COMPUTER ALGEBRA SYSTEMS (CAS) AS A TOOL FOR COAXING THE EMERGENCE OF REASONING ABOUT EQUIVALENCE OF ALGEBRAIC EXPRESSIONS

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Equivalence of algebraic expressions is at the heart of transformational work in algebra. However, we know very little about students' understanding of equivalence. This study is part of a larger project that explores the use of CAS as a didactical tool for promoting both technical and conceptual growth in high school algebra with tasks specially designed by the research team. We report on a class of 10th graders coming to grips with the underlying theoretical ideas related to algebraic equivalence and on the role that the CAS played in the shaping of these newly emerging notions. Two different notions of equivalence of expressions were found to emerge: one was purely numerical and entailed reasoning about expressions for which some but not necessarily all numerical substitutions would yield equal values; the other entailed both numerical and common-form reasoning. Interpretation of CAS outputs (the equality test, in particular) played a role in occasioning discussions that don't normally occur in algebra classrooms and led to clarifying distinctions about equivalence for many.

PAST RESEARCH IN THIS AREA

While computers and calculators enabled with symbol-manipulating capabilities have been considered quite appropriate for student use in tertiary-level mathematics courses (Heid, 1988), such has generally not been the case for the secondary-level. Many teachers have tended to stay away from such technology in their classrooms, preferring that their students first develop paper-and-pencil skills in algebra (NCTM, 1999). In contrast, graphing calculators have been adopted on a wide scale -- their use supported by the large number of studies providing evidence of the role that graphical representations can play in enhancing student understanding in algebra (Kieran & Yerushalmy, 2004). This has, of late, encouraged some researchers to begin to investigate whether and how CAS technology can contribute to student learning of secondary school algebra.

Recently, researchers (e.g., Artigue, 2002; Guin & Trouche, 1999; J.-B. Lagrange, 2000) have argued that these new technological tools promote both conceptual and technical growth in mathematics, as long as the technical aspects are not ignored. More specifically, Lagrange (2000) has elaborated the notion that technique is the bridge between task and theory. In other words, as students develop techniques in response to certain tasks, they engage in a process of theory building. The

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technological tools that students use in this theory building become instruments of mathematical thought (Verillon & Rabardel, 1995), permitting both conceptual amplification and reorganization (Pea, 1987). Instrumented techniques thus have an epistemic value, contributing to the understanding of the objects they involve. Instrumentation theory, which articulates the relation between tool use and conceptual development has been applied to the learning of not only calculus but also high school algebra. For example, Drijvers (2003), who focused on the relation between instrumented techniques in a CAS environment and the learning of parameters, noted that the obstacles raised by student difficulty in reconciling CAS output with expected results presented opportunities for learning, when addressed within whole-class demonstrations and discussions.

In another CAS study (Ball, Pierce, & Stacey, 2003) dealing explicitly with equivalence of expressions, researchers found that students could not recognize equivalent expressions, even simple cases, and noted that, "the ability to recognize equivalent forms of algebraic expressions is a central part of working with CAS and one that is likely to take on new importance in future curricula" (p. 4-16). Similarly, Artigue (2002) drew upon students' work involving the passage from one given form of an expression to another to illustrate how the research team paid specific attention to the fact that "equivalence problems arise which go far beyond what is usual for the classroom." She used the CAS as a "lever to promote work on the syntax of algebraic expressions, which is something very difficult to motivate in standard environments" (p. 265), adding that its use obliges students to face equivalence and simplification issues. According to Nicaud et al. (2004), "reasoning by equivalence is a major reasoning mode in algebra; it consists of searching for the solution of a problem by replacing the algebraic expression of the problem by equivalent expressions … identities allowing for the transformation of expressions while maintaining equivalence" (pp. 171-2). The importance accorded to the notion of equivalence of expressions, as well as students' reported difficulties in this area, suggests that we need to know a great deal more about the ways in which algebra learners think about equivalence of expressions. Wishing to build upon the recent work that has been initiated by researchers in this area, we designed a study that uses CAS as a lever for promoting student work, *and reflection,* on equivalence of expressions.

THE STUDY

The study reported in this paper is part of an ongoing three-year project involving five intact classes of 10th graders (15-year-olds). These students have been following an integrated program of mathematics since the 7th grade, which means that algebra is part of the course of study each year. One of these five classes is featured in this report. Students in this class had learned basic techniques of factoring and the solving of linear and factorable quadratic equations during the previous year and had used graphing calculators on a regular basis; however, they had not had any experience with the notion of equivalence or with symbol-manipulating calculators. They were quite skilled in algebraic manipulation, as was borne out by the results of a pre-test

administered at the outset of the study. It was during the algebra part of their 10th grade mathematics course, when the activities designed by the research team, accompanied by CAS technology (TI-92 Plus calculators), were integrated into their regular program of mathematics and taught by the classroom mathematics teacher.

The Design of the Activities

Of the eight activities created by the research team, each one designed to take up about two 65-minute-long class periods, three dealt with equivalence of expressions. Each activity was punctuated by parts, each part including presentation of student work and discussion of the main issues raised by the tasks in the given part. Tasks were of three types that involved either work with CAS, or with paper/pencil, or were of a reflective nature. For every activity, there was an accompanying teacher version that included suggestions for classroom discussion. In designing these tasks, we took seriously both the students' background knowledge and the fact that these tasks were to fit into an existing curriculum; but we also moved to ensure that they would unfold in a particular classroom culture that reflected a certain priority given to discussion of serious mathematical issues.

Activity 1: Equivalence of Expressions
  Part I (with CAS): Comparing expressions by numerical evaluation
  Part II (with paper/pencil): Comparing expressions by algebraic manipulation
  Part III (with CAS): Testing for equivalence by re-expressing the form of an
      expression – using the EXPAND command
  Part IV (with CAS): Testing for equivalence without re-expressing the form of an
      expression – using a test of equality
  Part V (with CAS): Testing for equivalence – using either CAS method
Activity 2: Continuation of Equivalence of Expressions
  Part I: Exploring and interpreting the effects of the ENTER button, and the
      EXPAND and FACTOR commands
  Part II: Showing equivalence of expressions by using various CAS approaches
Activity 3: Transition from Expressions to Equations
  Part I (with CAS): Introduction to the use of the SOLVE command
  Part II (with CAS): Expressions revisited, and their subsequent integration into
      equations
  Part III (paper/pencil): Constructing equations and identities
  Part IV (with CAS): Synthesis of various equation types

Figure 1: Outline of content of the three activities related to equivalent expressions

In these activities, *equivalent expressions* were defined as follows: "We specify a set of admissible numbers for *x* (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces *x*, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values." As seen in Figure 1, which displays in outline form the content of the three activities related to equivalence, numerical evaluation of expressions served as the entry point for discussions of equivalence. The impossibility of testing all possible numerical replacements in order to determine

equivalence motivated the use of algebraic manipulation and the search for <u>common forms</u>. Discussion included attention to restrictions on equivalence. The relation between equivalent/non-equivalent expressions and equation solutions was then explored in both CAS and paper/pencil tasks.

Classroom Set-Up and Data Sources

Two video cameras were set up in the classroom, one in the front and fixed in position, and a second one in the rear that pivoted and zoomed. Two ceiling-mounted microphones provided sound that was mixed and directed into each camera. One or two researchers took field notes during each class period. Two students were interviewed (and videotaped) after each class period in order to further probe their thinking. A post-test involving CAS use was administered after the three sets of activities had been completed. Thus, data sources for the segment of the study analyzed for this report include the DVD-captured videotape of all the classroom lessons involving the three activities, individual interviews with nine students, transcripts of selected video segments, the completed activity sheets of all students, written pre- and post-test responses, and researcher field notes.

RESULTS AND DISCUSSION

Responses to one of the post-test questions in particular, Question 5(iii) (see Figure 2), suggested that students' emerging ways of thinking about equivalence were being confounded with notions of equality. While for part (i) 93% of the students were coded with a correct score, and 100% of them used a valid CAS method to show that there are no other solutions in part (ii), only 60% of them correctly answered part (iii). Examples of students' correct answers to Q.5(iii) were:

- "They are not equivalent, because when x stays as x and both sides are factored out they are not identical. These expressions can be equal when x is replaced by 2 or 2/3, because the two sides would be identical."
- "They are not equivalent, as only when 2 and 2/3 are plugged in as values of x are the expressions equal. They cannot be put into common form."

Incorrect answers included the following:
- "They are equivalent when the numbers you are substituting are $x = 2$ or $x = 2/3$. Not any other numbers."
- "The expressions are equivalent since they both have the same solutions."

While it could be argued that students' incorrect responses were merely an indication of linguistic error (i.e., using "equivalent" instead of "equal"), we wondered whether the problem might be deeper than that and might reflect an interpretation of equivalence purely in numerical terms and an absence of the relevance of the idea of algebraic form as a tool of algebraic thinking. Furthermore, we asked ourselves whether the ways in which they were using the CAS tool or whether the ways in which the CAS displayed its outputs could be serving to reinforce numeric interpretations to the exclusion of those related to form.

> **Q.5** The following equation has $x = 2$ and $x = 2/3$ as solutions:
> $$x(2x-4)+(-x+2)^2 = -3x^2+8x-4$$
> (i) Precisely what does it mean to say that, "the values 2 and 2/3 are solutions of this equation"?
> (ii) Use the CAS to show that: (a) the two values above are indeed solutions, <u>and</u>
> (b) there are no other solutions.
> What I entered into the CAS:
> What the CAS displays <u>and</u> my interpretation of it:
>
> (iii) Are the expressions on the left- and right-hand sides of this equation equivalent? Please explain.

Figure 2: Post-test question (see part iii) that revealed interpretations of equivalence

Correct answers to Q.5(iii) had generally incorporated two components, that of being able to represent equivalent expressions in some common form and that of producing the same numerical value for equivalent expressions on substitution by each of an infinite set of values. Incorrect answers did not refer to common form. Of the 60% of those who were successful in answering Q.5(iii) on the non-equivalence of the two given expressions, 88% of them used some type of common-form argument in their reasoning. (See Saldanha & Kieran, 2005, for further discussion of this result.)

We note also that only two students used the equality test to determine whether the two given expressions are re-expressible in a common form. Indeed, our analysis of students' work sheets for the activities preceding the post-test, as well as the interview data, revealed that many students were puzzled by the equality test.
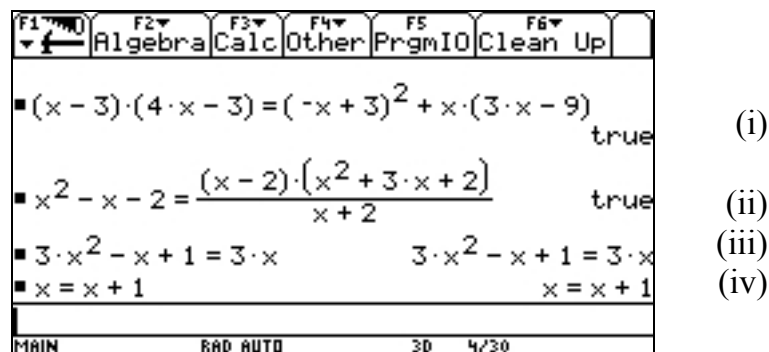

Figure 3: Output from the TI-92 Plus for four cases of equivalent and non-equivalent expressions using the equality test

Figure 3 illustrates the TI-92 Plus output when the equality test is used for: (i) two equivalent expressions with no restrictions, (ii) two equivalent expressions with one restriction, (iii) two non-equivalent expressions that are equal only when $x = 1$ or $x = 1/3$, and (iv) two non-equivalent expressions that are never equal. As can be seen, this test produces *true* when an equation is entered that is formed from two equivalent expressions (note, however, that the restriction to the equivalence is not displayed); on the other hand, the CAS simply displays the input-equation when the left- and right-hand-sides of the equation are not equivalent expressions. Thus, with its two

distinct outputs – either *true* or the input-equation -- the equality test can serve as a test for equivalence of expressions.

The latter output was, however, difficult for students to interpret. This is illustrated by the following excerpt, drawn from an interview with a student (S) immediately after classroom work on Activity 1:

S: When I see "true", I figure that that there aren't any exceptions [having entered: *(3x–1)(x²–x–2)(x+5) = (x²+3x-10)(3x-1)(x²+3x+2) / (x+2)*]. Like, I figure if it says "true" all the time, it would always, no matter what you put *x* as, it would be equivalent.

I: Ok. Let's go on to the next part here. [turning page] Alright, do you remember that one?

S: Uhm, I entered the problem [$(x^2 + x - 20)(3x^2 + 2x - 1) = (3x - 1)(x^2 - x - 2)(x + 5)$] and it gave me pretty much the same problem back, but rearranged, it's the same answer. When you think that the other one said "true," it is kind of puzzling. ... The answer that it gave me. I figure that that's this statement, like the first expression equals the second expression is true. … When I see an equal sign, I figure they are equivalent, the same.

I: Is there anything you might do to check whether your hunch about what that means is right indeed?

S: I'd expand it. Then you see all the parts.

I: Did you actually do that as a follow up?

S: I did, myself, but I don't think we did, as a class.

I: And did it corroborate what you

S: No, because when you expand them, they're different.

I: Would you revise your response to this in light of that?

S: Yes, but I still don't understand why it would tell me that it's equivalent.

I: So, if I understand you correctly, to you it's sort of mysterious why

S: It's misleading.

While this student evidently thought that, when a statement with an equal sign is returned, such a display should still signal equivalence of the two given expressions, others did not know what to make of such output. They thought that the CAS should display *false* if the two expressions that formed the equation were not equivalent, especially for cases such as *x = x+1*, which they argued "could never be true." They had seen *false* displayed when they had substituted certain values into equations containing restrictions, and thus expected *false* for certain cases when using the equality test. Naturally, there was considerable classroom discussion around this issue when it was first encountered during Activity 1. Teacher and students alike both referred to some of the prior numerical substitutions they had made with these expressions, pointing out that equations made from such expressions were sometimes true and sometimes false, depending on the numerical value being substituted. If an equation formed of two expressions was not always true (subject, of course, to a few restrictions), they argued, then the expressions forming that equation were not equivalent and never could be. But because there were sometimes exceptions to the equivalence (in the case of inadmissible values, which were not always obvious to the students), some believed there might be exceptions to the "non-equivalence of expressions," which they were similarly not seeing. Thus, the fragility of students'

emerging knowledge of equivalence was being exposed by their difficulty with interpreting some of the CAS displays.

Activity 3 introduced students to the SOLVE command and its use in interpreting the equivalence of the given expressions of an equation. Students seemed more at ease with the CAS outputs of SOLVE: *true*, some solutions, or *false*. They were on more familiar ground; but it was a numerical ground (solutions that included all real numbers, only some real numbers, or no real numbers). Thus, those equations that had been returned "as is" by the equality test, and which had been uninterpretable by many, were now seen in a clearer light. However, as illustrated by some of their post-test responses, the expressions on the two sides of an equation for which there were some solutions were being considered as equivalent expressions, that is, "equivalent for certain values of *x*." With their purely numerical interpretations of equivalence, they tended to lump into the category of equivalence both those pairs of expressions whose equations were true for all, as well as those with only a few solutions. In contrast, those students whose numerical interpretations were accompanied by "common-forms" reasoning (i.e., only those pairs of expressions forming equations that were true for all could be expressed in common form) correctly distinguished equivalent from non-equivalent expressions. The ramifications of the former kind of reasoning on students' understanding of the process of equation solving where sub-expressions of the equation are to be replaced by equivalent expressions must surely be obvious.

While the CAS equality test, with its "mysterious" output, alerted students to something that they needed to think deeply about and brought their conceptual difficulties to the surface, clearly not all of them were resolved. As students continue to work with equation-solving procedures during the year and more attention is paid to the role of common-form arguments in talking about equivalent expressions, equivalence of expressions may slowly become disentangled from equality of expressions. As students gain more experience with the CAS, along with and in interaction with their emerging knowledge on equivalence of expressions, they may come to use the equality test, among others, with confidence. Ball, Pierce, and Stacey (2003) have pointed out that the students in their CAS study had a great deal of difficulty in recognizing equivalent expressions. To compensate for this difficulty, we suggest that students need to be adept at using various CAS tests in order to determine equivalence of expressions, but they also must have a clear idea of what is meant by equivalent expressions. The report of this part of our study has described our initial efforts toward understanding students' grappling with both these issues.

CONCLUDING REMARKS

In this study, CAS was not intended to replace paper/pencil work as a technical tool. Nor was it to be simply a means for checking paper/pencil work. It was, however, intended for use as a didactical tool for coming to grips with underlying theoretical ideas in algebra. Some of the components of our activities that made CAS a

didactical tool were questions that used the machine to occasion discussions that don't normally happen in mathematics classes. Tasks that asked students to write about how they were interpreting their work and the related CAS displays bring mathematical notions to the surface, making ideas and distinctions much clearer, in ways that simply "doing mathematics" may not require. If CAS is to be effective at the high school level, it is precisely this kind of usage that needs to be considered.

Acknowledgments

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