10.4 Partitions of unity on general manifolds

Definition 39 *A partition of unity on M is a collection $\{\varphi_i\}_{i\in I}$ of smooth functions such that*

- $\varphi_i \geq 0$
- $\{\operatorname{supp}\varphi_i : i \in I\}$ *is locally finite*
- $\sum_i \varphi_i = 1$

Here *locally finite* means that for each $x \in M$ there is a neighbourhood U which intersects only finitely many supports $\operatorname{supp}\varphi_i$.

Theorem 10.8 *Given any open covering $\{V_\alpha\}$ of M there exists a partition of unity $\{\varphi_i\}$ on M such that* $\operatorname{supp}\varphi_i \subset V_{\alpha(i)}$ *for some $\alpha(i)$.*

Proof: (by exhaustion – !)

1. M is locally compact since each $x \in M$ has a neighbourhood homeomorphic to, say, the open unit ball in $\mathbf{R}^n$. So take U homeomorphic to a smaller ball, then $\bar{U}$ is compact. Since M is Hausdorff, $\bar{U}$ is closed (compact implies closed in Hausdorff spaces).

2. M has a countable basis of open sets $\{U_j\}_{j\in\mathbf{N}}$, so $x \in U_j \subset U$ and $\bar{U}_j \subset \bar{U}$ is compact so M has a countable basis of open sets with $\bar{U}_j$ compact.

3. Put $G_1 = U_1$. Then

$$\bar{G}_1 \subset \bigcup_{j=1}^{\infty} U_j$$

so by compactness there is $k > 1$ such that

$$\bar{G}_1 \subset \bigcup_{j=1}^{k} U_j = G_2$$

Now take the closure of G_2 and do the same. We get compact sets $\bar{G}_j$ with

$$\bar{G}_j \subset G_{j+1} \qquad M = \bigcup_{j=1}^{\infty} U_j$$

4. By construction we have

$$\bar{G}_j \backslash G_{j-1} \subset G_{j+1} \backslash \bar{G}_{j-2}$$

and the set on the left is compact and the one on the right open. Now take the given open covering $\{V_\alpha\}$. The sets $V_\alpha \cap (G_{j+1} \backslash \bar{G}_{j-2})$ cover $\bar{G}_j \backslash G_{j-1}$. This latter set is compact so take a finite subcovering, and then proceed replacing j with $j+1$. This process gives a countable locally finite *refinement* of $\{V_\alpha\}$, i.e. each $V_\alpha \cap (G_{j+1} \backslash \bar{G}_{j-2})$ is an open subset of V_α. It is locally finite because

$$G_{j+1} \backslash \bar{G}_{j-2} \cap G_{j+4} \backslash \bar{G}_{j+1} = \emptyset$$

5. For each $x \in M$ let j be the largest natural number such that $x \in M \backslash \bar{G}_j$. Then $x \in V_\alpha \cap (G_{j+2} \backslash \bar{G}_{j-1})$. Take a coordinate system within this open set and a bump function f which is identically 1 in a neighbourhood W_x of x.

6. The W_x cover $\bar{G}_{j+1} \backslash G_j$ and so as x ranges over the points of $G_{j+2} \backslash \bar{G}_{j-1}$ we get an open covering and so by compactness can extract a finite subcovering. Do this for each j and we get a countable collection of smooth functions ψ_i such that $\psi_i \geq 0$ and, since the set of supports is locally finite,

$$\psi = \sum \psi_i$$

is well-defined as a smooth function on M. Moreover

$$\operatorname{supp} \psi_i \subset V_\alpha \cap (G_m \backslash \bar{G}_{m-3}) \subset V_\alpha$$

so each support is contained in a V_α. Finally define

$$\varphi_i = \frac{\psi_i}{\psi}$$

then this is the required partition of unity. □

10.5 Sard's theorem (special case)

Theorem 10.9 *Let M and N be differentiable manifolds of the same dimension n and suppose $F : M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N. In particular, every smooth map F has at least one regular value.*