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The use of spreadsheets within the mathematics classroom

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This paper examines the potential of the use of spreadsheets in the teaching and learning of mathematics. Consideration is given to the teacher's role in structuring computer-based learning environments so that pupils are encouraged to reflect on mathematical processes. Crucial aspects of the software which pupils need to learn in order to use the spreadsheet as a mathematical tool are identified, and a number of introductory activities aimed at addressing these aspects are included. Examples of spreadsheet activities related to different types of mathematical problems are also presented. Finally we include an example of pupils using a spreadsheet as a context for formalizing mathematical generalizations.

1. Introduction

Computers are gradually becoming more accessible within mathematics classrooms and this has encouraged a search for computer applications which are appropriate for the teaching and learning of mathematics. Recently there has been an upsurge of interest in the use of spreadsheets and articles on their classroom use are appearing with increasing regularity [1–7]. We are beginning to investigate systematically the potential of the spreadsheet for learning mathematics and this article will draw together the threads of our recent work.

There are several different types of spreadsheet packages but they all look similar to the one shown in Figure 1; this shows a worksheet from the spreadsheet package EXCEL for the Apple Macintosh computer. We explain our rationale for choosing this spreadsheet for our classroom work in the Appendix. A spreadsheet is used for presenting and manipulating data (both text and numbers).

In some ways a spreadsheet can be thought of as similar to a programming language, in that there are sets of problems for which a spreadsheet is a useful problem-solving tool. For the purposes of our work we have concentrated on the idea of entering and replicating rules relatively. In many spreadsheets it is also possible to define and copy absolute references (e.g. by naming the cell). Other possibilities include the construction of loop structures and the use of conditional statements. When a rule containing a relative reference is copied, the physical relationship between the cell containing the reference and the referenced cell will be preserved. This is illustrated by the rule in Figure 2 (*b*) which will generate the sequence shown in Figure 2 (*a*). In general the value produced by the rule is automatically calculated and displayed in the cell; however it is usually possible to re-examine the rule by selecting the cell in which it is contained.

We describe the process of generating these types of rules in more detail later in this paper.

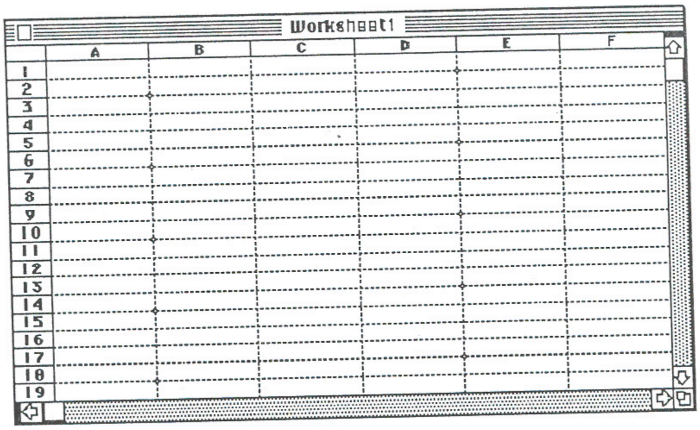
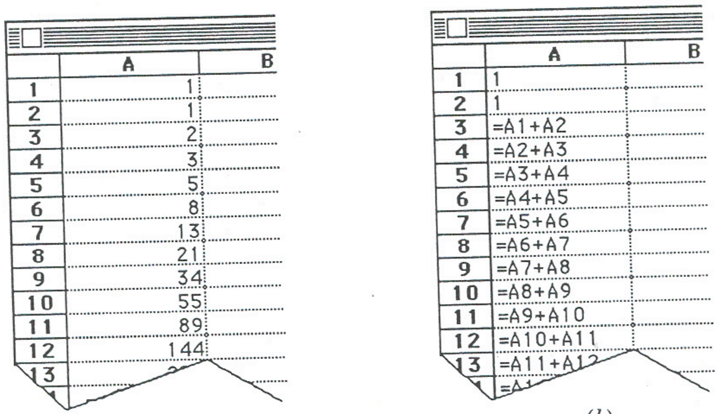


Figure 1



(*a*) (*b*)

Figure 2

2. Developing a teaching/learning sequence

In developing a framework for our teaching/learning environment we have been influenced by the ongoing computer-based research carried out at the Institute of Education [8–10]. Although this work was mainly concerned with the computer programming language Logo it does provide a valuable background from which to analyse the important factors relating to teaching and learning within a computer-based environment.

First and foremost, the teacher's role is crucial. Not only must teachers find a way of structuring the learning environment so that pupils are appropriately challenged and motivated by the problems which they are being asked to solve on the spreadsheet but we suggest that within a computer environment it is also important for pupils to:

(*a*) use mathematics as a problem-solving tool;
(*b*) reflect on processes for themselves;
(*c*) learn to debug their conjectures and representations.

We wanted pupils to use mathematics as a problem-solving tool, i.e. be introduced to the use of a spreadsheet within the context of solving a problem. We needed to think of two separate, if related, issues. First we wanted pupils to know how to use the software; thus we needed to identify a set of interesting problems through which we could introduce necessary features of the spreadsheet. We also needed to present problems which confronted the pupils with the specific mathematical ideas under study.

If pupils do not begin to reflect on processes for themselves in a computer environment without always turning to the teacher for help, then many of the advantages of such an environment are lost. Pupils' first introduction to the computer-based activity is crucial. Non-reflective practice appears to be linked to the attribution of some sort of magical power to the computer. Results of the Logo Maths Project suggest that presenting pupils with previously written programs (that is, in effect problems which have already been solved) at the beginning stages of learning to program does not encourage them to reflect on process for themselves. This project also found that too rapid exposure to different control processes can sometimes lead to confusion. For this reason we decided to plan spreadsheet tasks such that pupils would always construct their own rules within the spreadsheet environment. This contrasts with many of the existing ways of introducing pupils to spreadsheets [11, 1, 5] but is consistent with the approach used in the previously referenced Logo studies. We refined our spreadsheet introductory tasks many times before we considered them to be both sufficiently challenging and sufficiently self-explanatory for pupils to work on them without needing too much initial teacher support. The aim was to balance the introduction of the powerful structures (which are themselves the motivating factors for using a spreadsheet) with an emphasis on an understanding of process at every stage. We also asked the pupils to work in pairs because this is another factor which makes it more likely that they will discuss and reflect on processes for themselves without turning to the teacher for help.

In any computer-based activity debugging is an important problem-solving strategy. Pupils need to know both that they are allowed to and how to debug, as this is not necessarily part of their normal school practice. This is another reason why we believe that it is important for pupils to construct and debug their own rules from the beginning stages of learning to use a spreadsheet.

3. Learning to use the spreadsheet

There appears to be a minimum level of syntax which pupils need to know before they can start to use a spreadsheet. They need to be familiar with the layout of the spreadsheet and they need to be able to enter both text and numbers. In order to use the spreadsheet as a mathematical tool they need to be able to define rules, and to understand the way in which these rules can be constructed from values or rules in other cells, and replicated into other cells.

Once pupils are confident in defining rules and copying them using relative referencing, they then have the necessary tools to attempt a variety of mathematical problem-solving activities in a spreadsheet environment.

Because pupils need to learn these crucial aspects before they can effectively use the spreadsheet as a tool, we found that it was not beneficial for pupils at the beginning stages to be asked to devise their own problems. We also discovered that if we left pupils in too open-ended a situation they then lacked the motivation to learn about the spreadsheet. Alternatively a very prescriptive sequence of activities turns out to be too constraining and tends to lower pupils' levels of motivation and engagement. During the development of our work, our aim has always been to present pupils with interesting and motivating introductory problems.

The activities which we presented to pupils can be divided into two groups: entering data and rules; and replicating rules. In the Excel environment it is possible to use the mouse physically both to select and reference cells when defining a rule. Pupils always made use of this facility when carrying out the introductory activities. The handouts we prepared to introduce pupils to entering and replicating rules using the mouse to reference cells are included in Appendix 2.

The first set of activities (Figures 3–6) aimed to introduce pupils to:

(i) the convention of labelling cells (e.g. D7);
(ii) the entering of text and numbers into cells;
(iii) the entering of rules into cells with an emphasis on 'making sense' of how these rules worked.

The sheet we prepared to introduce pupils to defining rules using the mouse to reference cells is included in Appendix 2.

The most successful of all the introductory activities was 'Guess The Rule' (Figure 5) because it encouraged pupils to reflect on the way to represent their rules. It also provoked pupils to discuss issues such as when two rules can be considered to

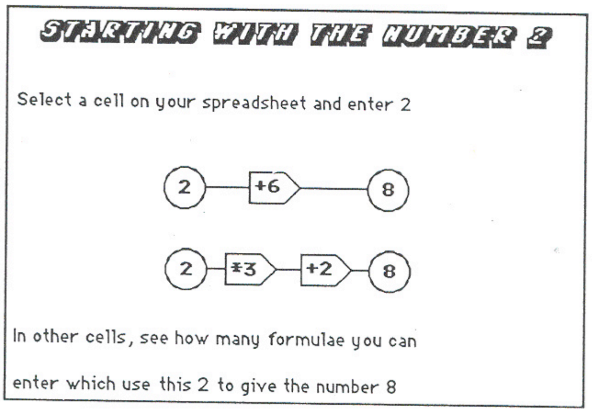


Figure 3

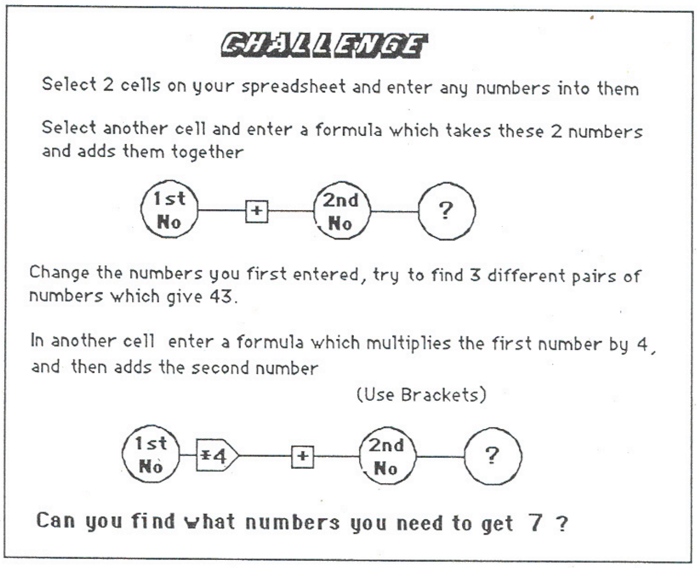
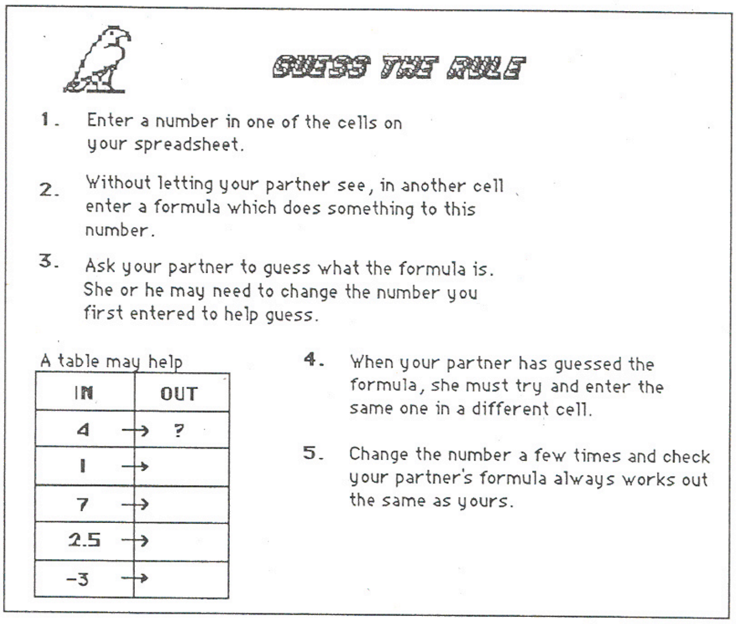


Figure 4



GUESS THE RULE

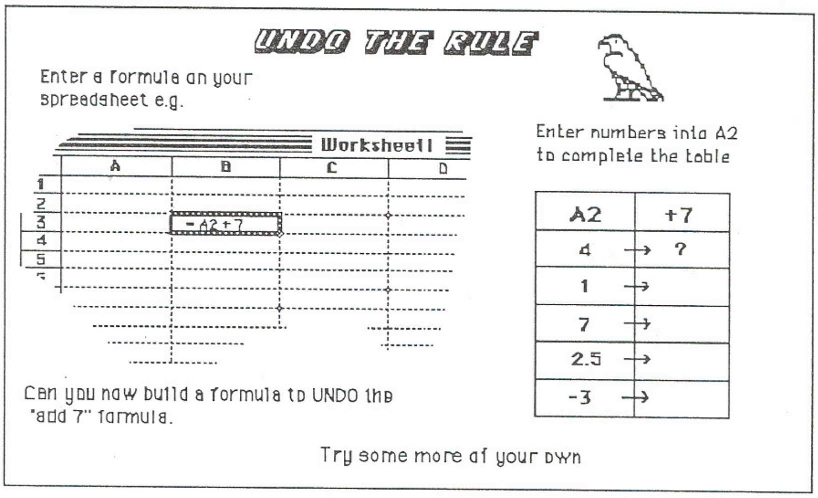
1. Enter a number in one of the cells on your spreadsheet.
2. Without letting your partner see, in another cell enter a formula which does something to this number.
3. Ask your partner to guess what the formula is. She or he may need to change the number you first entered to help guess.

A table may help

IN	OUT
4 →	?
1 →	
7 →	
2.5 →	
−3 →	

4. When your partner has guessed the formula, she must try and enter the same one in a different cell.
5. Change the number a few times and check your partner's formula always works out the same as yours.

Figure 5



UNDO THE RULE

Enter a formula on your spreadsheet e.g.

Enter numbers into A2 to complete the table

A2	+7
4 →	?
1 →	
7 →	
2.5 →	
−3 →	

Can you now build a formula to UNDO the "add 7" formula.

Try some more of your own

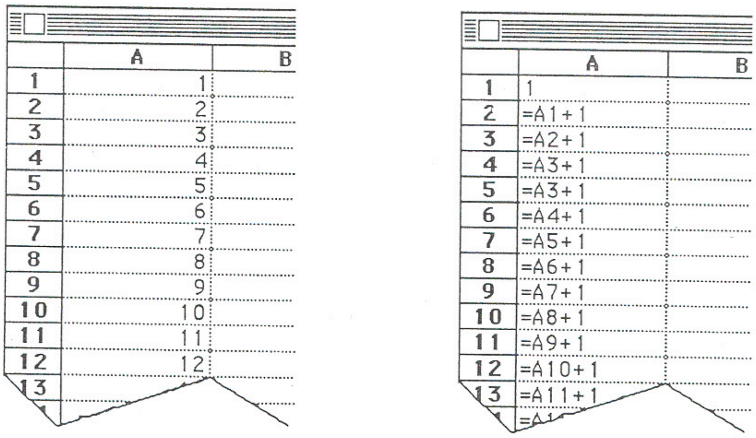
Figure 6

be the same. When using 'Undo The Rule' (Figure 6) pupils were often confronted with the need to use brackets to represent their inverse rule.

The time spent on the tasks in Figures 3–6 varied greatly between pupil pairs. As mentioned above, we discovered very early on that if we did not give pupils enough time to experiment with entering and reflecting on rules, then the subsequent introduction of replication was not well understood. Conversely, spending too much time before introducing the replicating feature of the spreadsheet may leave pupils unaware of the spreadsheet's potential as a problem-solving tool.

The second set of activities aimed to introduce pupils to the idea of relatively replicating a rule. Without exception, the pupils were introduced to replication by being asked to generate the sequence of natural numbers (see Figure 7).

Originally we prepared a handout which took pupils step by step through the replication process (see Appendix 2). In practice, however, we found that the teacher



(*a*) (*b*)

Figure 7

had a very important role at this stage in encouraging pupils to reflect on and make sense of what was happening to a rule when it was replicated relatively. (For example, by suggesting that pupils try a range of different starting values for their replicated rule and attempt to predict the outcome of such a change.) The 'prepared sheet' could not be used without substantial support from the teacher in the form of discussion on process.

The language used by the teacher was a crucial factor in helping pupils to develop a framework for the idea of replication. For example, as a pupil was entering a rule of the form 'A1 + 1' this would be referred to as

'*take the number from the cell above and add 1*'

or

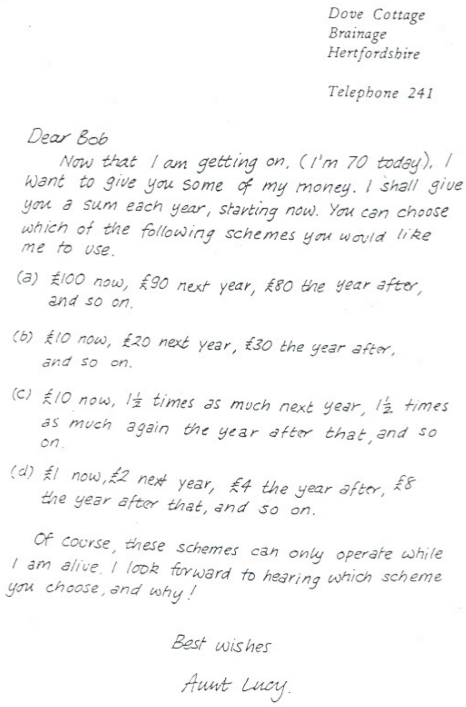
'*number above add 1*'

This use of language encouraged pupils to focus on the 'generic' nature of the rule, as opposed to the specific formula which was being entered. As we have said previously, rules were always entered dynamically with the mouse movement and this we believe provided a framework for the ultimate generalizing of the rule. With this type of support pupils did not seem to find the actual process of replicating difficult. We believe this is because the menu- and mouse-driven Excel environment is easily accessible.

Once pupil pairs had been introduced to relative replication, they were presented with a range of problems aimed at consolidating their understanding of replicating (see for example 'A Rich Aunt' in Figure 8, 'Number Patterns 1' in Figure 9). Asking pupils to generate the Fibonnaci sequence turned out to be an important activity, in that this involved writing a rule which was related to values in two previous cells (see Figure 2).

4. The spreadsheet as a mathematical tool

We are beginning to develop a picture of the type of mathematical problem for which a spreadsheet might be a useful problem-solving tool. This derives from our own work with pupils, from feedback from teachers and from previously referenced


Dove Cottage
Brainage
Hertfordshire

Telephone 241

Dear Bob

Now that I am getting on, (I'm 70 today), I want to give you some of my money. I shall give you a sum each year, starting now. You can choose which of the following schemes you would like me to use.

(a) £100 now, £90 next year, £80 the year after, and so on.

(b) £10 now, £20 next year, £30 the year after, and so on.

(c) £10 now, $1\frac{1}{2}$ times as much next year, $1\frac{1}{2}$ times as much again the year after that, and so on.

(d) £1 now, £2 next year, £4 the year after, £8 the year after that, and so on.

Of course, these schemes can only operate while I am alive. I look forward to hearing which scheme you choose, and why!

Best wishes

Aunt Lucy.

Figure 8

articles. The feedback has come from teachers attending the 'Developing Computer-based Microworlds for Mathematics' course at the Institute of Education, London University [9]. In particular many good ideas used in this article have come from the work of John Eyles, Richard Hale, Jane Harris, Adelaide Lister, Betty Lumley and Teresa Smart. We present an overview of these problems without suggesting that the way in which we have grouped them is in any way mutually exclusive or exhaustive.

Modelling problems. Spreadsheets are a valuable tool for modelling 'real life' situations (e.g. population growth, financial problems). The SMILE (Secondary Mathematics Independent Learning Experience) task (card no. 1425) 'A Rich Aunt' (Figure 8) is a good introductory modelling problem. In a 'paper and pencil' context pupils often focus on the laborious calculations needed to solve the task, as opposed to using the calculated data as a basis for making appropriate decisions.

Numerical solutions of equations. The facility to carry out, and display a number of calculations at one time makes the spreadsheet a useful tool in problems which can be solved iteratively (for example, an iterative solution to finding square roots or cube roots, the bisection method or the Newton–Raphson method for solving an equation). Whereas it is possible to solve these types of problems in most computer programming languages, performing these calculations on a spreadsheet has the added advantage that the 'step by step' calculations can be displayed in a way which facilitates analysis of the iterative process (for example, whether or not the process converges; what is an appropriate approximate solution). Using Excel to display the iterative calculations graphically could also provide another mathematical dimension.

Investigation of 'chaotic' processes. Only in the last few decades have scientists and mathematicians discovered that chaotic behaviour can result from seemingly ordered non-linear feedback systems of the form [12]

$$x_{n+1}=f(x_n, c)$$

Rodick provides a useful introduction to this idea [13]. He considers the equation

$$t_{n+1}=1-kt_n^4$$

which can be used to model mathematically a variety of biological and physical systems, and shows how, depending on the value of k and the starting value of the iteration, the iterative processes exhibits surprisingly different behaviours. A spreadsheet provides an opportunity to investigate this behaviour. Although the need to solve these equations may have derived from 'real life' problems, investigation of their behaviour now constitutes an active branch of mathematics from which the extraordinary computer-generated fractals illustrated in [14] are derived.

Generating sequences of numbers. A spreadsheet is a valuable environment in which to consider representations for generating sequences of numbers (e.g. arithmetic progressions, geometric progressions, Fibonnaci series). Many pupils have difficulty in understanding that some sequences can be generated by both a recurrence relation and by a general rule. So for example the sequence

4 7 10 13 16...

can be generated on a spreadsheet by the recurrence relation

$$a_1=4$$

$$a_n=a_{n-1}+3$$

where n is a natural number, or by the general rule

$$a_n=3_n+1$$

where n is a natural number.

'Number Patterns 1' (Figure 9) is an example of a task which we have used both to introduce pupils to this idea and to consolidate their understanding of the replication process. This activity was influenced by the DIME materials [15].

Maximization/optimization problems. The spreadsheet environment makes it possible for students with no experience of calculus to investigate maximization/optimization problems, again using iterative strategies. For example Wright describes [5] two sessions with a class in which the spreadsheet was used to model a situation which involved finding the maximum volume of a box made from a sheet of paper of a specified size (SMILE card no. 1441). A spreadsheet can also be used to solve the following tent design problem:

> *Design a tent out of a piece of A4 card (say 30 × 20 cm). What shape gives you most space inside your tent?*

5. Context for generalizing and formalizing

Finally we present an example of two pupils, Nadine and Penny, working together on a spreadsheet. This example is taken from our work on peer group

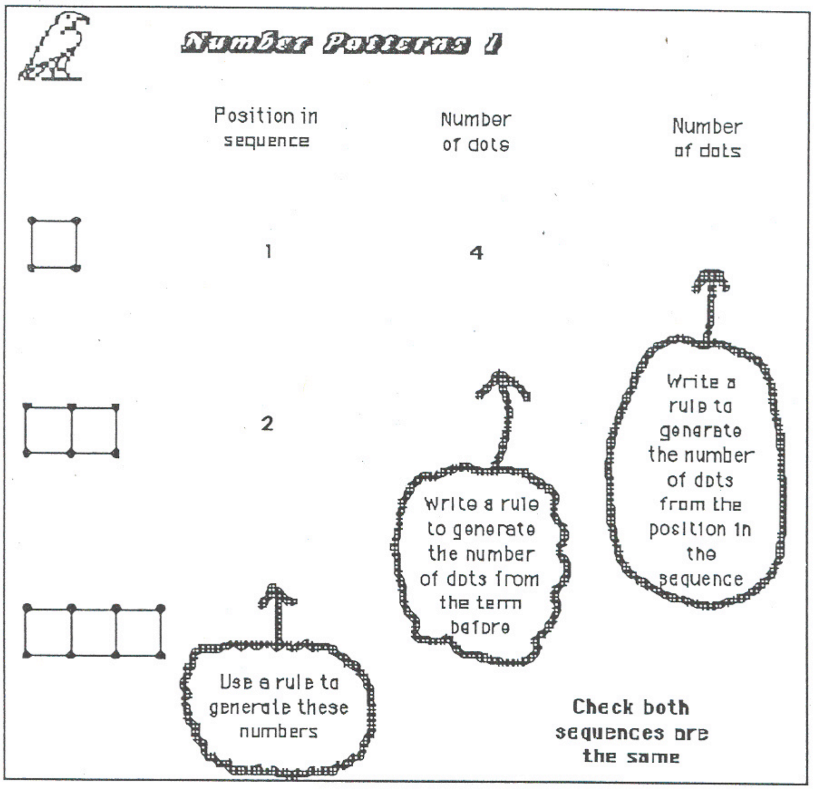


Figure 9

discussion in a computer environment†. As part of this project we have been concerned with investigating how a spreadsheet environment provides a context for generalizing and formalizing within the mathematics classroom. One aim of this project is to investigate the inter-relationship between pupils' discussion of a general method in natural language, and their formal representation of this method on a computer.

Penny and Nadine: Polygon patterns. Penny and Nadine are two second-year pupils from a mixed North London comprehensive school. During their third session of using spreadsheets (after about two and a half hours experience of Excel) they were asked to generate the polygon numbers (Figure 10). They started with the problem of generating the triangle numbers.

Without any difficulty they entered the natural numbers into their spreadsheet by replicating the rule 'add 1 to the cell before' which they had already used during their previous spreadsheet sessions (see Figure 11). Their initial strategy is to attempt to define a recurrence relation in which each term of the triangle numbers is generated solely on the basis of the term before it (e.g. the 5th term is defined in terms of the 4th term).

† The Role of Peer Group Discussion in a Computer Environment, 1988–89, based at the Institute of Education University of London, funded by the Leverhulme Trust and in conjunction with Professor Celia Hoyles.

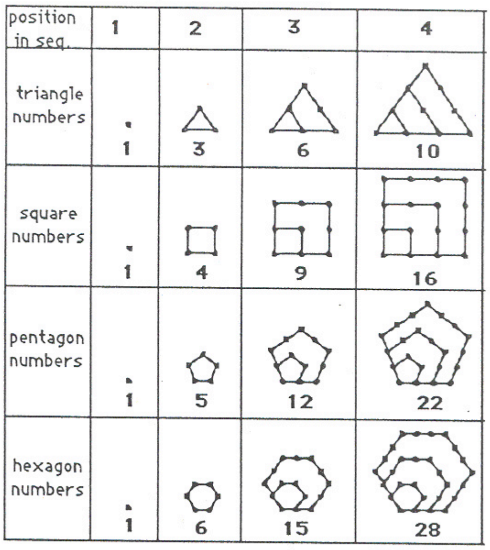


Figure 10

Worksheet1

	A	B	C	D	E	F
1	position	1	2	3	4	5
2	triangle no.	1				
3						
4						
5						

Figure 11

Penny: *Umm, what's, now we want the connection between those numbers here, and the numbers before them, so if we do 1 add 2 of course, no it doesn't work cos 3 add 2 isn't 6*

They continue to work on this strategy

Nadine: *yeah times 3 takeaway 3 . . . I think*

Penny: *where?*

Nadine: *2 times 3 is 6 then takeaway 3 . . . 6*

Penny: *2 times 1 is 2 takeaway 1 is*

Nadine: *no*

After unsuccessfully looking for a rule using this strategy, Penny starts to try out rules which are derived from the position of the triangle number in the sequence.

Penny: *let's think about anything else it could be 2 times what, 2 times 2 minus 1 equals 4, umm 2 times 2 equals 4 minus 1 equals 3, 2 times 1 equals 2 minus 1 equals 1, 2 times 3 equals 6, tchh, those 2 work, 2 times 2, 1 times 1 equals 1, 2 times 2 equals 4, minus 1, err . . . oh hang on maybe we can find a pattern that works for all of these but not the first one*

Nadine: *yeah that works, well then, 2 times 3 takeaway 3, 2 times 3, no 2 times 6 takeaway 3 no*

As Penny continues to try out these rules, moving between looking at the term before and the position in the sequence, she suddenly stops in the middle of a sentence, pauses, and finally announces her discovery of a pattern.

Penny: *it's 3, 2 times 6 takeaway 3, yeah it's 3 hang on, 2 times 2 minus 1 equals 3, 3 times 2 minus 1, na, 3 times 3 equals 9, minus itself, no . . . 1 2 3, hey look, look, I've found something look, here to get the next number you add 2, here to get the next number you add 3, and there to get the next one you add 4, how do we write that into a formula . . . ah I've got, it's this one add this one, that one add that one equals that one, that one add that one equals that one*

Nadine: *so how do we put it in?*

As Penny enters the rule '=B2+C1' (see Figure 12) Nadine asks her to explain her strategy more clearly before this rule is copied across.

Nadine: *yeah but then it'll be doing that (points at B2) by that (points at C1) all the time (then carries on the pattern, pointing at C2 and D1, D2 and E1, E2 and F1), is that what you want?*

This shows that Nadine has a clear idea of the way rules replicate on a spreadsheet, but she is not sure why this particular rule will produce the triangle numbers. Penny explains again:

Penny: *yeah cos look, look at it, this one add the top one which is the way we've set it out see, this one add this one equals that one, and then this one add this one*

Worksheet1

	A	B	C	D	E	F
1	position	1	2	3	4	
2	triangle no.	1	=B2+C1			
3						
4						
5						

Figure 12

Worksheet1

	A	B	C	D	E	F
1	position	1	2	3	4	
2	triangle no.	1	3	6	10	
3						
4						
5						

(*a*)

Worksheet1

	A	B	C	D	E	F
1	position	1	=B1+1	=C1+1	=D1+1	=E1+1
2	triangle no.	1	=B2+C1	=C2+D1	=D2+E1	=E2+F1
3						
4						
5						

(*b*)

Figure 13

equals that one, this one add this one equals that one, you understand, so it'll always be the one there and the one there, the one there and the one there, so that's what we want

After prompting Penny to make explicit the local details contributing to the construction of her final plan, Nadine is then able to express this as a general rule:

Nadine: number before, add cell, I mean position

Figures 13 (*a*) and 13 (*b*) show this rule copied across.

Without any teacher intervention they write down on paper the following general rule:

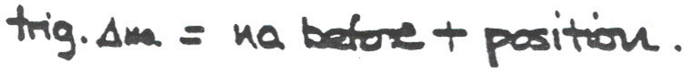


Figure 14

At this stage they have written down the rule using their own representation system. We suggest that it would not be difficult for them eventually to use the equivalent algebraic representation:

$$T_n = T_{n-1} + n$$

where T_n is the nth triangle number. Having generated the triangle numbers they were able to work out rules to generate all the other polygonal numbers. Their final spreadsheet is shown in Figures 15 (*a*) and 15 (*b*).

6. Conclusions As spreadsheets become increasingly used within the mathematics classroom we hope that pupils will begin to decide themselves when a spreadsheet may or may not be a useful problem-solving tool. In order to do this, pupils will first have to experience successful spreadsheet use and this will almost certainly require considerable support from the teacher.

The following excerpt illustrates that Penny and Nadine have started to reflect on the spreadsheet as a tool.

Worksheet1

	A	B	C	D	E	F
1	position	1	2	3	4	
2	triangle no.	1	3	6	10	
3	square no.	1	4	9	16	
4	pentagon no.	1	5	12	22	
5	hexagon no.	1	6	15	28	

(*a*)

Worksheet1

	A	B	C	D	E	F
1	position	1	=B1+1	=C1+1	=D1+1	=E1+1
2	triangle no.	1	=B2+C1	=C2+D1	=D2+E1	=E2+F1
3	square no.	1	=(B3+C1)+B1	=(C3+D1)+C1	=(D3+E1)+D1	=(E3+F1)+
4	pentagon no.	1	=(3*B2)+C1	=(3*C2)+D1	=(3*D2)+E1	=(3*E2)+
5	hexagon no.	1	=(4*B2)+C1	=(4*C2)+D1	=(4*D2)+E1	=(4*E2)+

(*b*)

Figure 15

Penny: *it is easy though . . . 'cos you don't have stupid words to try and write and explain them . . .*

Nadine: *yeah*

Penny: *because I used to, it used to be awful, I used to try, trying to explain is really hard, but if you've got this it's much easier don't you think? I wish I has one of these at home that I could do my investigations in, it'd be a lot easier . . .*

Nadine: *now you know how to think it out like a computer would . . . so it should be easier*

Penny: *oh well, next time you see I'll put all my numbers like this, cos it's a lot easier . . . makes sense*

It seems as if the spreadsheet, in providing them with a new problem-solving tool, is also providing them with a new framework within which they can begin to think about how to generalize and formalize. We hope that our future work will help us understand more about these processes.

Appendix 1

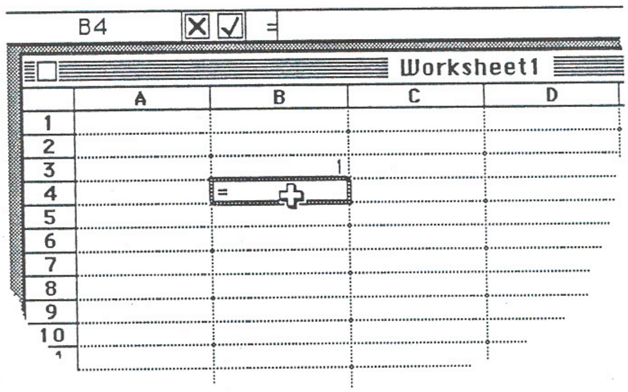
Choice of hardware and software. We must stress that our ultimate focus is mathematics. Nevertheless, at this particular stage of developing the educational potential of spreadsheets it seems important to present a rationale for our choice of spreadsheet package. We evaluated both Viewsheet for the BBC Acorn computer and Multiplan for the RML Nimbus machine, both of which are spreadsheets for the commercial market. We know that teachers are beginning to use this software within their mathematics classrooms and although 'good enough' neither package provides a very accessible spreadsheet environment. Although very different in nature, both these spreadsheet packages perpetuate a certain computer mystique in which the user can only obtain access to the software by knowing sets of meaningless and poorly documented codes. Our aim was to minimize the difficulties associated with learning how to use the software. This aim lead to our final choice of the Apple Macintosh computer because the menu and the mouse environment turned out to be important in enabling pupils to gain access to spreadsheet use.

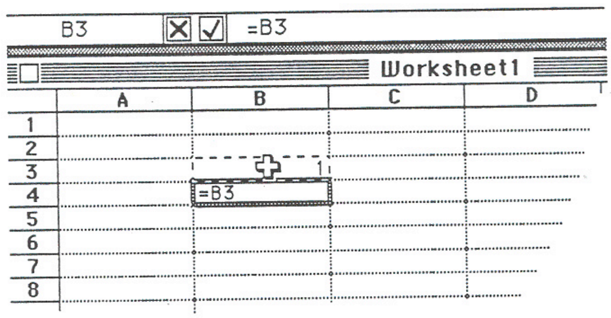
Having decided on the Apple Macintosh, we next evaluated both Multiplan and Excel, both widely used spreadsheets within the international business community. These seem to represent two distinct spreadsheet environments characterized by the approach to relative and absolute referencing. One very important difference between these is the way they refer to individual cells. In Multiplan cells are labelled R1C1 and formulas symbolized relatively R[-1]C[-5], whereas in Excel the default is to label using the convention B6 (where the letter is the row and the number is the column) although there is an option of choosing the other convention. There are also fundamental differences between the way rules are replicated within Multiplan and Excel. We decided that the method of cell referencing within Excel was simpler than that used by Multiplan and consequently chose Excel for our classroom work. Another feature of Excel (which we have not yet adequately pursued) is that it can be used both to produce graphs and to produce a database.

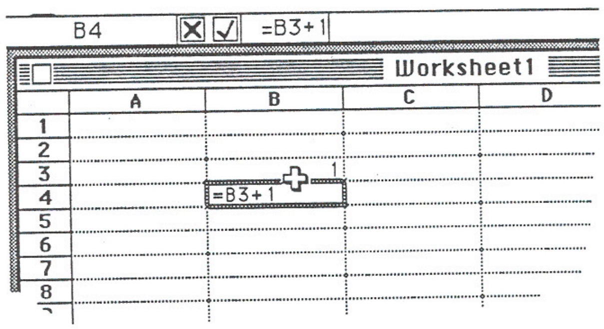
Appendix 2

Entering formulae. Entering formula to 'Take the number from the cell above and add 1':

(*a*) Enter 1 in cell B3
(*b*) Click mouse on B4
(*c*) Type =
(*d*) Click mouse on B3 (the cell with 1 in it)
(*e*) Type +1. (The value of B4 is automatically calculated by the formula shown)
(*f*) Press enter





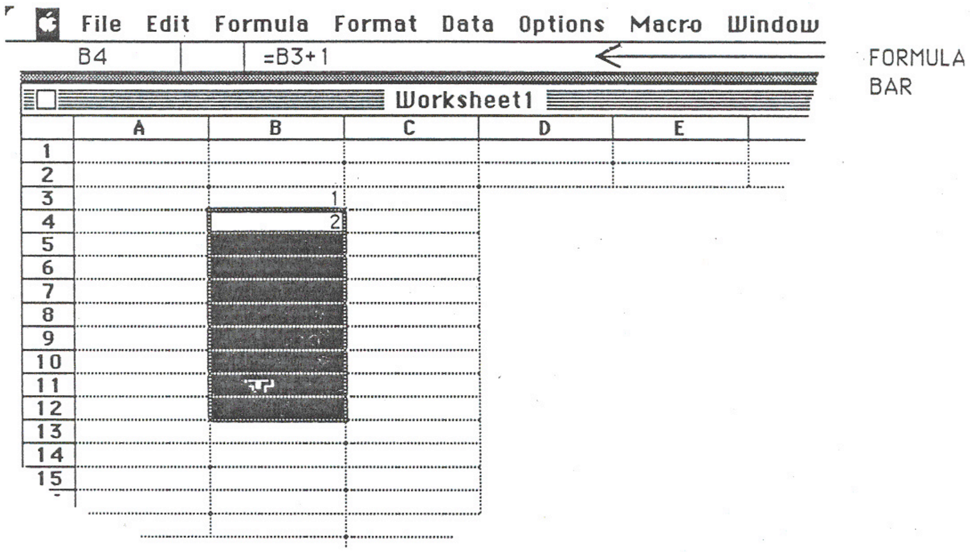


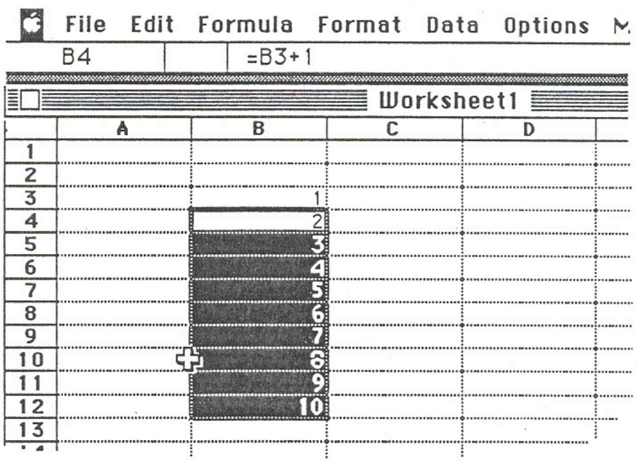
What happens if you change the number in B3? Try changing the formula in B4.

Copying rules down

(*a*) Click the mouse on the cell containing the formula B4. (The value is shown in the cell and the formula in the formula bar) (*b*) Without taking your finger off the mouse button, drag the mouse down. Release button.

(*c*) Go up and click on the Edit menu; and without taking your finger off the button move the mouse until Fill Down is highlighted. Release the button.





The rule is copied and the values are automatically calculated.

References

[1] CATTERALL, P., 1985, *Micromath.*, **1** (3), 14.
[2] CATTERALL, P., 1986, *Micromath.*, **2** (2), 26.
[3] PEASEY, D., 1985, *Micromath.*, **1** (1), 44.
[4] SCOTT, H., 1987, *Micromath.*, **3** (2), 38.
[5] WRIGHT, S., 1987, *Micromath.*, **3** (2), 42.
[6] FUTCHER, D., 1988, *New Educ. Comput.*, June.
[7] WADDINGHAM, J., and WIGLEY, A., 1985, *Algebra and Algorithms*, Secondary Mathematics with Micros in-service pack.