs between the voice of practice and the voice of theory. The practice of the pupils consists in the experience of drawing, evoked by:

- concrete objects, such as drawings, realised by paper and pencil, ruler and compass.
- computational objects such as Cabri figures or Cabri commands.

Geometry theory, imbedded in the Cabri microworld, is evoked by the observable phenomena and the commands available in the Cabri menu.

Figures and commands may be considered external signs of the Geometric theory, and as such they may become instruments of semiotic mediation (Vygotskij, 1978), as long they are used by the teacher in the concrete realisation of classroom activity to introduce pupils to theoretical thinking.

3. THE FIELD OF EXPERIENCE OF GEOMETRICAL CONSTRUCTIONS IN THE CABRI ENVIRONMENT

3.1. *Reference culture*

The reference culture is that of classic Euclidean Geometry. Euclidean Geometry is often referred to as 'ruler and compass geometry', because of the centrality of construction problems in Euclid's work. The fundamental theoretical importance of the notion of construction (Heath, 1956, p. 124 segg.) is clearly illustrated by the history of the classic *impossible* problems, which so much puzzled the Greek geometers (Henry, 1994).

Despite the apparent practical objective, i.e. the drawing which can be realised on a sheet of paper, geometrical constructions have a theoretical meaning. The tools and rules of their use have a counterpart in the axioms and theorems of a theoretical system, so that any construction corresponds to a specific theorem. Within a system of this type, the theorem *validates* the correctness of the construction: the relationship between the elements of the drawing produced by the construction are stated by a theorem regarding the geometrical figure represented by the drawing.

The world of Geometrical Construction has a new revival in the dynamic software Cabri-géomètre. As a microworld (Hoyles, 1993), it embodies Euclidean Geometry; in particular, as it is based on the intersections between straight lines and circles, Cabri refers to the classic world of 'ruler and compass' constructions.

However, compared to the classic world of paper and pencil figures, the novelty of a dynamic environment consists in the possibility of direct manipulation of its figures and, in the case of Cabri, such manipulation is conceived in terms of the logic system of Euclidean Geometry. The dynamics of the Cabri-figures, realized by the dragging function, preserves its intrinsic logic, i.e. the logic of its construction; the elements of a figure are related in a hierarchy of properties, and this hierarchy corresponds to a relationship of logic conditionality.

Because of the intrinsic relation to Euclidean geometry, it is possibile to interpret the control 'by dragging' as corresponding to theoretical control

– 'by proof and definition' – within the system of Euclidean Geometry. In other words, it is possible to state a correspondence between the world of Cabri constructions and the theoretical world of Euclidean Geometry.

Advantages and pitfalls of using Dynamic Geometry Environments[1] have been largely discussed in the literature. On the one hand, the possibility of making an explicit distinction between 'drawing' and 'figure' has been pointed out (Laborde, 1992); on the other hand, many studies have shown that students' interpretation of the drag mode is not obvious and cannot be taken for granted (Straesser, 1991; Noss et al., 1993; Hoelz, 1996). In the case of the Geometer's sketchpad, Goldenberg and Cuoco (1998) present and discuss alternative interpretations of the observable phenomena, generated by dragging on the screen.

As explained in the following, in our project, the development of the Field of Experience is based on the potential correspondence between Cabri construction and Geometric theorems. Once a construction problem is solved, i.e. if the Cabri-figure passes the dragging test, a theorem can be proved with a geometric proof. Thus, solving construction problems in the Cabri environment means accepting not only all the graphic facilities of the software, but also accepting a logic system in which its observable phenomena will make sense. The explicit introduction of this interpretation and the continuous reference to the parallel between Cabri enviroment and Geometry theory constitutes the basis of our teaching project.

3.1.1. *The theoretical dimension of mathematical knowledge*

Although it is important to distinguish between the intuitive construction of knowledge and its formal systematisation, one must recognise that the deductive approach, primarily set up in the Euclid Elements, has become inherent in mathematical knowledge. Even when stress is placed on the heuristic processes, it is impossible to neglect the theoretical nature of mathematics, as the following passage, quoted by Schoenfeld (1994), clarifies.

> What does mathematics *really* consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Gödel's proof of undecidability)? Definitions (such as the Menger definition of dimension)? [...] Mathematics could surely not exist without these ingredients; they are essential. It is nevertheless a tenable point of view that none of them is the heart of the subject, that mathematician's main reason for existence is to solve problems and that, therefore, what mathematics *really* consists of is problems and solutions. (Halmos, 1980, p. 519)

As often pointed out, the distance between the theoretical and the intuitive level raises great difficulties (Fischbein, 1987), but theoretical organisation according to axioms, definitions and theorems, represents one of the basic elements characterising mathematical knowledge. In particular, as far as theorems are concerned, it is worth reminding (Mariotti et al., 1997) that any mathematical theorem is characterised by a statement and a proof and that the relationship between statement and proof makes sense within a particular theoretical context, i.e. a system of shared principles and inference rules. Historic – epistemological analysis highlights important aspects of this complex link and shows how it has evolved over the centuries. The fact that the reference theory often remains implicit leads one to forget or at least to underevaluate its role in the construction of the meaning of proof. For this reason is seems useful to refer to a 'mathematical theorem' as a system consisting of a statement, a proof and a reference theory.

3.2. *The external context*

The external context is determined by the 'concrete objects' of the activity (paper and pencil; the computer with the Cabri software; signs – e.g. gestures, figures, texts, dialogues).

In the following sections of this paper, we shall focus on the particular 'objects' offered by the Cabri environment, which however must be considered in a dialectic relationship with all the other objects available.

The activity starts by revisiting drawings and artefacts which belong to the pupils' experience. Such objects are part of physical experience, the compass being a concrete object the use of which is more or less familiar to the pupils. In any case, the students are familiar with the constraints and relationships which determine possible actions and expected results; for instance, the intrinsic properties of a compass directly affect the properties of the graphic trace produced.

Revisitation is accomplished by transferring the drawing activity into the Cabri environment, so that the external context is moved from the world of ruler and compass drawings to the virtual world of Cabri figures and commands.

In a software environment the new 'objects' available are Evocative Computational Objects (Hoyles, 1993; Hoyles and Noss, 1996, p. 68), characterised by both their own computational nature and the evocative power caused by their relationship with geometrical knowledge:

- the Cabri-figures realising geometrical figures;
- the Cabri-commands (primitives and macros), realising the geometrical relationships which characterise geometrical figures;
- the dragging function which provides a perceptual control of the correctness of the construction, corresponding to the theoretical control consistent with geometry theory.

The development of the field of experience is based on the practice carried out through activities in the context of Cabri: construction tasks, interpretation and prediction tasks and mathematical discussions. Such development however, also concerns the practice of ruler and compass constructions, which become both concrete referents and signs of the Cabri figures. The relationship between drawings (accomplished on paper, using ruler and compass) and Cabri figures constitutes a peculiar aspect of the external context, which presents a double face, one physical and the other virtual.

3.3. *The internal context of the pupil*

3.3.1. *Intuitive and deductive geometry*

Traditionally in the Italian school, deductive geometry is introduced at the beginning of the 9^{th} grade (entrance to high school). In the previous grades geometry is studied, but usually at an ‘intuitive’ level: this means that Geometry is presented to pupils as a collection of ‘definitions’, naming and describing geometrical figures, and ‘facts’ stating particular properties. Most of these facts have a high degree of evidence, and in any case the arguments eventually provided by the teacher have the specific aim of constructing such evidence. Moreover, pupils are never asked to justify their knowledge, the truth of which is considered immediate and self-evident, i.e. *intuitive* (Fischbein, 1987). As a consequence, at the beginning of high school, pupils generally have an intuitive geometrical background which must be reorganised according to a deductive approach.

The delicate relationship between intuitive knowledge and its theoretical systematisation is usually very difficult to manage: pupils fail to grasp the new with respect to the old. Actually, it is very difficult to understand why well known properties should be put into question and long arguments used to support their truth, which is so evident.

In the pupils’ experience, justifying is the prerogative of the teacher and, rather than providing pupils with a basis for a deductive method, it aims at convincing one of the ‘evidence’ of a certain fact. However, when a statement reaches the status of evidence, any argument becomes useless and ready to be forgotten. According to its nature, *intuition* (Fischbein, 1987) contrasts the very idea of justification, and in this respect, intuitive geometrical knowledge may constitute an obstacle to the development of a theoretical perspective.

In summary, when the deductive approach is introduced, a crucial point is that of changing the status of justification.

A deductive approach is deeply rooted in the practice of justification (de Villiers, 1990; Hanna, 1990). Proving consists in providing both *logically enchained arguments* which are referred to a particular theory, and an *ar-*

gumentation which can remove doubts about the truth of a statement. This twofold meaning of proof is unavoidable and pedagogically consistent (Hanna, 1990). It is often pointed out and commonly accepted, however, that arguing and proving do not have the same nature; arguing has the aim of convincing, but the necessity of convincing somebody does not always coincide with the need to state the theoretical truth of a sentence.

> ...il y a une très grande distance cognitive entre le fonctionnement d'un raisonnement qui est centré sur les valeurs épistémique liées à la compréhension du contenu des propositions et le fonctionnement d'un raisonnement centré sur les valeurs épistémique lieées au statut théorique des propositions.
>
> (Duval, 1992–93, p. 60)

The distance between these two modalities explains the reason why arguing can often become an obstacle to the correct evolution of the very idea of proof (Balacheff, 1987; Duval, 1992–93).

To sum up, in the case of a deductive approach, two interwoven aspects are involved: on the one hand, the need for *justification* and on the other hand the idea of a *theoretical system* within which that justification may become a *proof.*

3.4. *The internal context of the teacher*

The presence of the computer and of particular software certainly represents a perturbation element in the internal context of the teacher. The teacher has to elaborate a new relationship to mathematical knowledge, augmented by the whole set of relations which link it to the computer in general and cabri in particular. At the same time, the teacher has to adapt his/her role of mediator taking into account the new elements offered by the software.

Specific issues arise that are related to the nature of the mediation instruments, with particular regard to problems referring to the 'computational transposition of knowledge' (Balcheff and Sutherland, 1994; Balcheff, 1998), i.e. resulting from implementation in a computer.

Up until now, no specific analysis of these aspects has been carried out, although there is some evidence of an interesting development in the teachers' conceptions, which will require further investigation.

4. The activities

4.1. *Construction activity*

According to the general hypothesis of our project, construction problems constitute the core of the activities proposed to the pupils. The main objective was the development of the meaning of 'construction' as a theoretical procedure which can be validated by a theorem.

The construction task was articulated as follows:

a) providing the description of the procedure used to obtain the requested figure;
b) providing a justification of the correctness of that procedure.

Pupils worked in pairs, and each pair had to provide a written text drawn up in common. These written reports constituted the basis on which the collective discussions were organised.

Within the Cabri environment, as soon as the dragging test is accepted, there is a need to justify one's own solution. The necessity of a justification for the solution comes from the need to explain why a certain construction works (that is, passes the dragging test). Such a need is reinforced during the collective discussion, when different solutions are compared, by validating one's own construction in order to explain why it works and/or to foresee whether or not it will function.

Experimental evidence (Mariotti, 1996) shows that, as far as the paper and pencil environment is concerned, the theoretical perspective is very hard to grasp. When a drawing is produced on a sheet of paper, it is quite natural for the validation of the construction to be focussed on the drawing itself and on a direct perception. Ambiguity between drawings and figures may represent an obstacle: although the question is about the drawing, its sense concerns the geometrical figure it represents.

A theoretical validation concerns the construction procedure rather than the drawing produced: each step of the procedure corresponds to a geometrical property, and the entire set of properties given by the procedure constitutes the hypothesis of a theorem proving the correctness of the construction.

When a construction problem is presented in the Cabri environment the justification of the correctness of a solution figure requires an explanation of why some constructions function and others do not. This implies shifting of the focus from the drawing obtained to the procedure that produced it. The intrinsic logic of a Cabri-figure, expressed by its reaction to the dragging test, induce*s* pupils to shift the focus onto the procedure, and in doing so it opens up to a theoretical perspective.

4.2. *'Mathematical Discussion' activities*

Among the classroom activities, 'Mathematical discussions' (Bartolini Bussi, 1991, 1999) take up an essential part in the educational process, with specific aims, which are both cognitive (construction of knowledge) and metacognitive (construction of attitudes towards learning mathematics).

In the literature, the role of discussion in the introduction to the idea of proof has been analysed. Both aspects of continuity and rupture have been clearly described; the most radical opinion is held by Duval, who claims that

> Passer de l'argumentation à un raisonnement valide implique une décentration spécifique qui n'est pas favorisée par la discussion ou par l'interiorisation d'une discussion. (Duval, 1992–93, p. 62)

Our proposal refers to a specific type of discussion, mathematical discussion, which is not a simple comparison of different points of view, not a simple contrast between arguments. The main characteristic (Bartolini Bussi, 1999) of this kind of discussion is the cognitive dialectics between personal senses (Leont'ev, 1976/1959, p. 244) and general meaning, which is constructed and promoted by the teacher. In this case, the cognitive dialectics takes place between the sense of justification and general meaning of mathematical proof.

Different senses of justification correspond to possible different goals of the discussion, whereas moving from one goal to another corresponds to the evolution of the sense of justification, which is the main motive of discussion.

The role of the teacher is fundamental, in order to direct the goal of the discussion and to guide the evolution of personal senses towards the geometrical meaning of a construction problem, and more generally to the theoretical perspective.

This corresponds to two different types of motivation which determine the discussion activity and interrelate and support each other.

On the one hand, justifications must be provided within a system of shared principles.

On the other hand, the methods of validation must be shared and the rules of mathematical argumentation made explicit.

In the following two sections, after a brief general summary of the results coming from the experimental data, we shall discuss the process of semiotic mediation with respect to the evolution of the pupils' internal context. Some examples will be analysed, drawn from the transcripts of collective discussions and from the written reports provided by the pupils.

5. Evolution of the Internal Context

The analysis of the experimental data highlights the development of the meaning of justification emerging for both the construction tasks and the Mathematical Discussions; such evolution may be broken up into a sequence of steps.

a) Description of the solution

From the very beginning the pupils are asked to provide a written report of the solution process. The main objective consists in communicating to the teacher and the classmates one's own reasoning; as a consequence, the main objective in writing the report is that of being *understood.*

b) Justifying the solution

The solutions described in the reports are proposed by the teacher and form the basis of a collective discussion. Once the pupils have accepted the criterion of validation by dragging, attention is shifted from the product (the drawing) to the procedure followed to obtain it; thus the main intention in writing the reports becomes that of providing a justification for that procedure. After the first discussions, a new objective emerges: the reports must provide a clear description of the construction, but also a justification so that it can pass the collective trial, and this means providing good arguments which can make the solution *acceptable.*

c) Justifying according to shared and stated rules

In the collective discussions the main purpose is the defence of one's own construction, and for this reason it becomes necessary to state a number of rules to be respected; in other words an agreement on the acceptable operations must be negotiated in class. This may be considered the origin of a theoretical perspective: looking for a justification within the system of rules introduces pupils to a theoretical status of justification.

The collective discussion guided by the teacher according to a specific objective determines the meaning of the construction problem to be developed. In this process a basic role is played by Cabri, and the following sections will be devoted to discussing the process of semiotic mediation, accomplished by the teacher through the use of specific software elements.

5.1. *Semiotic mediation*

Historic and epistemological analysis confirms the productive interaction between theory and practice in the development of mathematical knowledge (Bartolini and Mariotti, 1999). A crucial element of the dialectic relationship between theory and practice is represented by *technical tools*.

Tools have a twofold function, the former, externally oriented, is aimed at accomplishing an action; the latter, internally oriented, is aimed at controlling the action. This distinction, commonly used in studies about the practice and development of technologies (e. g. Simondon, 1968), can be elaborated starting from the seminal work of Vygotskij (1978), who introduces the theoretical construct of *semiotic mediation*. Vygotskij distinguishes between the mediation function of what he calls *tools*[2] and *signs* (or *instruments of semiotic mediation*). Both are produced and used by human beings and are part of the cultural heritage of mankind. Although assumed in the same category of mediators, 'signs and tools' are clearly distinguished by Vygotskij (1978, p. 53).

> The basic analogy between sign and tools rests on the mediating function that characterises each of them. They may, therefore, from the psychological perspective, be subsumed under the same category. . . . of indirect (mediated) activity.
> (ibd. p. 54)

However, their function cannot be considered isomorphic; the difference between the two elements rests on "the different way that they orient the human behaviour". (ibd. p. 54). The *tool* function, *externally oriented*, is to serve as conductor of the human activity aimed at mastering nature. The *sign* function, *internally oriented,* is a means of internal activity aimed at mastering oneself.

The use of the term *psychological tools*, referred to signs internally oriented, is based on the analogy between tools and signs, but also on the relationship which links specific tools and their externally oriented use to their internal counterpart.

According to Vygotskij, the mastering of nature and the mastering of oneself are strictly linked, "just as man's alteration of nature alters man's own nature" (ibd., p. 55).

The *process of internalisation* as described by Vygotskij may transform tools into psychological tools: when internally oriented a 'psychological tool' will shape new meanings, thus functioning as semiotic mediator.

The example of the counting system, as discussed by Vygotskij himself (ibd., p. 127), shows the transition from external to internal orientation; a system of counting produced and employed to evaluate quantity may be internalised and function in the solution of problems, so as to organise and control mental behaviour.

An illuminating description of such a process of internalisation in the case of the compass and circle is given by Bartolini Bussi et al. (to appear). In this case, rather than through a historic reconstruction, the process is described as it was accomplished by the pupils of a primary school class, during a long-term teaching experiment.

> The geometric compass, embodied by the metal tool stored in every school-case, is no more a material object: it has become a mental object, whose use may be substituted or evoked by a body gesture (rotating hands and arms).
>
> (Bartolini Bussi et al., to appear)

5.2. *The process of semiotic mediation in the Cabri environment*

A microworld like Cabri is a particular case of cultural artefact, characterised by the presence of:

- an object constructed with the aim of achieving a result (drawing images on the screen), and for this reason it was designed to incorporate a certain knowledge, i.e. Euclidean Geometry;
- utilisation schemes, namely the modalities of action accomplished by using the artefact according to a specific goal (drawing a screen image which will pass the dragging test).

The dragging test, externally oriented at first, is aimed at testing perceptually the correctness of the drawing; as soon as it becomes part of interpersonal activities – peer interactions, dialogues with the teachers, but especially the collective discussions – it changes its function and becomes *a sign* referring to a meaning, the meaning of the theoretical correctness of the figure.

Many examples in the literature show the potentialities of a Computational-technology artefact (computer, microworld, graphic calculator ...) in terms of meanings construction, but at the same time the instability of the processes of meanings construction, related to the use of an artefact (for a full discussion see Mariotti, to appear).

The framework of Vygotskian's theory makes it possible to overcome this 'impasse'.

The functioning of an artefact in the development of meaning can be described taking into account the process of semiotic mediation which develops at different levels:

- The pupil uses the artefact, according to certain utilisation schemes, in order to accomplish the goal assigned by the task; in so doing the artefact may function as a semiotic mediator where meaning emerges from the subject's involvement in the activity.
- The teacher uses the artefact according to specific utilisation schemes related to the educational motive. In this case, as explained in the following examples, the utilisation schemes may consist in particular communication strategies centred on the artefact.

Thus, the artefact is exploited by a double use, with respect to which it functions as semiotic mediator. On the one hand, meanings emerge from the activity – the learner uses the artefact in actions aimed at accomplishing a certain task; on the other hand, the teacher uses the artefact to direct the development of meanings that are mathematically consistent.

The artefact incorporates a mathematical knowledge accessible to the learner by its use, but meaning construction requires the guidance of the teacher, who organises and directs specific activities in which the development of meanings can be recognised and accepted mathematically.

Meanings are rooted in phenomenological experience (actions of the user and feedback from the environment, of which the artefact is a component), but their development is achieved by social construction in the classroom, under the guidance of the teacher.

The following sections will be devoted to illustrating the process of semiotic mediation accomplished through the action of the teacher using the facilities offered by the Cabri environment.

In the rich collection of data, gathered during the teaching experiment, some examples were selected. They can be considered representative of what happened in the classroom and we hope they illustrate our results clearly. The difficulties related to the selection of examples can be easily imagined, since the description of a long term process taking place in a 'living' classroom is always a hard task.

5.3. *Mediation of the 'history' command*

The first example concerns a collective discussion; the analysis, carried out on the transcript of the record, shows how within the Cabri environment the teacher can find specific tools of semiotic mediation contributing to the development of the meaning of geometrical construction.

The episode (for more details see Mariotti and Bartolini Bussi, 1998) involved one of the experimental classes of the project (9^{th} grade – 15 years old – of a scientific high school (Liceo Scientifico)); 19 out of 23 pupils in the class participated in the activity which constituted the very beginning of the experimentation.

The first part of the activity takes place in the Computer room, where the pupils sit in pairs at the machine. They have general ability with the computer, and are allowed to explore the software freely for about half an hour, after which they are presented with the following task.

Construct a segment.

Construct a square which has the segment as one of its sides.

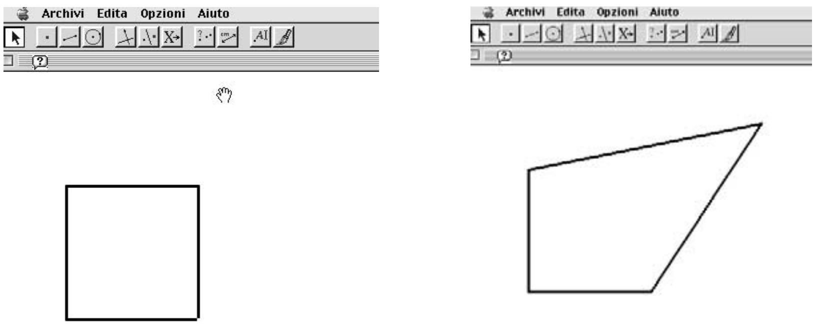


Figure 1. The construction does not pass the dragging test.

The pupils are asked to realise a figure on the screen and to write down a description both of the procedure and of their reasoning. The meaning of the term *construction* is not explained, leaving the pupils free to interpret the task. As expected, the protocols collected contain differently obtained solutions, some referring to geometrical properties, others to perceptual control. These solutions are differently transformed by using the dragging function, and such differences provide the basis for the following discussion.

The first solution proposed by the teacher is that given by Group1 (Giovanni and Fabio): four consecutive segments, perceptually arranged in a square.

When the teacher asks the pupils to judge this solution, everybody agrees that control must be exerted on the particular drawing; according to the well known definition of square, they suggest to measure the sides and the angles. The main elements arising from the discussion are the use of measure and the precision related to it. The pupils are stuck: all interventions show that the shared objective concerns the control of *the drawn square*.

When the teacher drags the figure (Figure 1) everybody agrees that it is no longer a square. At this point, another solution (Group 3, Dario & Mario) is proposed by the teacher.

‘T’ (= Teacher)

21 T: Well, I’d like to know your opinion about the construction made by Dario and Mario

22 Marco: They drew a circle, then two perpendicular lines. . .

23 T: Do you know where they started from?

24 Michele: We can use the command ‘history’.

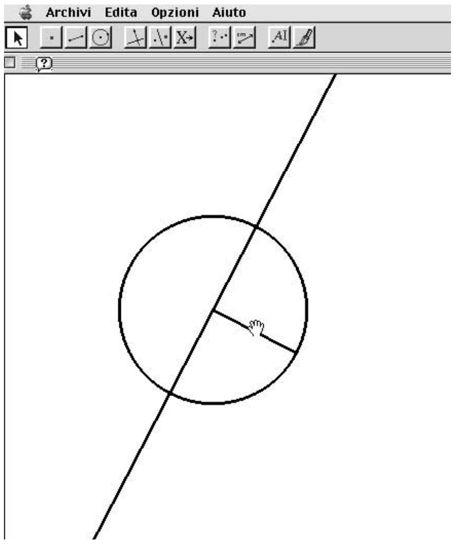


Figure 2. Group 3 Dario & Mario: The first steps of the construction.

25 T: Let’s do that. They took a segment, then they . . .
[*The step by step construction follows (Figure 2)*]
They drew a line perpendicular to the segment, then the circle . . . in your opinion, what is it for ? What is its use?

SILENCE

Is there a logic in doing so, or did they do it just because they felt like drawing a perpendicular line . . . a circle . . . Alex, what do you think? . . .

26 Alex: the measure of the segment is equal to the measure reported by the circle on the perpendicular line.

27 T: You mean that the circle is used to assure two equal consecutive segments, the first one and the other one on the perpendicular line . . . and the perpendicular . . .

28 Chorus: it is used . . . to obtain . . . an angle of 90°

29 T: I know that the square has an angle of 90° and four equal sides or three equal angles
. . . then let’s see if that is true . . . let’s go on.
Intersection between line and circle. They (Dario e Mario) determined the intersection point between the line and the circle . . . why did they need that point?

30 Chiara: the intersection point between the line and the segment . . .

31 T: and what should you draw from there ?

32 Chiara: a segment, perpendicular to the line

33 T: what else??

34 Chorus: parallel to the segment. . .

35 T: let's see what they did . . .

Let us analyse this episode.
A first suggestion comes from Michele: use the History command. The teacher catches the suggestion and activates the command. While the first steps of the construction are repeated, the teacher describes what has been done, until she interrupts the description and asks the pupils to detect the 'motivations' for these actions.

This intervention (which triggers what we call *interpretation game*) is aimed at provoking the first *shift from the procedure to a justification of the procedure itself.*

The move *from action to explaining the motivation of this action* is difficult and the pupils are pressed (25), until Alex (26) expresses the relationship between two of the segments according to the series of commands previously executed. The teacher (27) reformulates her statement in terms of motivations: 'You mean that the circle is *used to* assure two equal consecutive segments . . .'. The Chorus appropriates the teacher's expression '*used*. . .' and continues in terms of motivation.

The discussion goes on: a new game is activated (which we have called the *prediction game*): the pupils are now asked to foresee the next step, to motivate and then compare it with the step recorded in the history.

The discussion goes on, different solutions are compared, and finally together with the acceptance of a solution in terms of the dragging test, a new relationship to drawing is achieved. It becomes possible to explain the correctness of a construction controlling the 'logic' of the procedure.

As clearly shown by the previous analysis, the presence and availability in the computer of a decontextualised and detemporalised copy of the construction procedure allows the teacher to realise specific communication strategies (the interpretation game and the prediction game) consistent with the educational motive. Without explicit comments or implicit information (gestures, and so on) towards the expected answer, the pupils' attention moves from the drawing produced to the construction procedure, and at the same time the idea of a justification for the single steps of the construction is introduced.

It is interesting to remark that the History command remains a basic element in the process of theorems production. The sequence of construction steps may represent the temporal counterpart of the logic hierarchy between the properties of a figure. In the dynamics of the Cabri figure the relationships between the geometric properties are expressed globally, so that some relationships of logic dependence may be hidden and missed by the pupils. The History command reintroduces the temporal dimension and allows one to grasp the construction process in its development; in doing so, it supports the control of the logic relationships between the properties involved.

5.4. *Using commands and axioms*

The idea of proof involves two kinds of strictly interconnected difficulties. It is necessary to introduce on the one hand the general idea of justification, on the other the idea of a justification with regard to specific principles and accepted rules of inference, i.e. in order to be a proof, any justification must be referred to a theory.

As already discussed, the relationship between Cabri constructions and geometrical theorems is based on the link between the logic of Cabri, expressed by its commands, and the Euclidean Geometry expressed by its axioms, theorems and definitions. When the whole Cabri menu is available, the relationships between the main concepts and properties, available through the geometrical primitives of Cabri, remain largely hidden. As a consequence, the complexity of the corresponding geometric system may be too difficult to be managed by novices. In fact, because of the richness of the geometrical properties available, it is difficult to state what is given (axioms and 'old' theorems) and what must be proved ('new' theorems). Generally speaking, the richness of the environment might emphasise the ambiguity about intuitive facts and theorems, so that it can even represent an obstacle to grasping the meaning of proof.

Thus, taking advantage of the flexibility of the Cabri environment, instead of providing the pupils with an already-made Cabri menu, corresponding to an already stated axiomatization, we decided to construct our Cabri menu 'step by step', in parallel with the corresponding theory.

At the beginning an empty menu was presented and the choice of commands discussed, according to specific statements selected as axioms. Then, as new constructions were introduced and the corresponding new theorems enlarged the theory, new commands were added to the menu. In this way the geometry system was slowly built up with a twofold aim. The pupils participated (Leont'ev, 1976/1959) in the construction of both an axiomatization and a corresponding menu; at the same time, the complexity of the theoretical system increased at a rate which the pupils were able to manage.

It is impossible to go into the details of the protocol analysis. Nonetheless, the following example illustrates how the use of the commands may support the construction of the theoretical meaning of a justification. The task proposed to the pupils was the following:

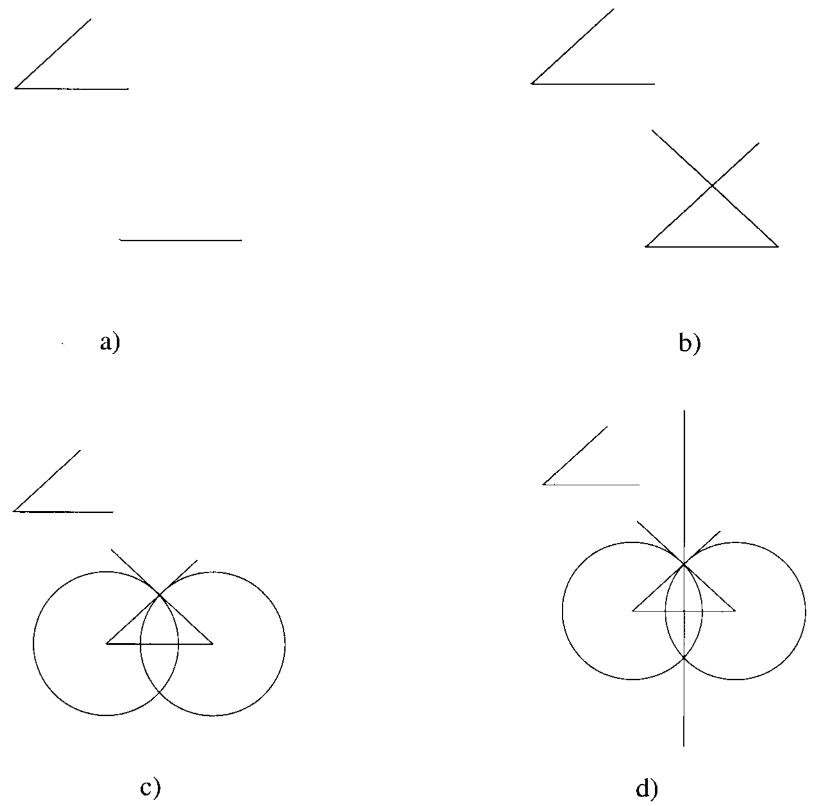


Figure 3. The construction of the mid-point as realised by G. & C.

Construct the mid point of a segment.

G. and C. (9th grade – 15 years old)

I create a segment through two points. I fix three other points on the screen and construct an angle with them. With (the command) 'report of an angle', I carry this angle on the edges of the segment and create the intersection of these two rays. Using (the command) 'circle (centre, point)', with the centre on the edges of the segment and point on the intersection of the rays, I create two equal circles. Joining the two intersections I find the mid-point O.

I did that because creating the equal angles on this segment an isosceles triangle is created. Using the equal sides of this triangle as radios of two new circles, I can construct two equal circles on the edges of the segment.

The protocol does not contain any drawing, and the pupils probably refer to the screen image. In the figure (Figure 3) the construction has been reproduced step by step to illustrate the procedure described by the pupils.

This protocol shows a good example of a first step in the development of the meaning of theoretical justification. The description of the construction and its justification are still mixed. This shows the difficulty of separating the operational aspect – realising the Cabri-figure – and the theoretical aspect which consists in identifying the geometric relationships drawn from the figure according to its construction. At the same time, however, the relationship between the use of the command and its theoretical meaning appears clearly: the protocol illustrates the process of internalisation through which the external signs – the Cabri commands – are transformed into internal tools related to theoretical control.[3] According to a classic axiomatization (Hilbert, 1899/1971; Heath, 1956, p. 229), the particular command 'report of an angle' corresponds to one of the axioms introduced in the theory.

Similar examples (see for instance the following protocol) can be found and provide evidence of the fact that the problem of construction has achieved a theoretical meaning. Similarly, examples can be found showing that the Cabri commands may function as external signs of the theoretical control, corresponding to using axioms, theorems and definitions. Let us now consider the following example:

> *Construct the line perpendicular to line t and passing through its relative point P, belonging to it.*

Sa and Si (9^{th} grade – 15 years old)

First of all we go to 'Creation' and we construct line t; in 'Creation' we also find point P and we constrain it on the line. In the same way we fix two other points A and B, so that P is located between them and we constrain them on the line. Then, in 'Construction', we choose the label 'angle bisector' of angle APB, which is also perpendicular to line t passing through P.

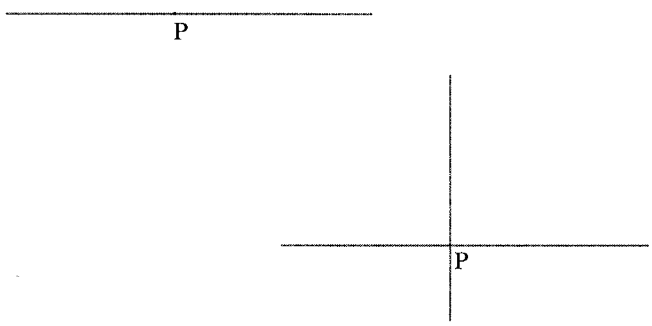


Figure 4. The drawings provided by Sa & Si.

Justification

According to the definition of perpendicular line which says that a line t is perpendicular to a line s if it is the angle bisector of a straight angle which has the vertex in a point of s, I can prove that the angle bisector of the straight angle APB is perpendicular to line t.

The explicit reference to the definition[4] of 'perpendicular line' allows the pupils to validate the correctness of the construction, based on the use of the command 'angle bisector'.

The fact that the commands available in the current menu are recognised as theoretical properties available in the theory, makes the construction procedure itself an external sign of a theorem of the theory. The command, expressed by a 'label – word', may function as external control, and refers to the possibility of making that theorem explicit. In the case of the example the command 'angle bisector' refers to a construction, previously realised, and corresponds to a theorem included in the Theory available.

According to Vygotskij, the process of internalisation of such signs determines the construction of the theoretical meaning of the construction problems and opens up to the theoretical perspective for geometrical problems in general.

In conclusion, according to our main hypothesis, pupils' introduction to contruction problems within the Cabri enviroment provided a key to accessing a theoretical perspective. The main point was the interpretation of the constraints of the environment in terms of geometrical properties and in terms of the mutual dependence between geometrical properties.

As the examples should have shown, that interpretation resulted from a lengthy process, carefully guided by the teacher, through specific activities among which collective discussions took a fundamental part. The following section is devoted to a micro analysis of the process of semiotic medation, centred on the use of the artefact Cabri.

6. INTERNALISATION OF DRAGGING AS THEORETICAL CONTROL

Interesting aspects related to the development of the meaning of mathematical theorem can be highlighted in the solution of open-ended problems, i.e. the particular type of task asking for a conjecture and its proof. In fact, exploration in the Cabri environment with the aim of producing a conjecture is based on the awareness of the theoretical interpretation of a cabri-figure. The data of the problem can be realized by constructing a Cabri-figure, so that the hierarchy of the construction realises the geometric relationship given by hypothesis; thus the interpretation of perceptive invariants in terms of geometric conjecture is based on the interpretation of the dragging control in terms of logic control. In other terms the use of Cabri in the generation of a conjecture is based on the internalisation of the dragging function as a logic control, which is able to transform perceptual data into a conditional relationship between hypothesis and thesis.

> . . . the changes in the solving process brought by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning.
>
> (Laborde and Laborde, 1991, p. 185)

Consciousness of the fact that the dragging process may reveal the relationship between the geometric properties embedded in the Cabri figure directs the ways of transforming and observing the screen image.

As a consequence, evidence of the process of internalisation of the theoretical control can be shown by the ways in which pupils construct and transform the image on the screen, when they are solving open-ended problems:

- the figure to be explored is constructed considering that the properties provided in the hypotheses are realised by the corresponding Cabri commands;
- the conjectures may emerge from exploration by dragging, but their validation is sought within the Geometric theory, i.e. in principle, conjectures ask for a proof.

The following example aims at analysing this process of internalisation.

6.1. *A particular case: the rectangle problem*

Let us consider the following activity, proposed to a class (9th grade – 15 years old), during a session in the computer lab. The pupils sit in pairs at the computer and are asked to perform the following task:

> *Draw a parallelogram, make one of its angles right and write your observations.*

Almost all the pupils correctly (although differently) construct a parallelogram and describe the construction. In some cases, the drawing, provided by the pupils in the report, reproduces the screen image; in many cases, in order to reproduce the complexity of the Cabri figure – an image controlled by the properties /commands – the drawing is completed with labels referring to the construction accomplished (for instance: 'line by two points', 'parallel line'). These elements testify to the intention of realising a Cabri figure incorporating the hypotheses related to the parallelogram (see Figure 5 below.).

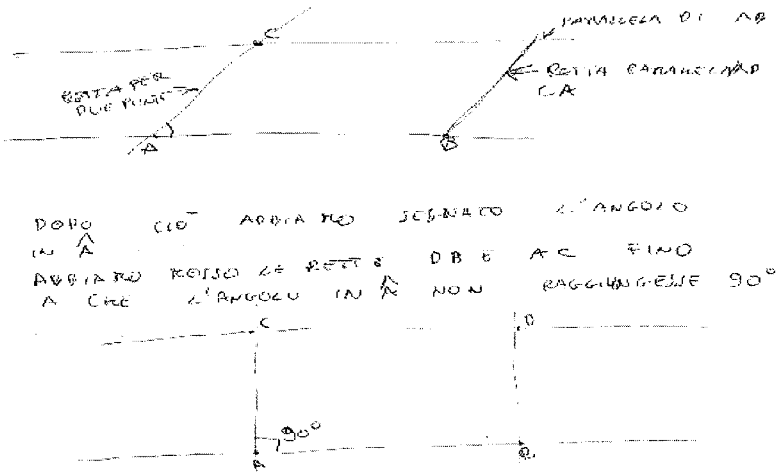


Figure 5. Use of command 'mark and angle' – Giovanni wrote: "After that, we marked angle A. We moved the lines DB and AC until the angle A reaches 90°".

Following the instructions given by the task, the pupils realise the new hypothesis ('*make an angle right*') and transform the figure by dragging, making an adjustment of the perpendicularity of the sides 'by eye'. The conjecture comes straightforward.

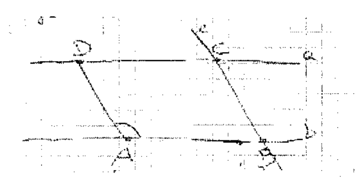
Because of the immediacy of the conclusion, the main difficulty concerns the production of the proof rather than the formulation of the conjecture (all conjectures were correct but there were some errors in the proofs).

The strategies followed in this part of the solution reveal the presence of particular 'signs' used in constructing the proof. Such 'signs' can be interpreted as means of external control (Bartolini Bussi et al., to appear) on the logic operations required to produce the proof.

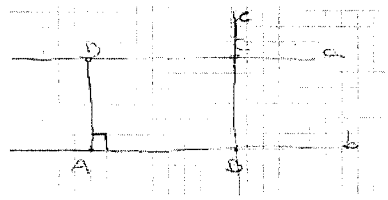
The signs are generated within the software environment and derive their semantics from that of the software, i.e. from the system of meanings which have so far emerged in the practice. Let us now analyse two of the observable signs.

6.1.1. *'Mark an angle' and 'perpendicular line'*

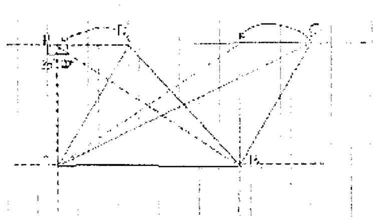
Before dragging the figure, some of the pupils use the command 'mark an angle'. This command, without which it is impossible to obtain the measure of an angle, is used as a means to control the hypothesis of 'one (only one) right angle', under which the exploration must be carried out and the conjecture formulated.



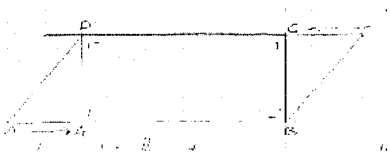
Michela & Elena 1st step



Michela & Elena 2nd step



Matteo & Alessio



Nicola

Figure 6. Examples of different signs supporting the identification of the hypothesis.

In those cases, the drawings in the reports reproduce a sequence of snapshots (photograms), showing the different phases of the solution and differently marked angles (Figure 5).

It is interesting to remark that the software distinguishes between the sign for ‘mark an angle’ and the sign for ‘right angle’. The same distinction is reproduced on the drawing.

Other pupils use a different type of ‘sign’ as follows: they construct a line passing through one of the vertices and ‘perpendicular’ to one of the sides of the parallelogram. Dragging one of the free vertices, one of the oblique sides is made to coincide with ‘the perpendicular line’. In their reports, the pupils try to express the dynamics of the Cabri figure; for instance pupils draw arrows connecting the vertex and the moving side to the ‘perpendicular line’ (Figure 6, Matteo & Alessio; Nicola).

Awareness of the fact that properties obtained by an adjustment of the figure ‘by eye’ do not grant the validity of the derived properties, leads pupils to look for a control, and they find it in an element obtained ‘by construction’, which for this very reason is reliable.

Strategy ‘by eye’ and strategy ‘by construction’ have definitely entered the practice and pupils use specific elements of the software (external signs) with the aim of keeping control of the two different meanings.

The previous example focusses on a basic aspect of the process of exploration in the Cabri environment: the need to keep control of the figure in terms of given properties (hypothesis) and derived properties (thesis). The phenomena observable on the screen often hide the asymmetry of the conditional relationship between properties, while in the dragging all the properties hold at the same time.

Keeping control of the conditional by dragging is possible, but can be very difficult: it requires that one

- relates the basic points (the only variable elements by dragging) and the observed properties, and
- expresses such a relationship into a statement, connecting hypothesis and thesis.

It is interesting to remark that the pupils feel the need to control the given properties on the figure and to look for a support. The semantics drawn from the interaction with the software allows the pupils to generate external signs to support the theoretical control which is not completely internalised.

7. CONCLUSIONS

This study, which is part of a collective research project on approaching theoretical thinking at different age levels, shows interesting similarities with other experimental results (Mariotti et al., 1997; Bartolini Bussi et al., to appear).

In spite of the differences, it is interesting to remark the common features characterising and explaining the process of introduction to theoretical thinking as it is accomplished in different fields of experience.

Cultural artefacts, either microworlds or mechanical devices, may offer similar support in the construction of meanings based on social interaction.

The field of experience of geometrical constructions in the Cabri environment provides a context in which the development of the meaning of Geometry theorem may be achieved.

The basic modification in which we are interested concerns the change of the status of justification in geometrical problems. This modification is strictly related to the passage from an ‘intuitive’ geometry as a collection of evident properties to a ‘theoretical’ geometry, as a system of relations among statements, validated by proof. According to our basic hypothesis, the relation to geometrical knowledge is modified by the mediation offered by the peculiar features of the software.

Mediation is a very common term in the literature concerning the use of computers in education. The term is not often explicitly defined, and simply refers to the vague potentiality of fostering the relation between pupils and mathematical knowledge. A few authors directly discuss the idea of mediation and, among others, Hoyles and Noss do this in relation to computers (Hoyles and Noss, 1996). The mediation function of the computer is related to the possibility of creating a channel of communication between the teacher and the pupil based on a shared language (ibd. p.6). The potentialities of the software environments are related to the construction of mathematical meanings that can be expressed by pupils through interaction with the computer.

Our analysis, which is based on the Vygotskian perspective and in particular on the notion of semiotic mediation, has tried to outline and explain the development of the meaning of proof.

The particular microworld offers a rich environment where different elements may be used by the teacher as tools of semiotic mediation.

The basic element is the dragging function; at the very beginning dragging is an externally oriented tool, that introduces a perceptual test controlling the correctness of the solution to the construction task. As it becomes part of interpersonal activities – mainly mathematical discussions – dragging changes its function and becomes a sign of the theoretical correctness of the figure.

More generally, the Cabri environment offers a complex system of signs supporting the process of semiotic mediation, as it can be realised in the social activities of the class.

The fact that a command is activated by acting on a label, identified by the 'name' of a geometrical property, determines the use of a sign functioning as control and as organiser of actions related to the task, i.e. construction procedures. Moreover, the fact that the commands available may be recognised as theoretical properties, corresponding to axioms or theorems of a theory, makes the construction procedure itself an external sign of a theorem. On the other hand the external sign (the word/command or the construction) may function as internal control, as long as it refers to the possibility/necessity of explaining a theorem (its statement and its proof).

The process of internalisation of such signs determines the construction of the theoretical meaning of a construction problem and opens the theoretical perspective for geometrical problems in general. The process of internalisation transforms the commands available in the Cabri menu – external signs – into internal psychological tools which control, organise and direct pupils' geometrical thinking, in producing both conjectures and proofs.

It is interesting to evidence the difficulty in the appropriation of the theoretical control related to the dragging function. The process of internalisation of the external control can be a slow process which must be supported by the mediation of new external signs (offered by the software), that pupils autonomously create and use with that specific pupose.

The evolution of the pupils' internal context is rooted in the construction activities, where different processes such as conjecturing, arguing, proving, systematizing proofs as formal deduction are given sense and value. Our experiments also show that the contradiction highlighted by Duval between everyday argumentation and deductive reasoning, between empirical and geometrical knowledge, can be managed in dialectic terms within the evolution of classroom culture.

In our experiment classroom culture is strongly determined by the recourse to mathematical discussion orchestrated by the teacher to change the spontaneous attitude of students towards theoretical validation.

The introduction of computer environments in school mathematical practice has been debated for years, in particular the potentialities of Cabri, and dynamic geometry in general, have been described and discussed. With this experimental research study we hope to offer a contribution in illustrating the cognitive counterpart of classroom activities with Cabri that allows an approach to theoretical thinking.

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NOTES

1. The following remark concerns types of software which share with Cabri the general feature of a 'drag mode'; I mean for instance Sketchpad or Geometric Supposer.
2. In this section I will employ the term 'tools' as used in the current English translations of Vygotskij's works. In current literature the terminology is rather confused, as different authors use the same words with different meanings.
3. This unusual construction, quite different from those which can be found in the textbooks, seems to provide a strong support to our interpretation.
4. After the construction of the angle bisector and the inclusion of the corresponding command in the available menu, the definition of perpendicularity was introduced.

REFERENCES

Balacheff, N.: 1987, 'Processus de preuve et situations de validation', *Ed. Stud. in Math.* 18, 147–176.

Balacheff, N.: 1998, 'Construction of meaning and teacher control of learning', in D. Tinsley and D.C. Johnson (eds.), *Information and Communications Technologies in School Mathematics*, Chapman & Hall, IFIP, London, pp. 111–120.

Balacheff, N. and Sutherland, R.: 1994, 'Epistemological domain of validity of microworlds, the case of Logo and Cabri-géomètre', in R. Lewis and P. Mendelson (eds.), *Proceedings of the IFIP TC3/WG3.3: Lessons from Learning*, Amsterdam, North-Holland, pp. 137–150.

Bartolini Bussi, M.G.: 1991, 'Social interaction and mathematical knowledge', in F. Furenghetti (ed.), *Proc. of the 15th PME Int. Conf.*, 1, Assisi, Italia, pp. 1–16.

Bartolini Bussi, M.G.: 1994, 'Theoretical and empirical approaches to classroom interaction', in Biehler, Scholz, Strässer and Winckelmann (eds.), *Didactic of Mathematics as a Scientific Discipline*, Kluwer, Dordrecht, pp. 121–132.

Bartolini Bussi, M.G.: 1996, 'Mathematical discussion and perspective drawing in primary school', *Educational Studies in Mathematics* 31(1-2), 11–41.

Bartolini Bussi, M.G.: 1999, 'Verbal interaction in mathematics classroom: a Vygotskian analysis', in A. Sierpinska et al. (eds.), *Language and Communication in Mathematics Classroom*, NCTM.

Bartolini Bussi, M.G. and Mariotti, M.A.: 1999, 'Instruments for perspective drawing: Historic, epistemological and didactic issues', in G. Goldschmidt, W. Porter and M. Ozkar (eds.), *Proc. of the 4th Int. Design Thinking Res. Symp. on Design Representation*, Massachusetts Institute of Technology & Technion – Israel Institute of Technology, III, pp. 175–185,

Bartolini Bussi, M.G. and Mariotti, M.A.: 1999, 'Semiotic mediation: from history to mathematics classroom', *For the Learning of Mathematics* 19(2), 27–35.

Bartolini Bussi, M.G., Boni, M., Ferri, F. and Garuti, R.: 1999, 'Early approach to theoretical thinking: Gears in primary school', *Educational Studies in Mathematics* 39(1–3), 67–87.

Bartolini Bussi, M.G., Boni, M. and Ferri, F. (to appear), 'Construction problems in primary school: A case from the geometry of circle', in P. Boero (ed.), *Theorems in School: From history and epistemology to cognitive and educational issues,* Kluwer Academic Publishers.

Baulac, Y., Bellemain, F. and Laborde, J.M.: 1988, *Cabri-géomètre, un logiciel d'aide à l'enseignement de la géométrie, logiciel et manuel d'utilisation*, Cedic-Nathan, Paris.

Boero, P., Dapueto, C., Ferrari, P., Ferrero, E., Garuti, R., Lemut, E., Parenti, L., Scali, E.: 1995, 'Aspects of the mathematics-culture relationship in mathematics teaching-learning in compulsory school', *Proc. of PME-XIX*, Recife.

De Villiers, M.: 1990, 'The role and function of proof in mathematics', *Pythagoras* 24, 17–24.

Duval, R.: 1992–93, 'Argumenter, demontrer, expliquer: continuité ou rupture cognitive?', *Petit x* 31, 37–61.

Fischbein, E.: 1987, *Intuition in Science and Mathematics*, Reidel, Dordrecht.

Halmos, P.: 1980, 'The heart of mathematics', *Mathematical monthly* 87, 519–524.

Hanna, G.: 1990, 'Some pedagogical aspects of proof', *Interchange* 21, 6–13.

Heath, T.: 1956, *The Thirteen Books of Euclid's Elements*, Dover, vol. 1.

Henry, P.: 1993, 'Mathematical machines', in H. Hanken, A. Karlqvist and U. Svedin (eds.), *The Machine as Metaphor and Tool*, Springer-Verlag, pp. 101–122.

Hilbert, D.: 1971, *Foundation of Geometry*, Open Court Classic (Original title *Grundlagen der Geometrie,* first published 1899).

Hoelz, R.: 1996, 'How does 'dragging' affect the learning of geometry', *International Journal of Computer for Mathematical Learning* 1.2, 169–187.

Hoyles, C. and Noss, R.: 1996, *Windows on Mathematical Meanings*, Kluwer Academic Press.

Hoyles, C.: 1993, 'Microworlds/schoolworlds: The transformation of an innovation', in C. Keitel and K. Ruthven (eds.), *Learning from Computers: Mathematics Education and Technology*, NATO ASI Series, Springer-Verlag.

Keitel, C. and Ruthven, K. (eds.),: *Learning from Computers: Mathematics Education and Technology*, Springer-Verlag.

Goldengerg, E.P. and Cuoco, A.: 1998, 'What is dynamic geometry?', in Lehrer and Chazan (eds.), *Designing Learning Environments for Understanding of Geometry and Space*, Erlbaum.

Laborde, C. and Capponi, Bs.: 1994, 'Cabri-géomètre constituant d'un mileu pour l'apprentissage de la notion de figure géométrique', *Recherches en Didactiques des Mathematiques* 14 (1.2), 165–210.

Laborde, C.: 1992, 'Solving problems in computer based geometry environment: The influence of the features of the software', *Zentralblatt für Didaktik der Mathematik* 92(4), 128–135.

Laborde, C.: 1993, 'The computer as part of the learning environment: the case of geometry', in C. Keithel and K. Ruthven (eds.), *Learning from Computers: Mathematics Education and Technology*, NATO ASI Series, Springer Verlag.

Laborde, J.M. and Strässer, R.: 1990, 'Cabri-géomètre: A microworld of geometry for guided discovery learning', *Zentralblatt für Didaktik der Mathematik* 90(5), 171–177.

Laborde, C. and Laborde, J.M.: 1991, 'Problem solving in geometry: From microworlds to intelligent computer environments', in Ponte et al. (eds.), *Mathematical Problem Solving and New Information Technologies*, NATO ASI Series F, 89, pp. 177–192.

Leont'ev, A.N.: 1976/1959, *Problemi dello sviluppo psichico*, Ed. Riuniti & Ed. Mir.

Mariotti, M.A.: 1998, 'Introduzione alla dimostrazione all'inizio della scuola secondaria superiore', *L'insegnamento della matematica e delle Scienze Integrate* 21B(3), 209–52.

Mariotti, M.A.: 1996, 'Costruzioni in geometria', su *L'insegnamento della Matematica e delle Scienze Integrate* 19B(3), 261–288.

Mariotti, M.A: 1997, 'Justifying and proving: figural and conceptual aspects', in M. Hejny and J. Novotna (eds.), *Proceedings of the European Research Conference on Mathematical Education*, Podebrady, Czech Republic.

Mariotti, M.A., Bartolini Bussi, M., Boero P., Ferri, F. and Garuti, R.: 1997, 'Approaching geometry theorems in contexts: from history and epistemology to cognition', *Proceedings of the 21th PME Conference,* Lathi, I, pp. 180–195.

Mariotti, M.A. and Bartolini Bussi, M.G.: 1998, 'From drawing to construction: teachers mediation within the Cabri environment', in *Proceedings of the 22nd PME Conference,* Stellenbosh, I, pp. 180–195.

Mariotti, M.A. (to appear), 'Influence of technologies advances on students' math learning', in L. English, M.G. Bartolini Bussi, G. Jones, R. Lesh and D. Tirosh (eds.), *Handbook of International Research in Mathematics Education,* Lawrence Erbaum Associates.

Noss, R. and Hoyles, C.: 1996, *Windows on Mathematical Meanings*, Kluwer Academic Publishers.
Simondon, G.: 1968, *L'invention et le développement des techniques*, Sorbonne, Paris, cours polycopié.
Schoenfeld, A.: 1992, 'Learning to think mathematically: problem solving, metacognition, and sense making in mathematics', in D.A. Grouws (ed.), *Handbook of research on Mathematics teaching and learning*, NCTM, pp. 334–366.
Straesser, R.: 1991, *Dessin et figure*, IDM, Universitaet Bielefeld, Bielefeld, (Occasional Paper 128).
Straesser, R.: 1992, *Students' Construction of Proof in a Computer Environment*, IDM, Universitaet Bielefeld, Bielefeld, (Occasional Paper 134).
Vygotskij, L.S.: 1978, *Mind in Society. The Development of Higher Psychological Processes*, Harvard University Press.

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