Name:

Activity 8: Systems of Equations

**Lesson 1**: **An introduction to systems of equations**

**Part I (with CAS): Using numerical evaluation to test solutions of given types of equations**

(A) First-degree equations in a single unknown

1. The following table contains an equation and some numerical values.

*Without solving* this equation, determine whether the values of the left-hand column are solutions of the equation (using the CAS). But before doing so, describe and justify the strategy you will use to determine whether a given number is a solution (write in the box below). What information will your CAS strategy provide you with?

A recommended strategy is to substitute the given value into the equation. If a given value produces the same result for each side, then this value is a solution. If the CAS displays “true” upon substituting a given value for *x*, this means that the results produced by this substitution are equal.

2. Now the work with the calculator (but no solving, please). Fill in the table below with the appropriate information.



|  |  |  |
| --- | --- | --- |
| Values for *x* | What I entered into the CAS | Result displayed by the CAS |
| *x* = -2 | *12x + 15 = 4(x - 2) + 7* **|** *x = -2* | true |
| *x* = 2 | *12x + 15 = 4(x - 2) + 7* **|** *x = 2* | false |
| *x* = -5 | *12x + 15 = 4(x - 2) + 7* **|** *x = -5* | false |

3. Are there other solutions for this equation? If so, please give one and justify your choice.

There are no other solutions to this equation because a first-degree equation in one unknown has but a single solution (why?).

(B) First-degree equations in two unknowns

1. The next table contains a new equation and some numerical values.

*Without solving* this equation, but again using the CAS, determine whether each pair of values in the left-hand column is a solution. But before doing so, describe and justify the strategy you will use to determine whether a given number is a solution (write in the box below). What information will your CAS strategy provide you with?

Substitute the pairs of values into the equation to see whether both sides of the equation produce the same result. The CAS should display “true” if the given pair of values is a solution.

2. Now the work with the calculator (but no solving, please). Fill in the table below with the appropriate information.



|  |  |  |
| --- | --- | --- |
| Values for the pair *x* and *y* | What I entered into the CAS | Result displayed by the CAS |
| *x* = 3 and y = 12 | ***|*** x = 3 and y = 12 | false |
| *x* = -3 and y = 4 | ***|*** x = -3 and y = 4 | true |
| *x* = -18 and y = -6 | **|** x = -18 and y = -6 | true |

3. Are there other solution pairs for this equation? If so, please give one and justify your choice

There are infinitely many solution pairs for this first-degree equation in two unknowns. One of these is *x = -9* and *y = 0*. This solution pair, when substituted into the equation, produces a left-hand-side value of 9, which equals the given right-hand-side value.

# (C) Systems of two first-degree equations in two unknowns

1. The next table contains a system of equations and a few ordered pairs of numbers.

*Without solving* this system of equations, determine whether each pair in the left-hand column is a solution of the system (using the CAS). But before doing so, describe and justify the strategy you will use to determine whether a given number is a solution (write in the box below). What information will your CAS strategy provide you with?

For a system of equations in two unknowns, a given pair of values is a solution

if and only if it satisfies both equations of the system. The CAS should display “true” for both equations if the given pair of values being substituted into the equation is indeed a solution.

2. Now the work with the calculator (but no solving, SVP). Please write in the table below what you entered into the calculator, as well as what the calculator displays as a result.



|  |  |  |
| --- | --- | --- |
| Values for the pair *x* and *y* | What I entered into the CAS | Result displayed by the CAS |
| *x* = 0 and *y* = 2 | 2x = 8 - 4y **|** x = 0 and y = 2  17x - 31y = 3 **|** x = 0 and y = 2 | true  false |
| *x* = 4 and *y* = 3 | (2x = 8 - 4y **|** x = 4 and y = 3) and  (17x - 31y = 3 **|** x = 4 and y = 3) | false |
| *x* = 2 and *y* = 1 | (2x = 8 - 4y **|** x = 2 and y = 1) and  (17x - 31y = 3 **|** x = 2 and y = 1) | true |

3. Are there other solutions for this system of equations? If so, please give one and justify your choice.

|  |
| --- |
| There are no other solutions for this particular system of two first-degree equations in two unknowns. [Note that a graphical representation of this system would indicate two lines intersecting at a single point] |

4. Are there any questions or ideas that occurred to you while you were working on these three types of equations? What were they?

Can we tell how many solutions an equation or a system of equations will have, prior to solving it?

###### Classroom discussion of Part I

[For note-taking during the class discussion]

Section A equation: 

Section B equation: 

Section C system of equations: 

**Part II (with CAS):**

**Interpreting CAS solutions for equations in one and two unknowns**

II (A) Solving an equation in one unknown.

Use the command “SOLVE” of the CAS to solve the following equation:

*4(3x-7) = 2(3-x)+5*

|  |  |
| --- | --- |
| What you enter into the CAS | What the CAS displays for the result |
| Solve(*4(3x-7) = 2(3-x)+5*, *x*) | *x = 39/14* |

II (B) Solving an equation in two unknowns.

The six following questions concern the equation: *2x+7 = 8y+11*.

1. What do you predict as a result if you use the CAS to solve this equation for *x* ?

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| --- |
| The CAS would display an expression for *x*in terms of *y*. |

2. Use the CAS to solve this equation for *x*:

|  |  |
| --- | --- |
| What you enter into the CAS | What the CAS displays for the result |
| Solve(*2x+7 = 8y+11*, *x*) | *x = 2(2y + 1)* |

3. How do you interpret the result displayed by the CAS?

|  |
| --- |
| One possible interpretation:  Since the equation contains two unknowns, and because the “solve” command specified a solution for one of these unknowns, the CAS displayed the “value” of this unknown expressed in terms of the other unknown. |

4. What do you predict for the result if you use the CAS to solve this equation for *y*?

|  |
| --- |
| The CAS would display the “value” of *y* in terms of *x*. |

5. Use the CAS to solve the equation *2x+7 = 8y+11* for *y:*

|  |  |
| --- | --- |
| What you enter into the CAS | What the CAS displays for the result |
| Solve(*2x+7 = 8y+11*, *y)* | *y = (x - 2)/4* |

6. How do you interpret the result displayed by the CAS?

|  |
| --- |
| The CAS did indeed produce the “value” of *y* in terms of *x*. |

II (C) Distinctions between solutions of equations in one and two unknowns

1. You probably noticed that, in Part II (A), the CAS displayed a numerical value as the solution for x. In contrast, for Part II (B), the calculator displayed the result for *x* in the form of an algebraic expression. How do you explain this difference?

|  |
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| The equation in II A contained only one unknown; thus the CAS is able to determine and display a numerical value as a solution. However, the equation in II B contains two unknowns, in which case the CAS can only display the relationship between the two unknowns by expressing the one it was asked to solve for in terms of the other. It is possible to obtain numerical values as solution pairs in this case also; this issue is addressed in the next question.  [Incidentally, the equation has an infinite number of solutions (why?)] |

2. How can we use these expressions displayed by the CAS to find numerical solutions of the equation *2x+7 = 8y+11*?

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| We can take either the value of *x* in terms of *y* or the value of *y* in terms of *x* and substitute a numerical value for it. This will generate a numerical value of the other member of the solution pair.  For example, in *y = (x - 2)/4*, if we replace *x* by 6, *y* will assume a value of 1. If we substitute the pair *x = 6* and *y = 1* into the equation, *2x+7 = 8y+11*, both sides will have the same numerical value result because *x = 6* and *y = 1* is a solution pair of the equation. Other numerical solution pairs can be generated in the same way. |

##### **Classroom discussion of Part II A, B, C**

II (D) Using CAS to generate and verify solutions to equations in two unknowns

1. Use the CAS to generate three solutions for each of these equations (keep track of what you enter into the CAS as you go along, and of what the CAS displays as a result at each step). Verify at least one solution for each equation (with the CAS).

(a) 

|  |  |
| --- | --- |
| What I entered into the CAS | Result displayed by the CAS |
| Solve(, x)  x = 15y - 22 **|** y = {1, 2, 3}  **|** x = 8 and y = 2 | x = 15y - 22  x = {-7, 8, 23}  true |

(b) 

|  |  |
| --- | --- |
| What I entered into the CAS | Result displayed by the CAS |
| Solve(, *y*)  *y* = -7(72x - 37)/24 **|** x = {1, 2, 3}  **|** *x =* 3 and *y* = -1253/24 | *y* = -7(72x - 37)/24  *y* = {-245/24, -749/24, -1253/24}  true |

2. Give at least one question or idea that occurred to you while doing Part II D above (for example, a question concerning difficulties you experienced).

**Lesson 2** (Parts IIIA, IIIB, IIIC)

**Part IIIA (paper & pencil): Review of Comparison and Substitution methods**

1. Here is the COMPARISON method for solving a system of linear equations (adapted from p. 126 of your textbook):

|  |  |
| --- | --- |
| COMPARISON METHOD  The algebraic comparison method consists of: | x + 3y = 5  7x + 6y = 20 |
| 1. Isolating the same unknown in each of the equations, thereby creating two expressions each in a common single unknown; | y = (5-x)/3  y = (20-7x)/6 |
| 2. Setting the two expressions obtained in step 1 equal to each other to construct an equation in one unknown; | (5-x)/3 = (20-7x)/6 |
| 3. Solving the resulting equation; | (5-x)/3 = (20-7x)/6  2(5-x) = (20-7x)  10-2x = 20-7x  7x-2x = 20-10  5x = 10  x = 2 |
| 4. Replacing the resulting value in one of the system’s equations to calculate the value of the other unknown of the solution pair. | y = (5 – x)/3 = (5 – 2)/3 = 1  Solution pair is therefore (x, y) = (2, 1)  Verify this! |

Question: Why do you think that this method is called the Comparison method (in other words, in what sense is a comparison being made in this method)?

We isolate the *y*-unknown in both equations above (Step 1) to produce two different expressions in terms of *x*. Then we compare these two expressions in *x* with each other, by means of an equation. That is, constructing an equation *is* making a comparison of equality between the two expressions. Then we search for a value of *x* that satisfies both of these expressions. In this way, we find a pair, (*x*, *y*), that satisfies both equations.

2. Here is the SUBSTITUTION method for solving a system of linear equations (adapted from p. 128 of your textbook)

|  |  |
| --- | --- |
| SUBSTITUTION METHOD  The algebraic substitution method consists of: | 2x + 3y = 25  5x + y = 30 |
| 1. Isolating, if necessary, one of the unknowns in one of the equations; | y = 30 – 5x |
| 2. Substituting the expression obtained in step 1 for the appropriate unknown in the other equation, thereby creating an equation in a single unknown; | 2x + 3(30 – 5x) = 25 |
| 3. Solving the equation obtained in step 2; | 2x + 90 – 15x = 25  -13x = 25-90  -13x=-65  x = 65/13 |
| 4. Substituting the value obtained in one of the system’s equations to calculate the value of the other unknown in the ordered pair solution. | y = 30 – 5(65/13)  = 65/13  Solution pair is (x, y) = (65/13, 65/13)  Verify this! |

Question: Why do you think that this method is called the Substitution method?

|  |
| --- |
| We isolate the *y*-unknown in one equation above (Step 1) to produce an expression for *y* in terms of *x*. We then substitute this expression for *y* into the other equation; that is, we replace the unknown *y* in that equation with this expression. In this way we eliminate all explicit references to the unknown *y* in the two equations. |

3. In what way do these two methods (the Comparison and the Substitution methods) allow us to reduce the given situation to one that we already know how to handle?

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| Both methods essentially enable us to “eliminate” references to one of the two unknowns. In this way we reduce the system to linear equations in a *single* unknown, which we can easily solve. |

# **Classroom discussion of Part IIIA**

### Part IIIB (with CAS): Using the Comparison method with the CAS

Here is a system of linear equations: 

1. With the CAS, use the Comparison method to solve this system (keep track of what you enter into the calculator as you go along, and of what the calculator displays in response to your commands).

|  |  |  |
| --- | --- | --- |
| The Comparison method involves: | What you enter into the CAS | What the CAS displays for the result |
| 1. Isolating the same unknown in each of the equations, thereby creating two expressions each in a common single unknown | Solve(*x - 8=2y+2, x*)  Solve(*3x+5y+3=0*, *x*) | *x=2y+10*  *x=-(5y+3)/3* |
| 2 & 3. Setting the two expressions obtained in step 1 equal to each other to construct an equation in one unknown; solving the resulting equation | Solve(*2y+10=-(5y+3)/3*, *y*) | *y=-3* |
| 4. Replacing the resulting value in one of the system’s equations to calculate the value of the other unknown of the solution pair | *x=2y+10* **|** y=-3 | *x=4* |

2. How do you verify with the CAS that your solution is correct?

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| --- |
| Use the test “(*x-8=2y+2* and *3x+5y+3=0*) **|** *x=4* and *y=-3”* and anticipate that the CAS will display “true” if the pair (4, -3) is indeed the solution to the system. |

3. In Step 4 of Question 1 above, you replaced the value obtained in Step 3 (for the first unknown) in one of the equations. Now substitute this same value obtained in Step 3 into the other equation. What do you notice? Why is this so?

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| --- |
| If we use the evaluate operator “*x=-(5y+3)/3* **|** *y=-3*”, the CAS responds by displaying “*x=4*”. Notice that this is the same value produced when replacing *y=-3* in the other equation that expresses *x* in terms of *y*. This should not be surprising: since if *y=-3* is indeed the second value of the ordered solution pair of the system, then its replacement in either equation (or equivalent equation) of the given system must produce the numerical value of the other member of the solution pair. This is so because the solution pair, by its very definition, satisfies *both* equations. |

# **Classroom discussion of Part IIIB**

### Part IIIC (with CAS): Using the Substitution method with the CAS

1. With the CAS, use the Substitution method to solve this system (keep track of what you enter into the calculator as you go along, and of what the calculator displays in response to your commands).



|  |  |  |
| --- | --- | --- |
| The Substitution method involves: | What you enter into the CAS | What the CAS displays for the result |
| 1. Isolating, if necessary, one of the unknowns in one of the equations; | Solve(*x+3y = 5*, *x*) | *x=5 - 3y* |
| 2. Substituting the expression obtained in step 1 for the appropriate unknown in the other equation, thereby creating an equation in a single unknown; | *7x+6y=20* **|** *x=5 - 3y* | *35 - 15y=20* |
| 3. Solving the equation obtained in step 2; | Solve(*35 - 15y=20*, *y*) | *y=1* |
| 4. Substituting the value obtained in one of the system’s equations to calculate the value of the other unknown in the ordered pair solution. | *x=5 - 3y* **|** *y=1* | *x=2* |

2. How do you verify with the CAS that your solution is correct?

|  |
| --- |
| Use a strategy analogous to that described in Part IIIB, Question 2. |

3. Which of the two methods (COMPARISON and SUBSTITUTION) do you prefer and

why?

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|  |

4. What do these two methods have in common (don’t just rewrite the steps of the two methods)?

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| These two methods are driven by a common overriding logic and goal: to reduce a given system of two equations in two unknowns to a situation involving an equation in one unknown, which we can easily solve and thus determine the value of one member of the solution pair. Then, the value of the other member of the solution pair can be determined by substitution in one of the other (equivalent) equations. |

# **Classroom discussion of Part IIIC**

### Homework

With the CAS, use the more appropriate method (comparison or substitution) to solve the following systems of linear equations:

(1) *y + 1 = x + 6*

*y – 4 = -x + 3*

(2) *3x + y = 23*

#### 2x + 3y = 48

(1) The Comparison method might be considered more appropriate for solving the first system (why?):

|  |  |  |
| --- | --- | --- |
| The Comparison method involves: | What you enter into the CAS | What the CAS displays for the result |
| 1. Isolating the same unknown in each of the equations, thereby creating two expressions each in a common single unknown | Solve(*y + 1 = x + 6, y*)  Solve(*y – 4 = -x + 3, y*) | *y=x+5*  *y=-x+7* |
| 2 & 3. Setting the two expressions obtained in step 1 equal to each other to construct an equation in one unknown; solving the resulting equation | Solve(x+5=-x+7, x) | *x=1* |
| 4. Replacing the resulting value in one of the system’s equations to calculate the value of the other unknown of the solution pair | *y=x+5* **|** *x*=1 | *y=6* |

To verify that the ordered pair (1, 6) is indeed the solution to the first system, we can replace these values wherever the appropriate unknowns appear in each equation to check that they satisfy both equations:

Replacing in the first equation, *y + 1 = x + 6*, gives 6+1=7 for the left-hand side and 1+6=7 for the right-hand side. Thus, this ordered pair of values satisfies the first equation.

Replacing in the second equation, *y – 4 = -x + 3*, gives 6-4=2 for the left-hand side and –1+3=2 for the right-hand side. Thus, this same ordered pair of values satisfies the second equation.

Hence, we have verified that (1, 6) is indeed the solution pair for this system of equations.

(2) The Substitution method might be considered more appropriate for solving the second system (why?):

*3x + y = 23*

*2x + 3y = 48*

|  |  |  |
| --- | --- | --- |
| The Substitution method involves: | What you enter into the CAS | What the CAS displays for the result |
| 1. Isolating, if necessary, one of the unknowns in one of the equations; | Solve(*3x + y = 23, y*) | *y=23 - 3x* |
| 2. Substituting the expression obtained in step 1 for the appropriate unknown in the other equation, thereby creating an equation in a single unknown; | *2x + 3y = 48* **|** *y=23 - 3x* | *69 - 7x=48* |
| 3. Solving the equation obtained in step 2; | Solve(*69 - 7x=48, x*) | *x=3* |
| 4. Substituting the value obtained in one of the system’s equivalent equations to calculate the value of the other unknown in the ordered pair solution. | *y=23 - 3x* **|** *x=3* | *y=14* |

We verify that the ordered pair (3, 14) is indeed the solution of the second system by using the same strategy described in the last question:

Substituting these values appropriately in the left-hand side of the first equation,

*3x + y = 23,* produces 3(3)+14=23, which is equal to the equation’s right-hand side.

Similarly, appropriate substitution into the left-hand side of the second equation, *2x + 3y = 48*, produces 2(3)+3(14)=6+42=48, which is equal to the equation’s right-hand side.

Hence, (3, 14) is indeed the solution pair of the second system of equations.