RICHARD NOSS

CONSTRUCTING A CONCEPTUAL FRAMEWORK FOR ELEMENTARY ALGEBRA THROUGH LOGO PROGRAMMING

ABSTRACT. This paper examines the extent to which Logo-experienced children are able to mobilise their Logo-based knowledge to construct meaning for elementary algebraic concepts. It reports the results of an exploratory study which was part of a longitudinal investigation of the mathematical environment created through Logo programming. The study aimed at gauging the influence of children's Logo learning in facilitating their conceptualisation of algebraic variable, and their ability to formalise in a non-computational context.

The evidence which has accumulated from a number of recent longitudinal studies has suggested that children learning to program a computer with the Logo language, have participated in an environment which is rich in mathematical activity (Papert *et al.*, 1979; Noss, 1983, 1984; Hoyles *et al.*, 1985). Such evidence is implicitly acknowledged by a recent report of the UK Schools Inspectorate (DES, 1985), which advises that if microcomputers are to be granted a role as "a powerful means of doing mathematics", children will need to program the machines, and that "if programming is not taught elsewhere, it should be included in mathematics lessons" (p. 35). In this paper, my starting point is that the symbolic representation of mathematical concepts in the form of computer programs, engages the learner in the *doing* of mathematics. The question is, what else may she be learning?

Papert (1972) has suggested that by learning to program in Logo, children may develop a 'Mathematical Way of Thinking' which can serve as a foundation for learning traditional mathematical content such as algebra and geometry. More daringly, and in contrast to traditional questions of transferability of taught skills, he has enquired whether it is possible to use algebra itself (embodied in a computer programming language) as a vehicle for teaching transferable concepts and skills. His conjecture is that programming in Logo offers the student an environment for mathematising, which may serve as an introduction to more formal mathematical structures:

> When mathematizing familiar processes is a fluent, natural and enjoyable activity, then is the time to talk about mathematizing mathematical structures, as in a good pure course on modern algebra. (Papert, 1972, p. 260).

In learning Logo, the student is not simply solving problems; she is solving problems in a mathematical domain. The objects and processes available to her for the construction of programs are themselves mathematical (itera-

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tion, recursion, variable, state, coordinate system etc.). The question is whether, in learning to program in Logo, the child may develop a bridge between the pseudo-concrete mathematical world of a computer screen, and the abstract world of mathematics. This paper will be concerned with only one aspect of mathematical activity, namely elementary algebra. It will report the results of an exploratory study aimed at investigating the extent to which learning Logo provided a small group of 10-year old children with a 'conceptual framework' (Feurzeig, Papert *et al.*, 1969) for the learning of algebra, and in particular, a basis for the understanding of the concept of variable.

RELATED RESEARCH

If algebraic abstraction, in the sense of symbolic representations of relationships, is central to mathematics, then the twin ideas of function and variable are central to algebra (Freudenthal, 1982). Yet it is precisely these ideas which children find such a major stumbling block in their learning (Tonnessen, 1980; Kuchemann, 1981; Wagner and Rachlin, 1981; Jensen and Wagner, 1982; Booth, 1984). These difficulties arise despite the role that natural language variables play in human cognitive processing (Davis *et al.*, 1978), although there are, as Wagner (1979) points out, important differences between natural language and mathematical usages.

The idea of function and variable are central to computing as they are to mathematics. Consider the Logo instruction FORWARD 50. The function FORWARD takes one argument (or 'input' in Logo jargon), which in this case is a number. Constructing a procedure which includes the command FORWARD :LENGTH on the other hand, utilises the possibility of turning the input into a variable. FORWARD :LENGTH will not be executed until the variable LENGTH has been given a value, as for example in the instruction:

MAKE "LENGTH 40.

Note that there is a clear distinction, not present in some other commonly used programming languages, between the name (or label) of the variable "LENGTH, and the *value* assigned to "LENGTH, :LENGTH.

Logo allows the composition of functions in a way which models mathematical usage quite closely. For example, the instruction FORWARD HALF HALF :LENGTH will send the turtle forward 10 units, provided that HALF has been defined appropriately e.g.:

```
TO HALF :AMOUNT
OUTPUT :AMOUNT / 2
END
```

Logo thus provides a model of the ideas of function and variable which is reasonably consistent with mathematical usage. However, there is no *a priori* reason why such structural features of the language should be enough to develop relevant conceptual frameworks. DiSessa (1982) has suggested that while, for example, FORWARD is structurally a function in the mathematical sense, it seems likely that young children are much more likely to interpret it as essentially an abbreviation for the English sentence "Go forward 100 steps". Samurcay (1985) points out that the algebraic and computational concepts of variable differ in subtle but important respects. Leron (1983), and Hillel and Samurcay (1985) have indicated that the notion of variable within a Logo context is certainly far from straightforward for many children.

The construction of computer programs raises a number of more general issues related to algebraic abstraction. Consider for example, the lack of closure (Collis, 1975) inherent in the Logo expression

```
FORWARD 5+10
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or even

```
FORWARD :LENGTH+5
```

A related issue is the interpretation of more than one variable such as occurs in, for example, the procedure:

```
TO POLYSPI :LENGTH :ANGLE :DELTA
FORWARD :LENGTH
RIGHT :ANGLE
POLYSPI (:LENGTH+:DELTA) :ANGLE :DELTA
END
```

Lawler (1980) has shown that programs such as this can provide, even for young children, a powerful environment in which to explore the interaction between several variables which, in developmental terms, is perhaps surprising (Lunzer, 1973; Collis, 1974; Halford, 1978; Karplus *et al.*, 1982). Such findings (see also Lawler, 1985, for a stronger developmental claim), seem to depend on the provision of a context in which the construction of children's own formalism becomes a necessary (and natural) component of the environment itself (Papert, 1975), rather than a mere agreement to play the game of mathematical rigour according to the whim of the teacher.

This issue relates the need for formal syntactic precision in a programming context, to the idea of algebra as a formal/symbolic system of generalised number. Booth (1984) reports that children had difficulty in formalising mathematical methods, even in the case of arithmetic. The relationship between formal and intuitive methods has been investigated by other workers in relation to algebraic notions (Petitto, 1979; Kieran, 1981; Clement, 1982; Rachlin, 1982).

Alongside this formal/logical component of mathematical rigour, programming involves a symbolic component. Adda (1982) has vividly illustrated how the *naming* of concepts (in the form of Logo procedures), can provide a way to overcome ambiguities of mathematical symbolism. She points out that the concept of variable is rich in potential confusion, with, for example, a letter standing for a parameter, a variable, or a specific unknown. Much the same point is made by Feurzeig and Papert who refer to the 'many roles of the "*x*" in algebra' (Feurzeig and Papert, 1969, p. 7). Even children who can grasp the notion of letters as numbers, often have difficulty in viewing a letter as standing for a range of numbers (Booth, 1984). A related problem is the tendency noted by Kuchemann (1981) to view a letter as representing an object rather than a number.

Such confusion is not of course restricted to the interpretation of the variable itself. For example, Kieran (1980) has illustrated the misconceptions which can arise over the equals sign, a misconception which seems to originate, at least in part, from the limitations of teaching strategy (Herscovics and Kieran, 1980). With respect to teaching strategies involving programming, it seems reasonable to avoid programming languages which employ symbolic representations which only add to the confusion such as BASIC's 'LET X=X+1'. More generally, it appears likely that the embodiment of mathematical relationships in the form of computer programs encourages a more dynamic view of the processes involved, and thus contributes to a decrease in some of the more common misconceptions (Soloway *et al.*, 1982; Ehrlich *et al.*, 1982).

The question of children's conception of algebra has been studied extensively by Booth (1984). She has suggested that the construction of formalised procedures constitutes a significant part of mathematical activity, and proposes that further investigation on this issue is merited. Booth's work did not involve the use of computers; the rationale for the formalisation was provided by a notional machine, complete with input pad, start button, processor, store locations and output pad. She claims that the advantages of such an approach are based on the need for explicit and precise representation of procedures, the scope for the provision of indeterminate answers, and the provision of a rationale for the use of letters as a means of instructing the machine.

The present study, while sharing much of Booth's rationale, differs in one major respect. The children in the study were actively engaged in the construction of computer programs. It thus becomes meaningful to investigate the possible implications for children's algebraic formalisation, of their participation in the processes of constructing formalised algorithmic (procedural) descriptions.

BACKGROUND TO THE STUDY

The study reported in this paper (referred to as the 'algebra study') was undertaken as part of a longitudinal investigation into the creation of a mathematical environment through Logo programming. The longitudinal study was concerned with identifying the mathematical/programming activities undertaken by children as they learned Logo, and in illuminating the relationship between computational and mathematical concepts and processes.

During the first year of the eighteen-month study, 118 children participated, aged between 8 and 11. They were distributed among five classrooms, one in each of five schools (one grade 3, one grade 4, and three grade 5). Each class spanned the full ability range within the school, and the schools were chosen to obtain a spread with respect to socio-economic class of the students and geographic location. During the last six months, the study focussed on a single class of children aged between 10 and 11; detailed case-study data was obtained and at the end of this period, the algebra study reported below was undertaken.

The Logo work was integrated into the curricular activities of each classroom, and children programmed in pairs throughout the school day. The teaching approach was loosely structured to allow children ample opportunity to pose and solve their own problems. Within this framework the teaching strategy employed was viewed as an issue for research, and for elaboration by the teachers and the researcher, rather than as an *a priori* component of the study. A full report of the study is contained in Noss (1985).

AIMS OF THE ALGEBRA STUDY

The general aim of the study was to examine the kinds of thinking which children who had learned Logo for 18 months (approximately 50 hours), could 'carry over' to an algebraic context. It was designed to investigate the extent to which the children could:

(a) construct meaningful symbolisations for the concept of variable
(b) construct formalised (algebraic) rules.

The study consisted of interviews with children as they solved a series of pencil-and-paper rule-formulation problems. It was not intended to compare the ability of 'Logo children' to conceptualise the notion of variable or to formalise, with that of 'non-Logo' children. The investigation was essentially exploratory in nature; it was concerned with illuminating areas of explicit linkage between Logo programming and algebraic concepts, and with identifying issues for further research.

The eight children who participated in the study were aged between 10 and 11 years, and none had studied any 'formal' algebra in their school mathematics. Each of the children had been the subject of case-studies referred to above, so that it was possible to accurately assess the kinds of Logo activities with which they were familiar.

The two issues of symbolisation of variable and formalisation, provided two specific questions on which the study was based as follows:

Question 1: How may children use the Logo ideas of (a) *naming* and (b) *inputs* to facilitate the conceptualisation and symbolisation of the concept of algebraic variable?

Question 2: In what ways are children able to use their Logo-based experience to assist the process of formalisation in a mathematical context?

METHODOLOGY FOR THE ADMINISTRATION OF THE RULE FORMULATION ITEMS

The rationale for the selection of items was based on the need for them to satisfy the following criteria:

1. They were appropriate for children who had not been introduced to algebraic notation.
2. They allowed scope for children to construct their own formalisation in the process of solution.
3. They allowed scope for children to construct their own notation for unknowns in the process of solution.

Four items from the Strategies and Errors in School Mathematics project (SESM, Algebra; Booth, 1984) were modified for the interview items. These dealt with the area classified by Booth as 'formalisation of method'. Comparison of the items given below (see Figure 1) with those of the SESM research will indicate that the focus of the items has been shifted to one of rule-formulation, rather than that of interpreting letters as unknowns, or recalling notational conventions. The key element in the 'solution' of each problem thus became the *construction* of a relevant formalism, rather than the *interpretation* of existing symbols.

No attempt was made during the children's programming activities by either the teachers or the researcher, to link the Logo work with algebraic conventions. Given that none of the children had been introduced to formalised algebraic notation (other than the syntactic requirements of Logo itself), the possibility of their utilising such notation spontaneously (i.e. without intervention by the researcher) could be effectively ruled out. It was therefore determined to adopt an approach in which the interviewer provided a series of prompts to assist in the solution of the rule-formulation problems.

The interview items were introduced by giving each child an 'initial problem card' to act as a stimulus for the rule-formulation problems (see Figure 1).

It will be seen that the initial problem card for the bridges item (Item 1) differs from that of the remaining items, in that it poses only a concrete problem rather than an abstract one (note that this item is derived from Instone (1982); a related item was employed by Booth, 1984). In the case of the bridges item, the problem of formulating a general rule for the number of green blocks when the number of red blocks was unknown, was posed verbally by the researcher.

All but one of the children were given Items 1, 2, and 3. Item 2a was given only to children who had managed Item 2 successfully, and were judged by the researcher to be capable of attempting it coherently. The interviews lasted between 25 and 35 minutes.

The three main items were presented in order of increasing abstraction, with Item 1 presented first in all cases. Each child was given red and green blocks with which to build the bridges. For Item 2, only the diagram on the card was presented. Item 3 consisted only of the written problem.

This progression from the concrete to the abstract was also mirrored within each of the Items 1, 2, and 3, by the presentation of 'incomplete iconic representations' (IIR's) of the problem when appropriate (see Figure 2). These were introduced as necessary, in order to motivate abstraction by presenting the problem in a form in which lengths or numbers of objects were indeterminate.

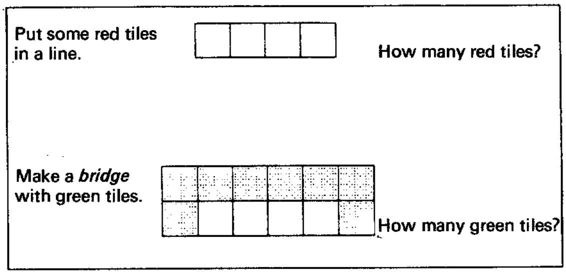

Fig. 1. Initial problem card: Item 1 – ‘Bridges’.

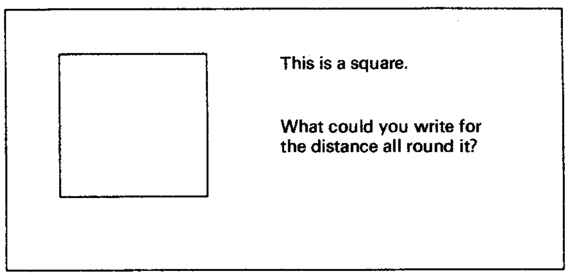

Fig. 1. (ctd). Initial problem card: Item 2 – ‘Square’.

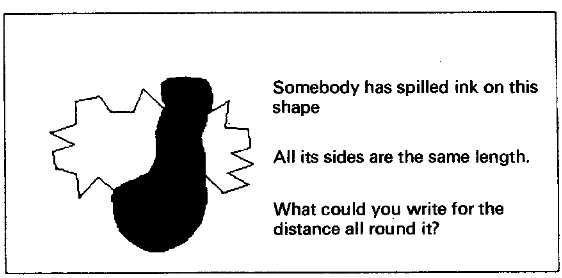

Fig. 1. (ctd). Initial problem card: Item 2a – ‘Ink Blot’.

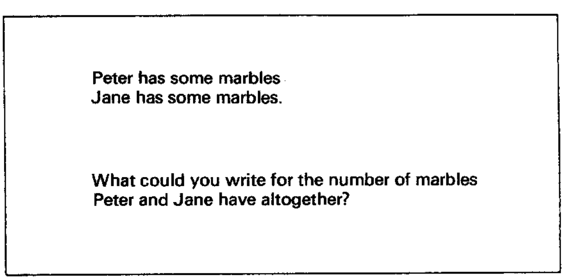

Fig. 1. (ctd). Initial problem card: Item 3 – ‘Marbles’.

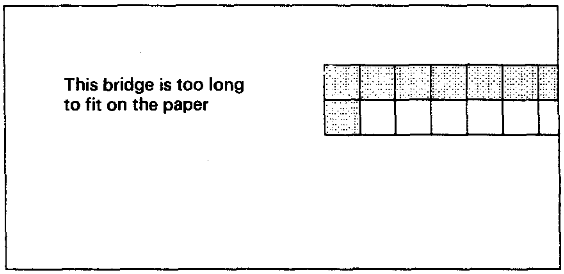

Fig. 2. ITR: Item 1 – 'Bridges'.

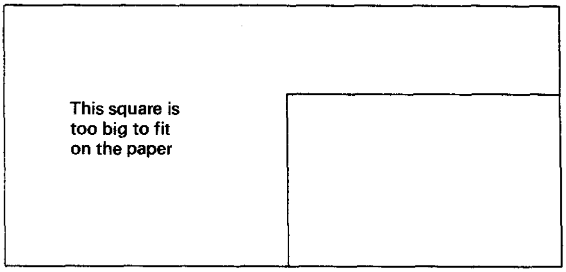

Fig. 2. (ctd). IIR: Item 2 – 'Square'.

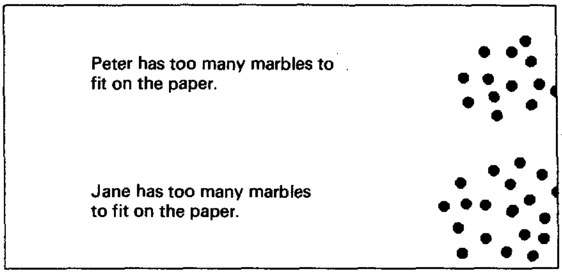

Fig. 2. (ctd). IIR: Item 3 – 'Marbles'.

Prior to the interviews, draft interview protocols were piloted with three children, in order to ascertain an appropriate order of presentation of the items, and the children's comprehension of the language employed. As a result, it was determined to situate each interview in the context of a 'story', in order to justify the need for conciseness in the rule-formulations. Students were told that the researcher worked with a group of 8-year old children who had learned Logo. The problem was to formalise rules for the individual items in such a way that these younger children could understand it, particularly as they 'couldn't read very well'. The rationale for the story was to encourage students to formalise their rules, and crucially, to deter them from deliberately choosing verbose descriptions of the rules. The story was effective in encouraging conciseness in all but one child. The exception was Anthony, who, after formulating a rather verbose rule, commented, in response to a prompt for formalisation:

A: "Umm... yeah. Well, you'd need it in quite a bit of detail so they could understand it".

Students were helped to establish rules by a sequence of questions designed to encourage generalisation. This approach is exemplified by the following sequence for the bridges item:

(a) Give blocks (red and green). (Note that the term 'blocks' and 'tiles' were used interchangeably by the researcher and the children).
(b) Help to construct the bridge.
(c) How many green blocks?
(d) How many green blocks, if there were 6, 8, 10 reds?
(e) How many if there were 27 red blocks?
(f) Can you make a rule if you don't know how many red blocks there are?

By this stage, all the children were able to propose a natural language rule e.g. 'Add 2 tiles on both ends of the reds and greens'. The remainder of each problem interview was concerned with probing the child's ability to construct a formalised rule and to employ a meaningful notation. Researcher interventions were made according to the scheme given in Table I.

TABLE I

1. Suggest unspecified notation (N1)
2. Suggest specified notation (N2)
3. Suggest formalisation of rule (F)
4. Suggest Logo connection (L)
5. Offer incomplete iconic representation (IIR)

Note: the codes were used in the analysis of the interview transcripts and will be employed in the subsequent report.

Interview Prompts

Points 1 and 2 of the schedule were used to suggest that some notation may be helpful. For example, 'Is there anything you could give a name to in this picture?', or "What could you call the number of red blocks?' would be examples of N1 and N2 interventions respectively.

Point 3 involved reminding the student of a possible Logo connection, e.g. 'Did I mention that the first years have learned Logo?' Other more overt references to Logo were made to some children at the end of the interviews, in order to probe their conceptions of points that had arisen in the course of the interview.

By formalisation of rule (Point 4) is meant a suggestion that conciseness was required e.g. 'The first years can't read very well', or 'Could you write that using less words?'. Point 5 refers to incomplete iconic representations which suggested the need for generality by emphasising the indeterminate nature of the unknowns.

The scheme of interview prompts was not adhered to in detail. Flexibility was employed in order to allow the researcher to follow up any points which arose in the course of the interview. The schedule thus provided a framework which was used to indicate the general direction that each interview should take. In addition, an attempt was made to intervene on as 'low' a level as possible in order to transform each problem from concrete to abstract (Krutetskii, 1976), and to allow students scope to make the abstraction his/herself if possible. For example, a student would be offered an N1 prompt before an N2 prompt. Similarly, a suggestion of a Logo connection (L) would not be made until other attempts at formalisation had failed. It should be noted that the interview prompts in Table I were employed cyclically rather than linearly; that is, the researcher returned to previously used prompts when necessary. For example, the introduction of an incomplete iconic representation might have necessitated suggesting a notation (N1 or N2), or a formalisation of the rule (F).

The eight interviews were audio-taped and transcribed. Each transcript was reviewed several times, and a classification of student responses was made. At the same time, a record of researcher-prompts was encoded onto each transcript using the coding scheme in Table I above.

FINDINGS

The findings are summarised in Tables II and III. Table II provides an overview of the notations employed; Table III provides a summary of the rule formulations. Discussion of the findings takes place under two headings corresponding to the two research questions posed earlier; the concept of variable, and the process of formalisation.

TABLE II
Overview of notations employed

	1 'Blocks'	2 'Square'	2a 'Ink blot'	3 'Marbles'
Anthony	A B C D E F G H I J 1 2 3 4 5 6 7 8 9 10	–		–
Julie	RBLOCK, GBLOCK	DIS	SIDE, DN	DN
Stephen	(i) REDS, GREENS (ii) :REDS, :GREENS (iii) R., S.	P., S.	L., A.	J, P
Daniella	:two, :tiles	:SIDE		
Mathew	(i) REDS, GREENS (ii) reds, greens, :NUM	(i) SIDE (ii) :SIDE, :NUM	–	WHOLE, LOTS
Nicola	R.B., G.B.	Side, Side 1 Side 2, Side 3	Side, Missing	:Peter, :Jane
Joanne	red, greens, number	–		marbles, marbles2
Bradley	Greens, Bricks Five			

'–' denotes no notation employed.
A blank entry denotes item not administered.

The Concept of Variable

There were two main aspects which emerged from the interviews on the question of variable. These concerned (i) the idea of naming, and (ii) the children's conception of variable as a generalised number. The convention adopted below is that children's written rule formulations are in bold type.

Naming: Of the eight children who participated in the interviews, six were able to suggest names for the unknowns in Item 1 (blocks), and to employ them in a rule which related the unknowns as variables. The two exceptions to this were Anthony and Bradley, neither of whom had used the idea of variable to any extent in the context of Logo. Indeed when prompted by the researcher for a connection with Logo, Anthony was able to say 'We make up procedure names', but was unable to make use of this idea of naming in the context of the naming of unknown values. His response suggests the possibility that any conceptual linkage in this respect on the part of the remaining children, is likely to be based on the idea of naming of inputs to procedures, rather than on the idea of naming itself.

The readiness with which children were prepared to name unknowns varied between items. For example, some children found the naming of the

TABLE III
Overview of rule formulations

	1 'Blocks'	2 'Square'	2a 'Ink Blot'	3 'Marbles'
Anthony	I added 4 blocks onto J so that I knew the amount of green blocks	–		–
Julie	rblock + 4 to gblock	DIS × 4	–	J + to P
Stephen	(i) IF :REDS = 10 [MAKE :GREENS + 4] (ii) G. = R. + 4	P. = S. × 4	L. × A.	P.M. + J.M.
Daniella	:two is :tiles + 4	The square is 4 × side		Count both set of marbles and add them up
Mathew	(i) If reds = :NUM + 2 greens at other ends (ii) Add 4 greens more than reds	(i) If :side = :num × :side 4 (ii) × :side by 4	–	WHOLE + LOTS
Nicola	R.B. + 4 = G.B.	Side × 4 = the whole thing	Side × missing = the whole thing	:Peter + :Jane
Joanne	(i) If red = Number + 4 to get greens (ii) Add 4 greens to red to get the total greens	The distance all round it is 4 × one side		Add marbles and marbles2
Bradley	Count the red bricks and put the green bricks on top			

'–' denotes no rule formulated.
A blank entry denotes item not administered.

unknowns problematic in Item 3 (marbles), even though they had succeess-fully used names for earlier items. Daniella for example, suggested 'guessing' the number of marbles that Peter had, and 'asking' him how many there were. Stephen's response reflects his initial confusion:

S: "I think it's like with the blocks ... yes. You can't do the same thing with the blocks, you can't have ... you know the top one's four, or um ... I don't know".

It is possible that the nature of the unknown itself – numbers of marbles rather than the length of side or 'length' of bridge (although note that this latter quantity was expressed in discrete terms i.e. numbers of blocks) – may have accounted for these responses, and may be related to the relatively restricted contexts in which unknowns were encountered within the Logo work. The children's experience of Logo inputs was generally confined to continuous quantities (length, angle), although it is an open question whether the children interpreted them in this way during their Logo work.

Similar difficulties were encountered with regard to Item 2a (Ink blot), which required the manipulation of more than one variable. This item was given to only four of the children, of whom only one (Stephen) was able to formulate anything like a generalised rule (he did not give a name to the total perimeter). Only two children suggested naming either the total number of sides (Stephen), or the number of missing sides (Nicola). Again, the naming of a discrete quantity (number of sides) caused some difficulty.

A rather different interpretation of the children's difficulties in Items 2a, and 3 may be based on the *unmeasurability* of the unknowns, rather than to their discreteness. Evidence of this possibility is exemplified by Julie's response, in her solution to Item 2a, the 'Ink-Blot' below:

Interviewer: "Could you call it something, the number of sides there are?"
J: "Um ... no".
I: "Why not?"
J: " 'Cos you don't know how many sides there are on it, 'cos it's covered by ink".

It is interesting to assess the extent to which researcher prompts for a Logo connection were instrumental in generating meaningful notations for the unknowns. It should be noted that investigation of the issue was confounded by the presence of the researcher, who can be assumed to have been identified in the minds of the children with their Logo work. Nevertheless, it was noteworthy that only one child (Daniella), adopted the Logo-style colon as a first attempt at a notation in response to a researcher prompt. Three others (Mathew, Stephen and Nicola), adopted the colon at some point in the interviews. The example of Nicola provides an illustration of an approach which was apparently not explicitly motivated by her Logo experience, but which she was able to link with that experience in the final item (marbles) as follows:

N: (presented with marbles item) "You could use the input again" (note that she had not previously referred to inputs).
I: "Alright, show me how".

N: (writes)

:Peter + :Jane = all the marbles

I: "Can you read it out?"
N: "Peter plus Jane equals all the marbles. You use those two as the inputs, with as many marbles as you want to".
I: "So what are the dots in front of Peter and Jane?"
N: "They're to represent that its an input".
I: "But this isn't a Logo program is it?"
N: "I know, but if it was ... just to say that it's an input."
I: "So what does the input actually mean there then?"
N: "That you can type in however size you want it or how many you want it. How ever many they want. How many they want Peter to have, and how many they want Jane to have".

The connection in the children's perceptions between their Logo experience and the rule formulations was not always as clear-cut as this. For example, the following exchange between the researcher and Stephen, in his solution to Item 2 ('Square'), suggests that he was in the process of making connections between his Logo experiences and the formalisation of a rule:

I: "Can you write down a rule?"
S: (writes)

THE PERIMETER IS FOUR TIMES SIDE

I: "O.K., and what does 'side' mean in that sentence there?"
S: "Well, side is, if you, in a repeat you say, repeat forward side which is an input, and right 90 close brackets. And it'll repeat forward side and right 90. And you can make side whatever you like, by saying the name of the program and then the number".
I: O.K. Now can you write it using less words?"
S: "Mm ... O.K." (writes)

P. = S. × 4

Variable as generalised number. The interviews provided a number of instances of children constructing names for variables which stood for a range of numbers. Although the exploratory nature of the study does not permit such instances to be interpreted as necessarily linked with the Logo work, such findings do appear to run counter to the 'natural' tendency referred to by Booth (1984) of children to interpret letters as specific numbers. One illustration was provided by Julie (Item 1), who employed the singular terms RBLOCK and GBLOCK to stand for the number of blocks, even though she pointed out that 'You don't know what they are'. She also proposed (unprompted) that she 'make a word for the distance' in Item 2 (which she called DIS). Having proposed a 'rule' (DIS × 4), she illustrated the mental link which she had formed with her previous experience as follows:

I: "Have you ever done anything like that before" (i.e. calling an unknown by a name).
J: "On the other page" (i.e. in Item 1).
I: "What about when you've been doing Logo?"
J: "Yes, when we did GAME we did it like that". (GAME was the first major Logo project in which Julie had used inputs some two months earlier).
I: "Can you remember what that was?"
J: "It did the distance around the people".

There were a number of examples of children utilising explicit computer-based metaphors to aid in the process of using a single variable to stand for a range of numbers. One illustration is provided by Nicola's remark above that her variables were based on being able to 'type in however size you want it'. A further illustration is provided by Mathew, replying to the interviewer's question to explain what his variable (:NUM) stood for:

M: "Random."
I: "Mm. But what is it? Just any number?"
M: "Well a number that's chosen by the computer."

Examples such as this suggest a possible way in which the Logo work may have helped the students to formalise; the metaphor of typing in a value at the keyboard can be viewed as a means of conceptualising a range of numbers while only necessitating the consideration of specific values (one at a time). In the context of inputs, Logo variables are assigned a single value at the time the procedure is executed, although the name of the input may, of course, stand for an infinitely large range of possible values.

Such a conception appears less abstract than the normal mathematical usage of the term variable. In the equation $y = x$, the relationship between x and y is the crucial factor, not specific examples of the relationship. It is the focus on the relationship which confers the power to conventional algebraic notation. Yet it is precisely this focus that, as Kuchemann (1981) points out, children find so difficult. It is worth conjecturing that the experience of using Logo inputs may have provided some children with a way of conceptualising this abstract idea by linking the assignment of specific values to the variables. Such an explanation must be viewed only as one possible interpretation; the conjecture is one which is open to further investigation.

It was evident that the children who experienced greatest difficulty with a generalised conception of the unknown were those who had least experience of the idea of Logo inputs or variable (Anthony and Bradley). Anthony, for example, displayed a specific number' misconception by assigning a 'code' whereby letters stood for individual numbers:

I: "Can you make up a name to put as a label on the number of red blocks?'
A: "Make up a name... can we put a name for how many blocks there are? I see what you mean..."
I: "Mm."
A: "Like shall we put... yes but its going to vary..." (pause). "Hey, I've got an idea... **Hang on I've got**"
(writes)

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

A: "Now... we can have a certain number up to... we can have any number up to... we can go up to 26."

Anthony's conception adds weight to the suggestion that the idea of inputs itself (rather than the naming of procedures) was an important facet of any linkage between the Logo work and the rule formulation items.

On the question of the distinction between the name and value of a variable in a Logo context, the interviews did not permit any conclusions to be reached; the unknown as object misconception (Kuchemann, 1981) was not greatly in evidence. Further investigation remains to be undertaken on this issue, particularly in relation to recent work on reformulating the syntactic and semantic basis of the Logo language (Allen and Davis, 1984).

The Formalisation Process

Using Logo-based contructs. Consideration of the rules formulated (see Table III), indicates that a number of children adopted certain aspects of Logo syntax in the construction of formalised rules (e.g. IF and MAKE). The case of Stephen illustrates a number of important issues. Stephen began by proposing two informal formulations, based on adding two more blocks 'at each end'. In response to a prompt to name one or more unknowns (prompt N1), he proposed naming the numbers of red and green blocks as 'REDS' and 'GREENS' respectively. This notation led to the following formulation:

IF :REDS = 10 [MAKE :GREENS 14]

The colons in the above rule were inserted as an afterthought. This formulation was the first in which Stephen assigned a value (10) to the unknown. He has adopted a formalised (Logo) notation to relate the unknowns, employing the Logo word IF in the role of 'suppose' i.e. 'suppose there were 10 reds'. In response to the researcher's query as to the role of the 10, Stephen replied that it was 'just a random number'.

In his next formulation, Stephen generalised from his assigned-value formulation, and at the same time moved away from a Logo-based notation:

MAKE :GREENS FOUR MORE THAN REDS

It was clear that this formulation was not intended to be a Logo statement, although the role of the colon in front of GREENS (but not REDS) was not clarified. In moving away from the formalised notation of his previous rule, Stephen was still employing the Logo idea of MAKE (assigning a value to a variable) as a means of relating the two unknowns in the above 'equation'.

Stephen's final formulation, proposed in response to a prompt for formalisation (prompt F), was as follows:

G. = R. + 4

There are three interesting aspects to this final rule. Firstly, Stephen has replaced the Logo 'MAKE' (essentially procedural/dynamic), with an equality symbol, seemingly as an equivalence. Secondly, he has introduced the addition symbol in a role which is clearly an operator on the unknown number of red blocks and the number 4. Thirdly, he has avoided the error of adding the 4 to the wrong side of the equation, which makes it unlikely that he had simply 'translated' his previous rule for natural language to an algebraic equation.

It is possible to discern, in the development of Stephen's three rules, a transition from an essentially descriptive, specific-number formulation, to a generalised, algebraic one, in which the adoption of aspects of Logo formalism may have provided a catalyst for the transition to a more generalised (algebraic) conception. While the data of the present study allows nothing more than the postulation of a relationship of this kind, examples such as Stephen's illustrate one possible basis of a conceptual linkage, and point the way to further avenues of research.

The use of arithmetic symbols. Six of the eight children employed an addition symbol in the rule formulated for Item 1. All six employed a multiplication symbol for Item 2. The exceptions were, once again, Anthony and Bradley. Three children, Nicola, Daniella and Stephen were prepared to operate on the unknowns. For example, Daniella formulated a rule as follows:

:two is :tiles + 4

In contrast, others employed arithmetic symbols rather as a shorthand for a verb (e.g. RBLOCK+4 to GBLOCK – Julie, Item 1). In such cases, the arithmetic symbol was interpreted as a shorthand for the natural language equivalent, rather than as an operation. In the children's Logo work, relatively little use was made of the Logo arithmetic operators; the evidence of the interviews is inconclusive on possible conceptual linkage in this respect. It would be interesting to investigate the effect of a more determined intervention strategy aimed at encouraging the use of arithmetic operators in children's Logo activities.

The children's use of the equality symbol was equally inconclusive. Julie ignored the equivalence between her 'total' and the other variables altogether; Daniella employed the word 'is' to convey the equality of two sets. Nicola, on the other hand, used the equality symbol for each of the four items. The only consistent trend was in the use of the equals sign as an equivalence in the instances of the use of 'IF' (see the formulations by Stephen, Mathew and Joanne; Table III). For example, consider Mathew's formulation for Item 1,

If reds = :NUM + 2 greens at other ends

which was read as 'If reds equals say a certain number, add 2 tiles at either end'.

It is perhaps noteworthy that this was almost the only context in which the equality symbol appeared in the children's Logo work (e.g. IF :SIDE = 0 [STOP]). There is thus a possibility that these children used the equal sign as an equivalence (as in Logo), although it would be incautious to suggest a generalised conceptual linkage between this aspect of their Logo work and their rule formulations.

IMPLICATIONS FOR RESEARCH

This study has indicated that the experience of Logo programming may provide children with a framework on which further learning may be based. It may be worth emphasising here what the study is *not* intended to illustrate: namely that children who have learned Logo for some time will *necessarily* have learned something about algebra in general or about the concept of variable in particular. The interpretation of the data offered here (and it should be emphasised that it is one possible interpretation), is that children may – under the appropriate conditions – make use of the algebra they have used in a Logo environment, in order to construct algebraic meaning in a non-computational context. What then, apart from the experimental conditions of a research study, constitutes appropriate conditions?

The question of linking disparate conceptions formed from different contexts has recently been illuminated by Lawler's (1985) notion of 'microviews'. The issue of forming conceptual linkages can in Lawler's terms be reformulated in terms of aiding the process by which disparate microviews can be synthesised into more general conceptions. Such a process may take a long time. As Vergnaud (1982) points out, understanding what a 15-year old does may involve knowing the kinds of 'primitive conceptions' which the child experienced at the age of eight or nine, and the ways in which such conceptions have been transformed over time. One way of viewing a role for Logo may be as an aid in forming primitive conceptions of algebraic notions (perhaps at an early age), which may then be integrated as part of a system of algebraic understandings – in Vergnaud's terms a conceptual field.

Viewed in this way, the task becomes one of designing a mathematics curriculum with overt links to the kinds of ideas which the Logo environment has encouraged. For example, the idea of *naming* of variables (that is, with meaningful names chosen by students) is almost unheard of in established introductory algebra curricular materials, presumably because of the difficulty of linking such an idea with children's experience. Substituting 'apples' for 'a', of course, only makes the problem worse (Kuchemann, 1981). Whether, in the future, we might substitute :apples is an open question.

There is another, more challenging possibility. This would involve constructing a syllabus which was based on Logo itself. That is, to design a syllabus in which the key ideas relating to the concepts of algebra were introduced *via* programming, i.e. to create a system of microworlds (or perhaps just one, providing it is rich enough) based on the ideas of algebra. In other words, algebraic concepts could be introduced via their symbolic representation within Logo programs.

It is apparent that such a curriculum would rely heavily on a pedagogical input; there seem no grounds for supposing that a purely technical construction in the form of the relevant Logo programs would be sufficient. This paper illustrates that through working with Logo, children may develop primitive conceptions to which they may link their further understandings. But it may also point towards a way to take a further step which would provide an environment for higher-order conceptions to be constructed. It should be possible to construct Logo-based environments which will allow access to the richness of algebra – in much the same way that the turtle primitives of Logo allow access to a wide range of geometric ideas (Abelson and DiSessa, 1980).

Microworlds of this kind would be based on the creation of an environment which allowed exploration and problem-solving within a sufficiently rich conceptual field (a single concept seems too poor for the construction of a microworld). At the same time, the challenge would be to find ways of constructing Logo environments which are sufficiently transparent and flexible to enable the learner to gain control over the embedded concepts. Preliminary work on the construction of such algebraic learning environments has been undertaken by Leron and Zazkis in Israel, and by Hoyles and Noss in the UK.

Almost certainly, such microworlds will involve non-elementary examples of Logo programming. At the very least programs will contain primitives which are not accidentally stumbled upon (e.g. OUTPUT). It is likely that they will involve ideas (such as recursion), which current research is indicating are far from trivial for most children. How to introduce such ideas must be an important area of investigation in the future.

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University of London,
Institute of Education,
20 Bedford Way,
London WC1H OAL,
U.K.