Activity 7: Factoring and Solving Equations Involving

Expressions with Radicals

This single-lesson activity will be introduced during the first sequence on algebraic manipulations («Reflection III» in the *Reflections* *IV* textbook) in secondary IV, at the end of the material on quadratic equations (Subject 7 in *Reflections* *IV*), and after students have already acquired basic factoring and equation-solving skills. The activity is intended to have students draw together various elements of the algebra content knowledge they will have acquired since the beginning of the school year.

*Primary idea*: Factoring (taking out a common factor) as a tool for solving equations, particularly when used in conjunction with the “zero product theorem”.

*Secondary ideas*:

• Factoring (taking out a common factor) can be applied not only to constants and variables, but also to algebraic expressions that can be taken as objects to operate upon.

* + Students must be able to recall/reactivate, at a moment’s notice, the methods learned for solving linear and quadratic equations. They should be able to bring these methods to bear when solving equations that are neither linear nor quadratic, per se;
* Simplifying an equation by dividing both sides by some factor may lead to a loss of solutions. In equations in which such simplifications are possible, the strategy of isolating terms on one side of the equation and using the zero product theorem is generally a more effective solving method;
* In equations involving variables under the radical sign, verification after solving is not only advisable, but necessary;

## Anticipated duration:

Roughly 20 - 25 minutes for each of items 1, 2, and 3, including whole-class discussions. Item 4 is left as a challenge problem.

### Pre-requisite knowledge and skills:

* Manipulation rules and properties of radicals and exponents:

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* Solving equations and numerical substitutions.
* Factoring (taking out a common factor) variables and algebraic expressions; recognizing the common structure that results from such factoring.
* Linear equations (items 2 and 3) and quadratic equations (items 1 and 4).
* CAS commands: SOLVE, FACTOR, ENTER, WITH OPERATOR (“**|**”), and understanding some of the limits of the CAS.
* Domain of definition of algebraic expressions involving radicals. Interpreting the meaning of “false” displayed by the CAS.
* Zero product theorem. Problems with division by zero introduced when multiplying both sides of an equation by a common factor for the purpose of simplification.

1. Suppose you were asked to solve this equation:

 (\*)

How would you proceed when faced with such a « monster »?

Using paper and pencil, see whether you can first solve the following equation that is somewhat analogous to the above monster:

(y-2)3 –10(y-2) = y(y-2) (\*\*)

*Hint*: Factoring (taking out a common factor) might be useful here.

Compare your solution with that obtained using the calculator’s SOLVE command. If the solutions obtained are different, verify your paper and pencil algebraic work. If the calculator produced an additional solution to the ones you found, determine which among the paper and pencil algebraic manipulations you used led to the loss of this additional solution.

**For discussion:**

In the course textbook, taking out a common factor is approached without a clear motivation or rationale for its use. Here, the aim of taking out the common factor y-2 is relatively easy to motivate, be it in the expression (y-2)3 –10(y-2) or the expression

(y-2)3 –10(y-2) – y(y-2).

In each case, taking out the common factor enables students to reduce the problem to one of solving a quadratic equation (having solutions y = 6 and y = -1), whether it be by factoring out y-2 on both sides of the equation (y-2) ((y-2)2 – 10) = y(y-2), or by invoking the zero-product theorem in the equation (y-2) ((y-2)2 – 10 – y) = 0. Moreover, the aim is to orient students to the possible “taking out of the common factor” involving the radical expression in the two subsequent items.

Among those students who take out the common factor y-2 on both sides of the equation, some are likely to «lose» the solution y = 2. Whether or not this be the case, however, on the basis of this example the teacher should conduct a classroom discussion about what precautions to take before canceling a factor common to both sides of an equation. In effect, for the values of a variable for which the common factor vanishes, this simplification is tantamount to division by zero! Those values of the variable must therefore always be treated (i.e., verified as possible solutions) one by one, before simplification. It is this very simplification, for which the solution y = 2, given by the calculator, is lost, that we hope students will retain.

The teacher can also help students see how to avoid this problem by using the strategy consisting of: bringing all terms to one side of the equation …



… and invoking the theorem “a product of two factors is zero iff either one of the factors is zero”.

2. On the basis of the strategies employed in solving the previous equation, use paper and pencil to find the solution set of the following equation:

 (\*\*\*)

Then, substitute the values you obtained as solutions for equation (\*\*\*) using your calculator’s “with operator” (“**|**”). What does the calculator display as a result? Are there any solutions that you would discard? Why or why not?

**For discussion:**

The factoring of  will be even easier to do once the solving of the last item has been discussed in class. Still, the teacher is not expected to take up the discussion of the solution *u =* 0 at this point, other than to mention that the problem of simplifying the factor  is the exact analog of that posed in question 1 with the factor *y* – 2.

Here, our aim is, rather, to turn the discussion toward the second solution *u* = –11/2, given by the calculator’s SOLVE command, and which many students will produce themselves. The issue is to have students realize that this solution is inadmissible (when working in R), given the square root. The issue is equally one of impressing upon students to take a critical stance toward the calculator, which produces *u* = –11/2 as a solution, yet displays “false” to the command:

 **|** 

The teacher could discuss the intended meaning of the “false” display. He/she should insist on the necessity of verifying all solutions to an equation that entails a radical, given that the calculator’s SOLVE command does not include such verification.

3. Continuing with paper and pencil, try to solve the original equation (\*). Determine first the condition under which the solutions are admissible, given the radicals. Then, compare your solution with that produced by the calculator and discuss the validity of each value displayed.

**For discussion:**

Students must now integrate various elements developed in the last two items. The condition we are asking them to determine is that *x* ≥ 4. The calculator will display the solutions *x* = 10/3 and *x* = 4 in response to the « SOLVE » command, the first of which must be discarded. But the falsity of  will similarly appear less obvious for students.

The effective management of factoring by taking out a common factor presents additional difficulties here. We expect many students to make errors of the following kind:







…

But verification with the calculator will serve as a method of control, and we expect that students who make such errors will be compelled to re-do their calculations after having verified with the calculator.

4. *A challenge:* solve the following equation using paper and pencil.



What solutions does the calculator display for this equation? Discuss the validity of these solutions.

**For discussion:**

The elements to be worked on in this problem are the same as before, but taken at a higher level of difficulty with regard to managing radicals and exponents



In addition, this problem certainly entails a higher level of calculational complexity, including, notably, solving the quadratic equation 4x2 – 6x – 4 = 0. This produces the two solutions *x* = -1/2 and *x* = 2, one of which is valid while the other must be discarded, given the presence of the square root in the initial equation. As before, the solution *x* = 1/2, which annihilates all the radicals, must be retained.