THE ROLE OF COMPUTER ALGEBRA SYSTEMS (CAS) AND A TASK ON THE SIMPLIFICATION OF RATIONAL EXPRESSIONS DESIGNED WITH A TECHNICAL-THEORETICAL APPROACH

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In this report we analyze and discuss the role of CAS with two 10th grade students on a task related to simplifying rational algebraic expressions. The theoretical elements adopted in this study are based on the instrumental approach. Results indicate that CAS and a technical-theoretical-oriented task provoked students to theorize on certain aspects of the simplification of rational expressions, thus illustrating the role of CAS in improving specific technical-theoretical components of algebra learning. However, the results also indicate that good task activity and CAS may not be enough for other related technical-theoretical understandings in this domain; the teacher's intervention may also be necessary.

Introduction

In the past few years, many research studies have reported the potential of calculators (e.g., Computer Algebra Systems, CAS) in facilitating symbolic manipulation in algebra learning (e.g., Kieran & Damboise, 2007; Kieran & Drijvers, 2006; Thomas, Monahan, & Pierce, 2004; among others). For instance, Kieran and Damboise (2007) point out how students who are weak in algebra can improve both technically and theoretically by means of a CAS experience involving the factoring of algebraic expressions such as $x^2 + px + q$, with p and q whole numbers. Taking into account Kieran and Drijvers (2006), it can be seen how the use of calculators and tasks that promote the interaction between CAS and paper-and-pencil environments leads to increasing the quality of students' algebraic thinking.

One concept that the algebra research community considers central to algebra learning is that of the equivalence of algebraic expressions. Kieran (2004), for example, views this concept to be a critical component of algebraic transformational activity. Not incidentally, it is well known (e.g., Davis, Jockusch & McKnight, 1978; Matz, 1980; among others) that students make frequent, common and persistent errors when they try to simplify rational expressions, appearing to neglect issues of equivalence. Some researchers have studied student thinking in this domain within CAS environments (e.g., Ball, Pierce & Stacey, 2003; Kieran & Drijvers, 2006). However, the majority of the reported research on algebraic syntax errors made by students has been carried out in paper-and-pencil environments (e.g., Booth, 1984; Davis, Jockusch & McKnight, 1978; Matz, 1980; Kirshner & Awtry, 2004; among others).

In the reports of these studies, different explanations are offered regarding the origins of many of these errors, as well as a variety of remedial teaching treatments (e.g., the use of different syntactical approaches) to help students overcome such errors. With respect to the use of technology-supported instructional treatments, among the few studies we note the research of, for example, Sutherland (1991, pp. 40-41) who points out that a computational environment influences students' algebraic conceptualization and helps them to overcome difficulties in the comprehension of algebraic symbolism. Similarly, Tall and Thomas (1991) argue that a computational environment helps students to overcome difficulties and errors due to their interpretation and notion of variable. Another related study on the influence of technology in the learning of algebraic concepts is by Guzmán and Martínez (2009), who point out that CAS is useful in the sense that it helps students to identify certain kinds of frequent and persistent algebraic errors. However, little is known about the influence of this kind of technology on students' thinking in relation to the simplification of rational expressions — a topic in which common algebraic syntax errors are made in a very persistent way due to a lack of conceptual comprehension on the part of students regarding the manipulation of this kind of expression.

Thus, the aim of this study is to answer the following research question: How do CAS and a task designed with a technical-theoretical approach influence students' thinking about the simplification of rational expressions?

Theoretical Framework

The instrumental approach to tool use has been recognized as a framework rich in theoretical elements for analyzing the processes of teaching and learning in a CAS context (e.g., Artigue, 2002; Lagrange 2003; Trouche, 2005; among others). The instrumental approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). According to Monahan (2005), one can distinguish two directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics. In the former, the focus is the development of mental schemes within the instrumental genesis process. Within this approach, an essential point is the distinction between artifact and instrument (for more details see Drijvers & Trouche, 2008).

In line with the anthropological approach, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology (such as CAS). This approach is grounded in Chevallard's anthropological theory. Chevallard (1999) points out that mathematical objects emerge in a system of practices (praxeologies) that are characterized by four components: *task* (expressed in terms of verbs), in which the object is embedded; *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology. Artigue (2002, p. 248) and her colleagues have reduced Chevallard's four components to three: *Task, Technique,* and *Theory*, where the term *Theory* combines Chevallard's *technology* and *theory* components.

Within this (Task-Technique-Theory) theoretical framework, not only does the term *theory* have a wider interpretation than is usual in the anthropological approach, so too does the term *technique* have a wider meaning than is usual in educational discourse. Here, a *technique* is a manner of carrying out a task; it is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values (Artigue, 2002, p. 48). For Lagrange (2003, p. 271), technique is a mixture of routine work and reflection; it is a way of doing a task and it plays a pragmatic (in the sense of accomplishing the task) and epistemic role. With regard to the epistemic value of technique, Lagrange (2003, p. 271) has argued that: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency". So, this epistemic value of techniques is crucial in studying students' conceptual reflections within a CAS environment. In our study, this T-T-T framework was taken into account in all aspects, including the designing of the tasks to be used, the conducting of the interviewer interventions, and the analyzing of the data that were collected.

The Study

This report is part of a wider research on common literal symbolic errors and the use of CAS. The analysis and results presented here are based on a pilot study that involved three sets of task activities, each one designed to take into account a different common error in algebraic syntax. Here, we analyze and discuss just a part of one of those activities.

Methodology

The Population

The participants were eight 10^{th} grade students (15 years old) of a Mexican public school. The selection of the students was made by their mathematics teacher, who believed that they were strong algebra students. It is also well known that students of this particular grade level make errors in trying to simplify rational expressions – the kinds of errors that we were interested in investigating. None of the students were accustomed to using CAS calculators; consequently, at the outset of the study, all the students received some basic training from the interviewer on how to use the calculator for basic symbol manipulation (use of the commands FACTOR, EXPAND, and SOLVE).

Task Design

The task proposed in this study concerns the simplification of rational expressions, both with paper and pencil and CAS. In this report, we use the term *activity* to refer to "the set of questions related to the task". For the design, theoretical elements from the instrumental approach were used. In other words, the activity was designed so that technical and theoretical questions were central to the task and, hence, that students would have the opportunity to reflect on both the technical and theoretical aspects in both paper-and-pencil and CAS environments. In the present report, only the following parts of the activity are reported: first, students' paper-and-pencil work (with technical and theoretical questions); second, their subsequent CAS work (technical question); and, finally, theoretical questions related to their work in both environments.

Implementation of the Study

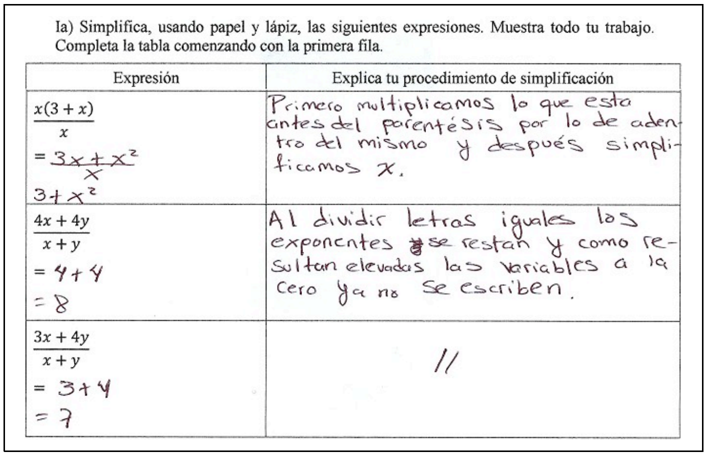
The data collection was carried out by the interview method, led by the researcher. Students worked in pairs; each work session lasted between two and three hours. Each team of two students had a printed activity as well as a TI-Voyage 200 calculator. Every interview was audio and video-recorded to register the students' performance during the sessions.

Analysis and Discussion of the Data

In this report, because of lack of space, we analyze and discuss only one team's work on a subset of the actual task questions that were proposed. This team was chosen (we'll call each member of the team student A and student B) for this report because we consider that their work is typical and represents the role played by both the CAS and the designed task. The analysis, which is qualitative in nature, is based on the team's work sheets, as well as the video-recorded interview. The analysis and discussion of the data is detailed below.

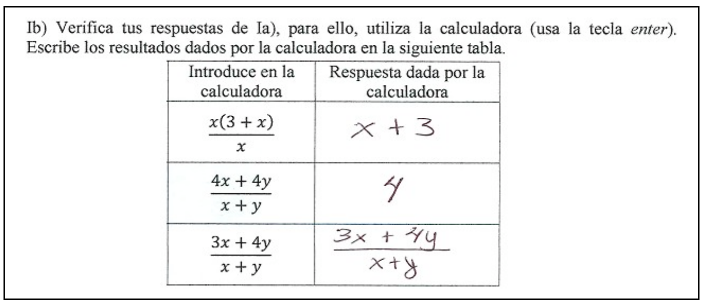
The Role of the Proposed Task

We take the first part of the task as solved by the students A and B. As per the task design, the first section of the activity helped us to know how the students simplify, with paper and pencil, the given rational expressions. Figure 1a illustrates the students' techniques and their explanations related to the simplification of rational expressions. From this, we confirm that, in this environment, students made the expected errors: they eliminated the 'literal components' that were common to both numerator and denominator, without taking into account whether these ‘literal components’ were, in fact, a factor of both the numerator and the denominator.


Ia) Simplifica, usando papel y lápiz, las siguientes expresiones. Muestra todo tu trabajo. Completa la tabla comenzando con la primera fila.

Expresión	Explica tu procedimiento de simplificación
$\frac{x(3+x)}{x}$ $= \frac{3x+x^2}{x}$ $3+x^2$	Primero multiplicamos lo que esta antes del paréntesis por lo de adentro del mismo y después simplificamos x.
$\frac{4x+4y}{x+y}$ $= 4+4$ $= 8$	Al dividir letras iguales los exponentes se restan y como resultan elevadas las variables a la cero ya no se escriben.
$\frac{3x+4y}{x+y}$ $= 3+4$ $= 7$	//

Figure 1a. Simplification of expressions: Paper and pencil work.


Ib) Verifica tus respuestas de Ia), para ello, utiliza la calculadora (usa la tecla *enter*). Escribe los resultados dados por la calculadora en la siguiente tabla.

Introduce en la calculadora	Respuesta dada por la calculadora
$\frac{x(3+x)}{x}$	$x+3$
$\frac{4x+4y}{x+y}$	4
$\frac{3x+4y}{x+y}$	$\frac{3x+4y}{x+y}$

Figure 1b. Simplification of expressions: CAS work.

This part of the task allowed students to describe their paper-and-pencil technique for simplifying these expressions. We note that their explanation uses the terminology of *dividing* (see the second example of Figure 1a, where the students wrote, “we divide the same letters”). We also note that, whenever there are parentheses, the students first expand the expressions of the numerator and denominator before cancelling (see the first example of Figure 1a). The fact that the students spontaneously expanded expressions and didn’t first observe them in terms of factors was something that hindered their theoretical reflection and seemed to lead them to make the kinds of errors that are reported in the literature.

Next, the students arrived at the part of the activity where they used the CAS calculator (Figure 1b). This allowed them to contrast their paper-and-pencil results with those obtained from the CAS. The differences between the two sets of results led them to wonder about their paper-and-pencil techniques and explanations. They began to question the theoretical underpinnings of their work. Hence, the design of the task (technical and theoretical questions in both environments) clearly led the students to confront their paper-and-pencil results (i.e., their initial techniques and theory) with the ones obtained from the calculator and promoted the search for other techniques in order to explain the CAS results.

Theoretical Reflection Promoted by the CAS in Students

From the analysis of the students’ conversation following the surprising CAS results, it was clear that the results given by the CAS (Figure 1b) provoked in students a conceptual change (the epistemic role of the techniques, in this case the CAS technique). In other words, the use of the CAS in the context of the designed task led the students to rethink their techniques and explanations and provoked a theoretical reflection that could explain for them the results given by the CAS. For the expressions that involve just one term in the denominator (as in the first example of Figure 1a), the students could see that their paper-and-pencil technique was not correct, but could also see how to fix it. As the following extract suggests, they were able to make a quick adjustment to this technique so as to eliminate the discrepancy between the results:

[1] Student A: What is it? [*She asked for the result given by the calculator for the first expression of Figure 1b*]
[2] Student B: *x* plus 3 [*the CAS result for the first expression of Figure 1b*]
[3] Student A: And we wrote 3 plus *x* squared [*She refers to the result which they got by paper and pencil for the first expression of Figure 1a*]
[4] Student B: Yes. We must've taken off only one *x* [*Meaning that they had to eliminate another x*]. No matter. What's next?

It seemed a minor matter to student B to make this kind of adjustment to their technique for handling the simplification of rational expressions with a single term in the denominator – an adjustment that called for cancelling each occurrence of the given term in the numerator. But it is noted that no accompanying theoretical justification seemed forthcoming. However, for the second and third examples, the students could not easily come up with a simple adjustment to their paper-and-pencil technique for simplifying rational expressions containing a binomial as the denominator and two terms in the numerator – that is, an adjustment that would allow them to arrive at the same result as that produced by the CAS. The following extract illustrates their bewilderment at the CAS result for the second expression:

[5] Student B: Four [*She refers to the result obtained by the calculator for the second expression of Figure 1b*]
[6] Student A: Uh! [*Expressing surprise*] We are wrong too [*Because it doesn't match up with their paper-and-pencil result for the second expression of Figure 1a*]
[7] Student B: But I don't know why. Here I don't really know why.
[8] Student A: Neither do I.

For this expression, the students had cancelled the *x* in the numerator and the denominator, as well as the *y* in both, which had left them with 4 + 4, which they simplified to 8. Yet, the CAS result had been 4. They were similarly unable to explain why, for the third expression, the CAS did not simplify at all, but merely returned the given rational expression:

[9] Student B: Yes, here [*Referring to the first expression of the Figure 1b*], it makes sense [*the result given by the calculator*] because the *x's* were taken off, it first multiplied and we missed taking off the two *x's*. [*She states the multiplication procedure that she thinks the calculator did, just as they had expanded the numerator of the first expression of the Figure 1b*]. But in here, I'm not quite sure why it's 4, neither the result in here [*Referring to the last two results (Figure 1b) given by the calculator*]. Why it is the same [*referring to the 3rd result*], I don't have any idea.

The two students continue to think about the discrepancies. While they could accommodate the result given by the CAS for the first example, the other two examples remained mysterious. They kept asking themselves if there were other ways to think about these simplifications. How might they justify the results offered by the CAS? The following extract underlines their dilemma, but then Student A suddenly had an idea:

[10] Student B: It's believed that in this case we should've taken off the *x* and the *y*, we take off both [*The repeated terms in the numerator and the denominator of the 2nd expression in Figure 1b*]. But why is it 4? [*The result given by CAS*]
[11] Student A: Let's see [*Pause*]. This is a division of polynomials!

In line [11], it is clear that the CAS result, for these two examples, has provoked a conceptual change in one of the students, which was understandable by the other. The theoretical reflection induced by the discrepant results moved the students from a technique involving eliminating literal symbols that are repeated in the numerator and the denominator to a technique involving division of polynomials. Thus, the numerator and denominator of the rational expression are now viewed as dividend and divisor respectively. But this new theoretical perspective, and its related technique of dividing the numerator by the denominator, is deemed necessary (by the students) only for rational expressions where the denominator is a binomial. For those cases where the denominator is a monomial, they continue to believe that the technique of cancelling the monomial of the denominator with all of its occurrences in the numerator is workable:

[12] Student A: In numbers 2 and 3 [*The last two expression of Figure 1a*], we didn't divide well.
[13] Interviewer: What do you mean by you didn't divide well?
[14] Student A: Ok, well, in here… we took off the *x's* and the *y's*. And [*Pause*].
[15] Interviewer: And can't we do that?
[16] Student A: Well [*Pause*] it would do if it were a division of a polynomial by a monomial.

The interviewer then asked the students to illustrate their new paper-and-pencil technique and to show (see Figure 2) how it helped them to avoid the errors they had made when simplifying the last two expressions of Figure 1a.

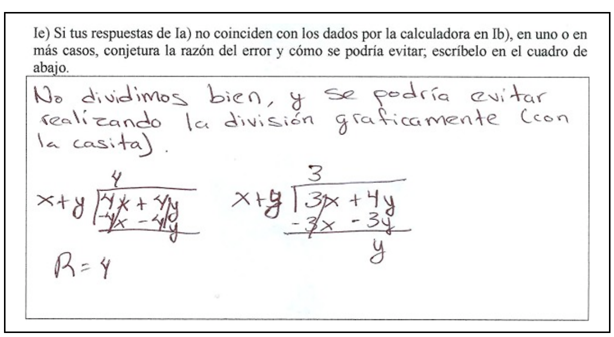


Figure 2. The new technique to simplify rational expressions.

Finally, based on their new technique (long division of polynomials) for simplifying rational expressions containing a binomial for the denominator, the students were able to explain the results given by the CAS. For them, to simplify such a rational expression came to mean that one should divide (numerator divided by denominator). They found, on their own, that if the quotient works out exactly, then the rational expression can be simplified -- the quotient of the division being the final simplification. But if the division is not exact, then the rational expression can't be simplified and the CAS calculator will give as the result the same expression. It's interesting to see how the students came to adapt their new technique and theory so as to make it also fit the case of rational expressions that could not be simplified. And so, through the technique of polynomial division, the students came explain the results given by the CAS. However, the connections between this technique and the one they used to simplify rational expressions where the denominator was a monomial were never made. For these connections, teacher intervention would clearly be necessary.

Conclusions

The present report shows that CAS and a task that promotes technical-theoretical thinking, within an environment involving the reconciling of paper-and-pencil and CAS results, can provoke in students a conceptual change regarding the simplification of rational expressions whose denominators are binomials. The change we observed involved moving from, on the one hand, simplifying expressions through the technique of cancelling out terms that were repetitive in the numerator and in the dominator to, on the other hand, the division of polynomials technique. The use of CAS to verify paper-and-pencil work, and the obtaining of surprising results by the CAS, led students to a theoretical reflection that provoked the use of a new technique and new theoretical explanations for both justifying the CAS results and rethinking their paper-and-pencil procedures of simplification. However, such theoretical reflection was not enough for them to understand the simplification of rational expressions in terms of cancelling out common factors of the numerator and denominator. These results thus suggest that, in spite of good tasks and the use of CAS, in order for students to more fully understand rational expressions and their simplification, including the relation between polynomial division and factored forms within rational expressions, the importance of teacher intervention is inescapable.

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