Activity 1: Equivalence of Expressions

Suggested point of insertion: beginning of school year, after students have become acquainted with basic calculator techniques listed below (see beginning of Part I).

[In Parts III-IV of this activity, students will learn to use two CAS algebraic commands and tests of equivalence]

**Lesson 1**

# Part I (with CAS): Comparing expressions by numerical evaluation

**Aim:** A numerical approach as a basis for discussion about equivalence of expressions.

Note to teacher: this activity deliberately avoids using the term “equivalence” or “equivalent expressions” until the class discussion following Part III.

We presume that students will have acquired these basic CAS techniques for the work that follows:

1. Inserting brackets for numerator and denominator of rational expressions;

2. Inserting the explicit multiplication operator (\*) when multiplying two variables, or when multiplying one variable in the leading position by a constant or some other expression.

3. Knowing how to use the “with operator” ( **| )** to evaluate expressions for any given value of x;

4. Knowing how to use arrow and delete keys to change selected parts of text in the entry line;

5. Knowing how to replace text in the entry line with whatever expression is in the “history area” of CAS screen;

6. Knowing how to clear the entry line or any other line of the history screen of the CAS;

7. Habit of verifying, by visual inspection, expressions entered in the entry line.

I (A) Individual work (25 minutes, presuming above skills)

The table below displays five algebraic expressions and two possible values for *x*.

Using the two given values of *x* (i.e., 1/3 and –5) and two others of your own choosing, calculate the resulting values for each expression by means of the evaluation tool of your calculator [i.e., the “with operator”, (**|**)].

Record your choice of additional *x* values in the table’s top row, and record the results in the appropriate cells below.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| For x =  | 1/3 | -5 |  |  |
| Expression | Result | Result | Result | Result |
| 1. *(x*–*3)(4x*–*3)* |  |  |  |  |
| 2. *(x2+x*–*20)(3x2+2x*–*1)* |  |  |  |  |
| 3. *(3x–1)(x2–x–2)(x+5)* |  |  |  |  |
| 4. *(-x+3)2 +x(3x–9)* |  |  |  |  |
| 5.  |  |  |  |  |

I (B) Compare the results obtained for the various expressions in the table above. Record what you observe in the box below.

I (C) Reflection question:

Based on your observations with regard to the results in the table above (in (A)), what do you conjecture would happen if you extended the table to include the other values of *x*?

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## Classroom discussion of Part I A, B, C

Whole-class discussion (to be conducted after completion of Part I A, B, C, 20 minutes)

A point of departure for the discussion will be students’ responses to the reflection question. Teacher can begin by asking, “So what did you notice?” and “What did you conclude?”.

Intended discussion issues[[1]](#footnote-1):

* Some students might attend to the fact that not all cells contain same result (i.e., they might have looked for across-the-board equality). Other students might notice pair-wise equality of results. In response to the latter, ask: “Which pairs of expressions yielded equal results?”
* Suppose some students propose expressions 1 and 4 as yielding equal results on each of the given and their own values of x, then this question can be raised: “Did anyone else notice that this was the case for their choices of values of x (for that pair of expressions)? Are you surprised by this fact? Why or why not?” We anticipate that some students will be surprised by the fact that the two expressions have such different forms. Teacher should bring out students’ ideas in this regard: “In what ways are these expressions different?”
* Suggested continuing questions: “Was there any other pair of expressions for which different forms yielded equal results?”, “Which ones?”

Presumably, some students will propose expressions 3 and 5 (Ask: “Are you surprised by this fact? Why or why not?”). Ask also: “What about expression 2; what can we say about it relative to the other expressions?”

* The following is a rhetorical narrative that we envision the teacher engaging students with: “In the case of each of these two pairs of expressions (1 & 4 and 3 & 5), we noticed that for each value in a diverse set of values of *x* that we chose, expressions 1 and 4 always yielded equal results. The same was true for expressions 3 and 5. Do you think this will always be the case for these two pairs of expressions?” At this point the discussion bifurcates into these issues: (i) one dealing with the domain of an expression’s definition, and (ii) the other dealing with motivating algebraic techniques.

(i) Suggested questions at this point: “Can we choose *any* values of *x* for these expressions?”, “What is the domain of definition for each of the given expressions?” (here, we envision having a discussion about the constraints on *x* for these expressions).

(ii) “For each of these pairs of expressions, having taken account of the constraints on *x*, will any given value of x yield equal results? In other words, can you find a value of *x* for which a given pair of expressions will yield different results? How might we answer this question without having to test all possible values of *x*?” (Note: this last question is designed to motivate the use of algebraic methods and properties to test for equivalence of expressions)

* Even if some pupils propose re-expressing the *forms* of the expressions to obtain a common form for each, the teacher should intervene as follows, in order to help students make the transition from numerical to algebraic approaches:

The teacher says, while writing at the board at the same time, “We could multiply these two factors to put the expression into a different form:

(x+1)(x+2) = x(x+2) + 1(x+2)

 = x2 + 2x + x +2

 = x2 + 3x + 2

Notice that the original and final expressions (as well as all the intermediate expressions) have quite different forms. Now, select any numerical value for x and substitute it in all four expressions (allow a few minutes to have students do this). What do you notice? Why does it happen that all these expressions produce a same numerical value when *x* is replaced by any number?

Now, in a reciprocal way, we have seen in the above table of given expressions that some pairs always yielded equal values when we substituted a given value of *x* (i.e., expressions 1 and 4, and expressions 3 and 5). Drawing on our example above, can you use algebra to convert one expression of a pair into the form of the other expression, or each of them into some common form? **If we can do this, then we are proving that each numerical substitution of *x* always yields a same value for the two expressions.**

Big idea that the teacher should raise: “In multiplying the two factors above, we

re-express a given expression in a different form.” This idea is intended as a segue into Part II of the activity.

Note for the teacher:

If there are some pupils who have difficulty in understanding what is meant by “a common form”, the teacher could use the example above to illustrate that, for example, x2 + 3x + 2 is a common form for both (x+1)(x+2) and x(x+2) + 1(x+2).

**Part II (with paper and pencil, 20 minutes): Comparing expressions by algebraic manipulation**

**Aim:** Using algebra to obtain common forms for given expressions.

II (A) Based on your observations in Part I A and the subsequent classroom discussion, make a conjecture as to which of the above set of given expressions might, in fact, be re-expressed in a common form?

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II (B) To test your conjecture by means of paper and pencil algebra, re-express the given expressions below in another form (not the expanded form). Show all your work in the table’s right-hand column.

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| --- | --- |
| **Given expression** | **Re-expressed form of given expression** |
| 1. *(x*–*3)(4x*–*3)* |  |
| 2. *(x2+x*–*20)(3x2+2x*–*1)* |  |
| 3. *(3x–1)(x2–x–2)(x+5)*  |  |
| 4. *(-x+3)2 +x(3x–9)* |  |
| 5.  |  |

II (C) In Part I C, you made some conjectures based on numerical evaluations of expressions. Explain in what way the algebraic manipulations in Part II B supported (or not) each of those conjectures.

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For any conjectures of Part IC not supported by your algebraic manipulations in Part IIB, how do you account for the discrepancy?

**Activity 1: Equivalence of Expressions**

**Lesson 2: Testing for equivalence of expressions using CAS**

**Part III (with CAS, 20 minutes with discussion):**

**Testing for equivalence by re-expressing the form of an expression—using the EXPAND command**

**Aim**: To use the CAS as a tool to provide information on equivalence of expressions.

Two possible envisioned strategies: (i) Using the machine’s EXPAND command to re-express forms; (ii) Using CAS to test for equivalence without re-expressing form -- by means of an equality test.

The left-hand column of the table below contains the expressions from the previous lesson. Using your calculator, fill in the right-hand column with the expression produced by the EXPAND command (see F2 menu on the calculator).

Syntax: EXPAND(*expression*)

|  |  |
| --- | --- |
| **Given expression** | **Result produced by EXPAND** |
| 1. *(x*–*3)(4x*–*3)* |  |
| 2. *(x2+x*–*20)(3x2+2x*–*1)* |  |
| 3. *(3x–1)(x2–x–2)(x+5)*  |  |
| 4. *(-x+3)2 +x(3x–9)* |  |
| 5.  |  |

## Classroom discussion of Part III

Discussion questions:

1. “What does the EXPAND command seem to do?” (See page 131 of English manual for TI-92plus for description of the EXPAND command.)

2. Concerning expressions 1 and 4:

*a*) “Did you produce the same expanded form as did the calculator for expressions 1 and

4? “

*b*) “Take a close look at the given expressions 1 and 4. Can you factor expression 4 so that it

has the same form as expression 1?”

*c*) “Notice that we have arrived at common forms for the given expressions 1 and 4 in two

different ways : (i) expanding expressions 1 and 4 produced the common form 4x2-15x+9; (ii) factoring expression 4 enabled us to re-express it in the form of expression 1. What do you think of these two different methods for obtaining common forms?”

3. “Can the given expressions 3 and 5 be made to have the same form without expanding?” (by means of factoring or simplifying)

4. “Did your paper and pencil algebra work of the previous lesson and the CAS results just above produce similar findings? In what way?” (this question aims only to *broach,* but not develop, the issue that the forms produced by the CAS may be different from the forms produced by paper and pencil work)

5. Concluding remarks: “Based on our algebraic work and verification using the CAS, can we now conclude that expressions 1 and 4 (the same applies to 3 and 5) can be re-expressed in the same algebraic form?”

Definition of **equivalent expressions:**

We specify a set of admissible numbers for *x* (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces *x*, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.”

“Thus, we can conclude that, on the set of real numbers, expressions 1 and 4 are equivalent. Similarly, on the set of real numbers excluding –2, the expressions 3 and 5 are equivalent.”

**Part IV (with CAS, 20 minutes):**

**Testing for equivalence without re-expressing the form of an expression—using a test of equality**

**Aims**: to understand what happens when we enter into CAS two expressions that are

*a*) equivalent, and *b*) not equivalent.

It is possible to explore whether two expressions are equivalent without having to re-express their forms. An alternative approach is to use a CAS test of equality:

IV (A) Enter directly into your calculator’s entry line the equation formed of expressions 3 and 5:

*(3x–1)(x2–x–2)(x+5)* = 

1. What does the calculator display as a result?

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|  |

2. How do you interpret this result?

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3. Use your calculator’s “with” operator (**|**) to replace *x* by –2 in the above equation.

Interpret the result displayed by the calculator.

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## Classroom discussion of Part IV A

Brief discussion around Part IV A: “In displaying the result “true”, notice that the calculator did not take into consideration the inadmissible values for *x*.”The following distinction between Parts A1 and A3 should be stressed: note that Part IV A3 is not a test of equivalence. Rather, it is a test of numerical equality. Since one of the two expressions cannot be evaluated at *x* = -2, the equality does not hold. Part IV A1, on the other hand, is a test of equivalence.

IV (B) Enter directly into your calculator’s entry line the equation formed from the two given expressions 2 and 3:

 = 

1. What does the calculator display as a result?

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| --- |
|  |

2. How do you interpret this result?

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## Classroom discussion of Part IV B

Brief discussion around Part IV B: “What result does the calculator display?” “Why doesn’t the calculator display ‘false’ in the latter case” (i.e., in question B.2)? Reply: This equation formed of expressions 2 and 3 is neither always true nor always false. Rather, only some values of *x*, when substituted into both sides of the equation, yield equal values. Notice, once again, that in displaying ‘true’ or ‘false’, the calculator has not taken admissible values of *x* into account.

To summarize, we encountered 3 cases when employing tests of equivalence with the CAS:

|  |  |  |
| --- | --- | --- |
| **Case** | **What the CAS displays** | **Remark** |
| Two expressions that are equivalent without any restrictions | “true” |  |
| Two expressions that are equivalent with restrictions | “true” | CAS does not take inadmissible values of *x* into consideration |
| Two expressions that are not equivalent | Same equation input into entry line |  |

**Part V (with CAS, 20 minutes):**

**Testing for equivalence—using either CAS method**

Here is a new set of expressions.

|  |
| --- |
| Given expression |
| 1.  |
| 2.  |
| 3.  |
| 4.  |

V (A) Use your CAS to determine which of these expressions are equivalent. Use whichever CAS method you prefer. Show all your CAS work in the table provided below:

(Note to teacher: This question is deliberately posed in a relatively open manner. The research team is interested in obtaining information on students’ “natural orientations” at this point of the activity -- in particular, whether numerical substitution methods are seen by them as adequate for determining equivalence of expressions.)

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| --- | --- |
| **What you enter into the CAS** | **Result displayed by CAS** |
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V (B) Based on your work above, which are the equivalent expressions (don’t forget to specify the set of admissible values for *x*)? Please explain your decisions about equivalence.

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1. Note to research team: we are attempting to address many deep issues in this discussion. Let us be attuned to the possibility that we may well have to create follow-up activities to ensure that these issues are considered/developed by students in the manner that we desire. We will need to be VERY attentive in class during this activity. [↑](#footnote-ref-1)