DEVELOPING TEACHER AWARENESS OF THE ROLES OF TECHNOLOGY AND NOVEL TASKS: AN EXAMPLE INVOLVING PROOFS AND PROVING IN HIGH SCHOOL ALGEBRA

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This research report focuses on the teaching practice of a 10th grade teacher, who participated in a research project involving the use of CAS technology and algebra tasks that were novel to this teacher. The classroom lesson that is analyzed centered on a proving problem embedded within a factoring task that had been engaged in the day prior. A two-fold analysis is presented, the first one focusing on the proving activity, the second one drawing on and connecting the classroom observations with the content of a follow-up interview with the teacher. His reflections during the interview highlight both the new awarenesses that emerged for this teacher during his teaching, as well as the factors that enabled these new awarenesses.

In their review of the emerging field of research in mathematics teacher education, Adler, Ball, Krainer, Lin, and Novotna (2005) have argued that we need to better understand how teachers learn, from what opportunities, and under what conditions. The findings that we recount in this research report provide a compelling case for the particular opportunities and conditions under which the knowledge and teaching practice of one particular teacher of mathematics evolved.

THE CONTEXT OF THE PRESENT STUDY

When our research group developed the project underlying the present study, we decided that the use of new technologies (i.e., Computer Algebra Systems – CAS) for the teaching of algebra would be one of its principal components. Another was the design of novel tasks that would both take advantage of the technology to further the growth of algebraic reasoning and also focus on the interplay between algebraic theory and technique. The theoretical framework that underpinned the research project, one that we refer to as the *Task-Technique-Theory* frame (see Kieran & Drijvers, 2006, for details), draws upon Artigue's (2002) and Lagrange's (2002) adaptation of Chevallard's (1999) anthropological theory of didactics.

The project also involved collaboration with local teachers. The teachers were our practitioner-experts who, within an initial workshop setting, provided us with feedback regarding the nature of the tasks that we were conceptualizing. After modifying the tasks in the light of the teachers' feedback, we requested that, at the beginning of the following semester, they integrate the entire set of tasks into their regular mathematics teaching and that they be willing to have us act as observers in their classrooms. Throughout the course of our classroom observations, which

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occurred over a five-month period in each class, we also offered ongoing support to the participating teachers. In addition, we conducted interviews with some of them immediately after observing certain lessons that we had thought might be considered pivotal moments in their practice. The following narrative concerns one such pivotal lesson, taught by the teacher Michael.

MICHAEL'S STORY

Some Background

At the time of the present study, we had already observed 15 of Michael's classes – classes where he had integrated the CAS-supported tasks that had been created for the research project. Michael was a young teacher whose undergraduate degree and teacher training had been carried out in the U.K. He had been teaching mathematics for five years, but had not had a great deal of experience with using technology in his teaching, except for the graphing calculator. He was a teacher who, along with encouraging his pupils to talk about their mathematics in class, thought that it was important for them to struggle a little. He liked to take the time needed to elicit students' thinking, rather than quickly give them the answers.

Our observations of Michael's class had started at the very beginning of the Grade 10 school year. The students in his class had already learned the basic techniques for factoring a difference of squares and certain trinomials, and had solved linear and quadratic equations. While they had used graphing calculators on a regular basis in the past, it was only at the start of our project that they became familiar with symbol-manipulating calculators (the TI-92 Plus calculator). They had never before done any proving, either in geometry or in algebra. This report concerns the lessons that involved the 'x^n – 1 task', the last component of which was a proof problem. We observed, and videotaped, these lessons. The day after the close of the proving activity, the first author interviewed Michael.

The x^n - 1 Task

The design for this task was an elaboration of earlier work carried out by Mounier and Aldon (1996) with slightly older students. The first part of our task activity, which included CAS as well as paper and pencil, aimed at promoting an awareness of the factor $(x-1)$ in the given factored forms of the expressions x^2-1, x^3-1, and x^4-1, as well as leading to the *generalized* form $x^n-1=(x-1)(x^{n-1}+x^{n-2}+...+x+1)$. The next part involved students' *confronting* the paper-and-pencil factorizations that they had produced for x^n-1 (first with integer values of *n* from 2 to 6, and then from 7 to 13), with the completely factored forms produced by the CAS, and in *reconciling* these two factorizations (see Figure 1). An important aspect of this part of the activity involved reflecting and *forming conjectures* (see Figure 2) on the relations between particular expressions of the x^n-1 family and their completely factored forms. The final part of the activity (see Figure 3) focused on students' *proving* one of these conjectures. This proving activity is the central feature of our analysis.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 1. Task in which students confront the completely factored forms produced by the CAS

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x+1)$ as a factor?
Please explain.

Figure 2. Task in which students examine the nature of the factors produced by the CAS

Prove that $(x+1)$ is always a factor of $x^n - 1$ for even values of n.

Figure 3. The proving task

Our Classroom Observations of the Proving Component of the Task

After students had completed the first two parts of the x^n-1 activity, they were faced with the proving segment of the task. They worked mostly within small groups, for about 15 minutes. Some were using their CAS calculators. Getting students into this proving task was not straightforward for the teacher, as they had never before engaged in such activity. However, with Michael's encouragement, students did make progress. When he sensed that the majority of them had arrived at some form of a proof, he initiated whole-class discussion, with various students sharing their work.

Proof 1: A general approach based on the difference of squares. Paul was the first to be invited to come to the front of the class and to present his 'proof':

Paul: Ok. So, my theory is that whenever x^n-1 has an even value for n, if it's greater or equal to 2, that one of the factors of that would be x^2-1, and since x^2-1 is always a factor of one of those, a factor of x^2-1 is (x+1), so then (x+1) is always a factor.

The teacher then asked: "Is everyone willing to accept his explanation?"

Dan subsequently came forward with what he considered a counterexample, $x^{12} - 1$, to Paul's proof. He proceeded by factoring $x^{12} - 1$ as $(x^6 + 1)(x^6 - 1)$, the latter of which he refactored as $(x^3+1)(x^3-1)$. He then factored (x^3+1) – a sum of cubes – which yielded the sought-for (x+1) factor. He argued that the presence of x^2-1 was not a necessary component of the proof because he (Dan) had shown that, for some even values of *n,* the factoring of x^n-1 does not have to end up with a difference of squares. A sum of cubes could result, and it too would yield (x+1). This led immediately to many students' voicing disagreement. Many of the other students, including Paul, contended that Dan's was not a counterexample, after all. They argued that $x^{12}-1$ could, in fact, produce x^2-1 if it were factored differently:

Paul: Isn't x^6+1 a sum of cubes? … So couldn't you also do the x^6-1 as the difference of cubes [one student says "yeah"] and that's x^2-1.

Commentary on Proof 1. While Paul had seen that x^6-1 could be viewed as a difference of cubes, and thus that x^2-1 was a factor, he did not seem able to link this particular example with his general affirmation that for all even *n*s in x^n-1, one would always arrive at x^2-1 as a factor. Yet, he was quite close. Could he see that x^6-1 was equivalent to $((x^2)^3 - 1)$, even if he had never expressed it in quite this way? This might then have been generalized to expressing x^n-1 for even *n*s as $((x^2)^p$-1) where $n = 2p$. And so because x^n-1 has (x-1) as its first factor, similarly $((x^2)^p$-1) has x^2-1 as its first factor, and thus (x+1) as a factor.

Proof 2: A proof involving factoring by grouping. The second approach to the proving problem was offered by Janet. Janet's proof, which she and her partner Alexandra had together generated, was based on their earlier work on reconciling CAS factors with their paper-and-pencil factoring (for the task shown in Figure 1). They had noticed that for even *n*s, the number of terms in the second factor of x^n-1 (when factored according to the general rule) was always even. Janet argued, as she presented the proof, using x^8-1 as an example, that it would work for any even *n*:

Janet: When *n* is an even number

Teacher: Write it on the board, show it on the board.

Janet: [she writes "x^8-1" and below it: $(x-1)(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$]

Teacher (to the class): Ok, listen 'cause this is interesting, it's a completely different way of looking at it, to what most of you guys did. Ok, so explain it, Janet.

Janet: When *n* is an even number [she points to the 8 in the x^8-1 that she has written], the number of terms in this bracket is even, which means they can be grouped and a factor is always (x+1).

Teacher: Can you show that?

Janet: [she groups the second factor as follows, $(x^6(x+1)+x^4(x+1)+x^2(x+1)+1(x+1))$]

Commentary on Proof 2. Janet's proof, which was generic in that it embodied the structure of a more general argument and was a representative of all similar objects (Balacheff, 1988; Bergqvist, 2005), was one that seemed to be understood and appreciated by most of the students in the class (see Weber, 2008, in this regard). Janet had been able to explain how the terms of the second factor (the factor beginning with the x^7 term) could be grouped pair-wise, yielding a common factor of

$(x+1)$. Her proof appealed to her classmates' common experience in factoring by grouping and led to insights that had not occurred to them before.

Mariotti (2002, 2006) has argued that there is no proof without theory. Similarly, Mariotti and Balacheff (2008) have emphasized that the proving process necessarily starts with the production of conjectures before moving on to proof. In this regard, it is noted that the two-lesson sequence that was devoted to the x^n-1 task involved an interplay between theory and technique, with the development of student conjectures throughout. The ideas that the students generated during the proving task were those about which they had been conjecturing through the entire factoring activity.

The Subsequent Interview with the Teacher, Michael

A 35-minute interview with Michael took place at the close of the proving activity. It inquired into a range of issues, including his views on the research project, as well as his impressions of the most recent activity involving the x^n-1 task.

Low expectations at the start of the project – Extract 1.

Interviewer: Do you now see this technology as playing a different role in your class from the time before the project started?

Michael: Yes. Before it started, I hoped it would be good, but my expectations were not that high about it. I certainly have been very pleasantly surprised with what's happened.

The role played by the tasks and the technology – Extract 2.

Interviewer: How would you describe the impact on the students of this project both mathematically and technologically?

Michael: I think the biggest impact, and the thing I've been most happy with, is the way you guys have designed the activities. It's the way that we've challenged their [the students'] thinking and actually made them think about a process that maybe they knew how to do, but made them think about why they're doing it that way. And I think that's what the calculator has helped them to do and helped them to really, really look at whether they understand the material. … That's something we don't do enough of in mathematics; I think we should do and I really like to do it. … The learning through the technology was amazing. But, the technology is nothing by itself. The amount of work that you put into these activities; that's why they were so successful. … And it's been really good to see how the kids have developed these [the tasks] and worked with them.

Change in his teaching – Extract 3.

Interviewer: Has this project affected your style of teaching in any way?

Michael: I think it's made me think more, or made me realize that what I like is making them think a little bit more. And I think I did that anyway, … but it just made me, just consider a little bit more: Can I let them come through this themselves, let them try this out themselves a little bit more, which I think I always did – but just seeing these activities work, it's made me realize there's more scope to it than I have done in previous years. There is much more scope to let them really go and really know the material properly.

Pushing students to go farther mathematically – Extract 4.

Interviewer: Has the project altered your view of the nature of the mathematics content that can be taught at this level?

Michael: Yes. Because some of the things that you had in those activities I wouldn't have touched. Such as, especially the last activity [the x^n-1 task], you know there's no way I would have gone anywhere with that. It was way beyond anything that they need to know, but just doing that activity was such a fulfilling experience for, not just for me, I spoke with some of the kids afterwards, and they really enjoyed it. They really did! Just going way beyond what they needed to do [in the program] and they were all able to do it. The really nice thing about that activity is that, at the end of it, everyone had something. Even if they didn't all have as nice a little proof as Janet and Alexandra, all of them had worked someway along the lines to get to something. So, so yeah, it certainly opens up things and they couldn't have done that without the technology. So, so for sure is the answer to your question.

Increasing student involvement and promoting learning – Extract 5.

Michael: With this technology, learning goes much further, it is much more involved. … It gives them the extra level of ability, and it involves more students. It gets them into it a lot more. … they could discover things themselves. That is a valuable effect.

Michael had not had high expectations at the outset of the project. This makes the results all that much more interesting. Clearly, one of his strongest impressions of the project was the way in which the tasks and the technology pushed the students to go much farther in their mathematical thinking – so much so that he wanted to continue using the tasks and technology the following year. He also wanted to share the lesson videos of the x^n-1 task with colleagues, just so that they could see what is possible.

ANALYSIS AND DISCUSSION

As stated by Michael, it was his participation in the research project, a project involving technologies and tasks that were novel to him, that led to new awarenesses. These new awarenesses constituted change in his knowledge of mathematics and his knowledge of mathematics teaching and learning, both of which were reflected in his practice of teaching algebra. Mason (1998) has pointed out that it is one's developing awareness in actual teaching practice that constitutes change in one's 'knowledge' of mathematics teaching and learning. While Michael did participate in our professional development workshop prior to his integrating the novel tasks and technology into his teaching, it was his actual practice with these materials that had the greater impact regarding his 'developing awarenesses' regarding mathematics teaching and learning. As we gleaned from the interview, Michael developed at least three new awarenesses:

* An awareness of what students at this grade level can accomplish mathematically – given appropriate tasks (the task aspect was considered very important) – as well as the realization that they can go further mathematically than expected (Extracts 3, 4).
* An awareness of the role that technology can play in the mathematical learning of students (Extracts 1, 2, 4, 5).
* An awareness regarding the culture of the class: it changes when technology is present – students become more involved; they are more autonomous (Extract 5).

Several factors were found to enable the emergence of these awarenesses: a) access to the resources and support offered by the research group; b) use of technologies and tasks whose mathematical content differed from that usually touched upon in class; c) the quality of the reflections of his own students on these tasks; d) his disposition toward student reflection and student learning of mathematics; e) his attitude with respect to his own learning. The first two factors relate principally to the role played by resources 'from without', while the remaining three could be said to be 'from within' in that they concern the given teacher and his students. However, it was in the interaction of the two dimensions that teacher awareness and change were promoted.

Had it not been for the 'from-without' factors, that is, the access to the resources and support offered by the research group and, consequently, the use of technologies and tasks whose mathematical content differed from that usually touched upon in class, then the 'from-within' factors, such as, the quality of the reflections of his own students on these tasks, would not have been put into play. Similarly, had it not been for 'from-within' factors, such as Michael's disposition toward student reflection and student learning of mathematics, as well as his attitude with respect to his own learning, then the 'from-without' factors related to the research team's contributions would not have taken root and flowered. Both types of factors supported each other in a mutually intertwining manner.

This is of interest from a theoretical perspective. It suggests firstly that the integration of novel materials and resources that have been designed to spur mathematical learning is more likely to be successful when the teachers who are doing the integrating see clearly that these resources are having a positive effect on their students' learning. Secondly, the novel materials and resources have a greater likelihood of producing this positive effect on student learning when the teacher doing the integrating engages in teaching practices that encourage student reflection and mathematical reasoning. The synergy between the two types of factors was found to be a major force in the development of Michael's professional awareness, and one that constituted change not only in his knowledge of mathematics and mathematics teaching/learning, but also in his practice.

In conclusion, we wish to emphasize one issue. Much of the research related to teachers' learning from their own practice emphasizes teachers' planning of their interactions with students, followed by their subsequent reflective analysis of these interactions. Considerably fewer studies (exceptions include, e.g., Leikin, 2006) follow the path that we did where the majority of the planning of the instructional interaction with respect to the mathematical content and the task questions to be posed to the students is elaborated in advance by the research team in partial collaboration with the participating teachers. This, we feel, added a dimension to the study that does not often come into play in research on teaching practice. The positive nature of the reflections shared by Michael during the post-lesson interview suggests that the integration of resources coming from without can be a powerful stimulus to teachers' learning from their own practice.

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REFERENCES

Adler, J., Ball, D., Krainer, K., Lin, F-L., & Novotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *ESM, 60,* 359-381.

Artigue, M. (2002). Learning mathematics in a CAS environment. *IJCML, 7*, 245-274.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder & Stoughton.

Bergqvist, T. (2005). How students verify conjectures: Teachers' expectations. *JMTE, 8,*171–191.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*, 221-266.

Kieran, C., & Drijvers, P., with Boileau, A., Hitt, F., Tanguay, D., Saldanha, L., & Guzmán, J. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: CAS use in secondary school algebra. *IJCML, 11,* 205-263.

Lagrange, J.-b. (2002). Étudier les mathématiques avec les calculatrices symboliques. Quelle place pour les techniques? In D. Guin & L. Trouche (Eds), *Calculatrices symboliques. Transformer un outil en un instrument du travail mathématique* (pp. 151-185). Grenoble, Fr: La Pensée Sauvage.

Leikin, R. (2006). Learning by teaching: The case of Sieve of Eratosthenes and one elementary school teacher. In R. Zazkis & S.R. Campbell (Eds.), *Number theory in mathematics education: Perspectives and prospects* (pp. 115-140). Mahwah, NJ: Lawrence Erlbaum Associates.

Mariotti, M.A, (2002). La preuve en mathématique. *ZDM Math Education, 34,* 132-145.

Mariotti, M.A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds), *Handbook Res. on the Psych. Math. Educ.* (pp. 173-204). Rotterdam, NL: Sense Publishers.

Mariotti, M.A., & Balacheff, N. (2008). Introduction to the special issue on didactical and epistemological perspectives on mathematical proof. *ZDM Math Education, 40,* 341-344.

Mason, J. (1998). Enabling teachers to be real teachers. *JMTE, 1,* 243–267.

Mounier, G., & Aldon, G. (1996). A problem story: factorizations of x^n-1. *Int. DERIVE, 3*, 51-61.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *JRME 39*, 431-459.