RECONCILING FACTORIZATIONS MADE WITH CAS AND WITH PAPER-AND-PENCIL: THE POWER OF CONFRONTING TWO MEDIA[1]

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The study presented in this report is part of a larger project on the co-emergence of technique and theory within a CAS-based task environment for learning algebra, which also includes paper-and-pencil activity. In this paper, an example is presented in which 10th graders in Canada use handheld computer algebra to factor expressions of the form $x^n - 1$. The factorizations made by the computer algebra system are compared with the students' by-hand factorizations. This confrontation, and the students' wish to reconcile the two, turned out to be productive for the development of both theoretical understanding and paper-and-pencil and machine techniques. These findings are in line with the anthropological theory of didactics.

THE FOCUS OF THE STUDY

In school algebra, technique and theory used to collide. Both *technique* and *theory* are broader in meaning than *procedures* and *concepts* (Artigue, 2002). The availability of computer algebra systems (CAS) technology in schools, along with the development of theoretical frameworks for interpreting how such technology becomes an instrument of mathematical thought, have both contributed to a recent increase of attention to the interaction between technique and theory.

The research study, of which this report is a part, is an ongoing one. It has as a central objective the shedding of further light on the co-emergence of technique and theory within the CAS-based algebraic activity of secondary school students (Kieran & Saldanha, 2005). Because of space restrictions, this report will highlight the design and findings from one task set on the factorization of expressions of the form $x^n - 1$.

THEORETICAL FRAMEWORK

The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). An essential starting point in the instrumental approach is the distinction between an artifact and an instrument. Whereas the artifact is the object that is used as a tool, the instrument involves also the techniques and schemes that the user develops while using it, and that guide both the way the tool is used and the

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[1] Appearing in J. Novotná, H. Moraová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of 30th PME* (Vol. 2, pp. 473-480). Prague, Czech Republic, 2006.

development of the user's thinking. The process of an artifact becoming an instrument in the hands of a user -- in our case the student -- is called instrumental genesis. The instrumental approach was recognized by French mathematics education researchers (e.g., Artigue, 2002; Lagrange, 2003; Trouche, 2000) as a potentially powerful framework in the context of using CAS in mathematics education.

Chevallard's anthropological theory of didactics, which incorporates an institutional dimension into the mathematical meaning that students construct, describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. By *technology,* Chevallard means the discourse that is used to explain and justify techniques; he is not referring to the use of computers or other technological tools. In their adaptation of Chevallard's anthropological theory, Artigue and her colleagues have collapsed *technology* and *theory* into the one term, *theory*, thereby giving the theoretical component a wider interpretation than is usual in the anthropological approach. Furthermore, Artigue notes that *technique* also has to be given a wider meaning than is usual in educational discourse.

Lagrange (2003, p. 271) has elaborated this latter idea further: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is precisely this epistemic role played by techniques that is a focus of our study, that is, the notion that students' mathematical theorizing develops as their techniques evolve. It is noted, as well, that our perspective on the co-emergence of theory and technique is situated within the context of technological tool use, where the nature of the task plays an equally fundamental role. Thus, the triad Task-Technique-Theory (TTT) served as the framework not only for constructing the tasks of this study, but also for gathering data during the teaching sequences and for analyzing the resulting data.

METHODOLOGY

The research involves six intact classes of 10th graders (15-year-olds) in Canada and Mexico, as well as a class of older students in Oregon. Five of the 10th grade classes were observed during the 2004-05 school year; the sixth class, the following year. One of these 10th grade classes from the 2004 study is featured in this report. This class consisted of 7 girls and 10 boys, all of them considered by their teacher to be of upper-middle mathematical ability Their teacher had five years of experience, and, along with encouraging his pupils to talk about their mathematics in class, believes that it is useful for them to struggle a little with mathematical tasks. He elicits students' thinking, rather than quickly giving them answers. The students in this report had already learned a few basic techniques of factoring (for the difference of squares and for factorable trinomials) and had used graphing calculators on a regular basis; however, they had not had any experience with symbol-manipulating calculators (i.e., the TI-92 Plus CAS machines used in this project).

All project classes were observed and videotaped (12-15 class periods for each of the seven project classes). Students were interviewed, alone or in pairs, at several instances -- before, during, and after class. Thus, data sources for the segment of the study presented in this report include the videotapes of all the classroom lessons, videotaped interviews with students, a videotaped interview with the teacher, the activity sheets of all students (these contained their paper-and-pencil responses, a record of CAS displays, and their interpretations of these displays), and researcher field notes.

Guided by the TTT foundations of the study, we developed a priori descriptions of the techniques and theories that we considered might emerge among students participating in the study. These descriptions provided the lens for gathering and analyzing the data drawn from the classrooms, from student work, and from student interactions. The structure of this article in fact makes explicit these a priori descriptions of technique and theory that we generated, as well as the way in which they served as a focus for our analysis.

TASK, TECHNIQUE, AND THEORY

The activity that exemplifies this theme is inspired by a task described by Mounier and Aldon (1996), two teachers who presented, over the course of a few years, to their classes of 16- to 18-year-old students the task of conjecturing and proving the general factorizations of $x^n - 1$, in a first year with paper-and-pencil, and in a second and third year with computer algebra. However, as Lagrange (2000) pointed out, the CAS factorizations and the paper-and-pencil factorizations are not always identical, and reconciling them is a non-trivial task. This latter issue is central in this paper (see also Kieran & Saldanha, 2006).

First, students discovered the so-called ‘general factorization’ of $x^n - 1$:

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + x + 1)$$

Then, a next part of the activity focused on the differences between CAS factorizations and by-hand factorizations (see Figure 1). A final part of the activity, not discussed here, concerned conjecturing on the forms of the CAS factorizations and on proving these conjectures.

Which techniques play a role in this activity? Table 1 provides an overview, including both paper-and-pencil techniques and CAS techniques. It should be noted, that the relation between the CAS techniques and the corresponding paper-and-pencil techniques is not as close as it might seem. Furthermore, the task of reconciling paper-and-pencil factors and CAS factors can involve a combination of the techniques mentioned in Table 1.

The students' theoretical development includes, for example, the notion of complete factorization, which comes to the fore when students attempt to factor an expression with a non-prime even exponent, such as $x^4 - 1$, according to the general rule, and are confronted with a CAS factorization that they do not anticipate. Reconciling paper-and-pencil factors with CAS factors is considered by us, indeed, an important part of the process of extending students' theoretical views on factoring. Reflection on these issues is related to techniques 1 to 4.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 1. Task in which students confront the CAS factorizations with their paper-and-pencil factorizations (note that the task also included integral values for the exponents up to 13)

Technique	**CAS command (using TI-92 Plus)**	**Paper-and-Pencil variant**
1. Expanding an expression completely	Expand command	Expanding all of the expression by hand, combining like terms, and ordering final terms
2. Expanding a sub-expression	Expand command, using as argument the desired part of the expression	Expanding, by hand, usually two factors of the given expression
3. Factoring completely an expression (if factorable)	Factor command	Factoring by hand, often with a choice of several possible methods
4. Factoring a sub-expression	Factor command, using as argument the desired part of the expression. The CAS may not always succeed in this regard.	Factoring, by hand, a particular factor of a given expression, often with a choice of methods possible

Table 1. CAS and paper-and-pencil techniques for this activity

SOME RESULTS

The students' first surprise while working on the task in Fig. 1 arrived when they entered Factor (x^4-1) into their CAS, which yielded $(x-1)(x+1)(x^2+1)$, in contrast with $(x-1)\left(x^3+x^2+x+1\right)$, which all of them had written for their paper-and-pencil version. It did not take long before students could be heard commenting: "it can be factored further," "it's not completely factored," "it gives you all the factors".

To reconcile the CAS factors with their own paper-and-pencil factors for x^4-1, students did the following:

- Multiplied the second and third CAS factors to produce their second paper-and-pencil factor (Figure 2, half the students);
- Factored by 'grouping' the second paper-and-pencil factor to produce the second and third CAS factors (Figure 3, a little fewer than half the students);
- Refactored the given x^4-1 as a difference of squares (Figure 4, one student).

For some students, the first of the three methods of reconciliation had initially been carried out with the CAS (using Expand) and then transferred to paper. However, most of the reconciliation work was done with paper-and-pencil, as had been requested by the teacher.

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

Figure 2. Reconciling by multiplying the second and third CAS factors of x^4-1 to produce the second paper-and-pencil factor

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x^2(x+1)+1(x+1))$ $(x-1)(x^2+1)(x+1)$

Figure 3. Reconciling by 'grouping' the second paper-and-pencil factor to produce the second and third CAS factors of x^4-1

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$x^4-1=(x^2-1)(x^2+1)$ $(x-1)(x+1)(x^2+1)$

Figure 4. Reconciling by refactoring the given x^4-1 as a difference of squares

In the class discussion that followed the completion of the first set of examples for *n* from 2 to 6 in the factoring of x^n-1, some clarification of the notion of complete factorization took place, which included the teacher's comment that: "Sometimes, they can be factored further. What we did initially is not wrong, it's just not complete." The notion that expressions with even exponents greater than 2 could also be regarded as a difference of squares was not obvious for some students, as suggested by the remark uttered by one student: "I can't get [the factors of] x^4-1". Furthermore, while it was mentioned by a few students that x^6-1 could be treated either as a difference of squares, $(x^3)^2-1$, or as a difference of cubes, $(x^2)^3-1$, the upcoming task involving the factoring of x^9-1 was to provide evidence that *seeing* a difference of cubes was even more difficult for some students than seeing a difference of squares.

Based on their limited set of examples thus far, it was inevitable that most students conjectured that, for odd values of *n*, the general rule seemed to be holding. In other words, they thought that the complete factorization of x^n-1 had exactly two factors for odd ns, while for even values of *n*, it contained more than two factors, of which $(x+1)$ was one of them. The following conversation between two students illustrates how the CAS helped them realize that their conjecture regarding odd ns was incorrect.

Chris — The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can't be broken down anymore. 'Cause it always ends up being all positive. And uh, then, because, it's sort of hard to explain.

Peter — When the exponent is [pause], when the exponent is an even number you'll have more than two factors, but when the exponent is not an even number, you'll have exactly two factors all the time.

Chris — Yeah. [Types Factor(x^7-1) into the CAS]

Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] and [Types in Factor(x^9-1); the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$]

And, no!!! [a look of utter surprise on Chris's face]

In light of the classroom discussion related to confronting paper-and-pencil factors with CAS factors for the expressions from x^2-1 to x^6-1, some students began to adjust their paper-and-pencil factoring techniques with the aim of "playing a game" with the CAS. They tried to anticipate what it would produce as its factored form, for the expressions from $x^7 - 1$ to $x^{13} - 1$, and to thereby reduce the amount of reconciliation that would need to be done.

Within this part of the task where students were confronting their paper-and-pencil factors with the CAS factors, the CAS played a role that was quite different from that which it had played in other parts of the activity. The CAS technique of Factor, with its accompanying output, disclosed to the students that there were certain factoring techniques that they were missing from their repertoire. As a consequence, they wanted to learn these techniques. This need to understand the factored CAS outputs and to be able to explain them in terms of a certain structure, or by means of paper-and-pencil techniques that would produce the same results, seemed important to the students (and to us!).

In all, the confrontation of students' paper-and-pencil factors with the CAS factors led to the development of new theoretical ideas. In the process of making sense of the CAS factors, the students extended their view of the range of the difference-of-squares technique. They came to see that exponents that have several divisors can generally be factored in more than one way. They began to look at expressions in terms of multiple possible structures. Their understanding of the notion of complete factorization evolved. Finally, some students were even able to detect new patterns, and with the aid of the CAS, developed another general rule.

CONCLUSION

From our analysis we conclude that the emergence of theory among the students would not have been possible without the accompanying technical demands raised by the tasks. In fact, the development of CAS techniques, simple as they were in most cases, was vital to theoretical advances. The confrontation of the CAS factored forms with those that students produced by paper-and-pencil was found to be very productive for most students, but especially so for those who, upon realizing that they could not generate the same factors as had the CAS, insisted on finding out how to do so. These CAS encounters resulted in the evolution of not only students' paper-and-pencil factoring techniques but also their theoretic perception of the structure of expressions (e.g., seeing that $x^6 - 1$ could be viewed either as a difference of squares, or as a difference of cubes, or as an example of the general rule that they had earlier generated). The need to make sense of the CAS outputs, and the ability to coordinate these with existing theoretical notions and paper-and-pencil techniques, was fundamental to the students' theoretical and paper-and-pencil-technical progress.

The main finding of the study is that we clearly found evidence for the relation *theory – technique* within the setting of the designed tasks, which confirms the importance and productiveness of the TTT approach. Technique and theory emerge in mutual interaction. The observations show how techniques gave rise to theoretical thinking, and, the other way around, how theoretical reflections led students to develop and use techniques. This interaction proved to be very productive in cases of confrontation, or even that of conflict, between the techniques – particularly the CAS techniques – and the students' theoretical thinking. A tendency to reconcile CAS work and theory was observed; students seemed to strive for consistency, and used the CAS on several occasions as a means of checking their theoretical thinking.

Our research findings lead us to suggest that the epistemic value of CAS techniques by themselves may depend both on the nature of the task and on the limits of students' existing learning. When students cannot explain, in terms of their current theoretical and technical knowledge, that which a CAS technique produces, reliance on additional CAS techniques may not suffice. In such cases, the epistemic value of paper-and-pencil techniques would seem to play a complementary, but essential, role. Recent research that has used the TTT theoretical framework for analyzing the learning of mathematics in technological environments has tended to pay less attention to the role of paper-and-pencil in interaction with CAS techniques in promoting theoretical growth. Our results point to this as a fruitful area for research involving, in particular, young high school algebra learners.

Acknowledgments

The research presented in this report was made possible by a grant from the Social Sciences and Humanities Research Council of Canada – INE Grant # 501-2002-0132. We express our appreciation to the class of students, and teacher, who participated in the research. In addition, we are grateful to Texas Instruments for providing the TI-92 Plus calculators used in the study. It is noted that the affiliations of our collaborators are as follows: A. Boileau, F. Hitt, D. Tanguay: Université du Québec à Montréal; L. Saldanha: Portland State University; J. Guzman: CINVESTAV, Mexico.

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