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Orthogonal complex structures on certain Riemannian 6-manifolds*

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Abstract: It is shown that the Hermitian-symmetric space $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and the flag manifold $F_{1,2}$ endowed with any left invariant metric admit no compatible integrable almost complex structures (even locally) different from the invariant ones. As an application it is proved that any stable harmonic immersion from $F_{1,2}$ equipped with an invariant metric into an irreducible Hermitian symmetric space of compact type is equivariant. It is also shown that $\mathbb{C}P^1\times\mathbb{C}P^1\times\mathbb{C}P^1$ and $F_{1,2}$ with its invariant Kähler–Einstein structures are the only compact Kähler–Einstein spin 6-manifolds of non-negative, non-identically vanishing holomorphic sectional curvature that admit another orthogonal complex structure of Kähler type. A necessary and sufficient condition on a compact oriented 6-manifold to admit three mutually commuting almost complex structures is given; it is used to characterize $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and $F_{1,2}$ as Fano 3-folds admitting three mutually commuting complex structures which satisfy certain compatibility conditions.

Keywords: Orthogonal complex structures, twistor spaces, harmonic maps, Fano 3-folds.

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1. Introduction

Let (M^{2n}, g) be an oriented Riemannian manifold of dimension $2n$. A complex structure on M, viewed as an integrable almost-complex structure J, is *positive* and *orthogonal* whenever J induces the same orientation on M and J is g-skew-symmetric.

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Given a Riemannian manifold (M^{2n}, g) it is natural to ask [35] does there exist an orthogonal complex structure on (M, g)? If so, we would like to describe the set of all orthogonal complex structures. This question (which in fact concerns the conformal structure determined by g) can be asked either locally or globally, and the corresponding answers can be rather different in nature. While on an oriented (real) surface the complex structures and the Riemannian conformal classes coincide, when $n \geqslant 2$ the *local* existence of orthogonal complex structures imposes constraints on the (conformally invariant) Weyl curvature tensor of M, cf. [37]. Consider for example $\mathbb{C}P^n$, $n \geqslant 3$ with the Fubini–Study metric. It is well known that the unique globally defined positive orthogonal complex structure is the canonical one while locally there are infinitely many positive orthogonal complex structures since the Bochner tensor of the canonical Hermitian structure of $\mathbb{C}P^n$ vanishes, cf. [29].

If M is a (real, oriented) 4-dimensional manifold, then any positive orthogonal complex structure J is determined (up to a 4-fold ambiguity) by the self-dual Weyl tensor W^+ at any point where it is non-zero. More precisely, J is equal to a universal function of the eigenforms and the eigenvalues of W^+, operating on the bundle of self-dual 2-forms, cf. [34, 4] (the above mentioned ambiguity comes from the lack of a canonical orientation for the eigenspaces of W^+). In particular, if W^+ is not identically zero, there are (even locally) at most 2 distinct compatible positive complex structures [31]. (Here and henceforth, *distinct* means that there is a point where the complex structures are not equal up to sign.) On the other hand the anti-self-dual 4-manifolds admit locally infinitely many compatible complex structures [8]. Compact Riemannian 4-manifolds admitting two distinct *globally defined* positive orthogonal complex structures are called *bihermitian surfaces*. It follows from the results in [31, 6] that few of the complex surfaces could admit bihermitian structures, i.e., "generically" on a *compact* oriented Riemannian 4-manifold there is at most one globally defined positive orthogonal complex structure.

When the dimension of M is more than 4 the situation is more complicated (see [35] and the included references). However, the question of global existence of orthogonal complex structures has been successfully studied for some special classes of Riemannian manifolds as Riemannian (inner) symmetric spaces of compact type [13, 12], compact quotients of irreducible symmetric spaces of non-compact type [16], quaternionic manifolds (of dimension $4n$, $n \geqslant 2$) [30, 2, 3]. Unfortunately, for a general Riemannian manifold little is known for the set of orthogonal complex structures.

In this paper we are interested in 6-dimensional Riemannian manifolds, admitting three commuting orthogonal complex structures. This condition comes naturally from the geometry of the twistor space and insures the triviality of the twistor bundle (see Section 2). It is equivalent to the splitting of the tangent bundle into three two-dimensional subbundles and leads to a certain topological restriction on the manifold (Proposition 1).

The simplest example of such a manifold is the product $\Sigma_1 \times \Sigma_2 \times \Sigma_3$ of three Riemann surfaces Σ_1, Σ_2, Σ_3. It can be in fact characterized by the existence of four mutually commuting Kähler structures (Section 3.1) and we observe in Theorem 3 that if the Gauss curvatures k_i, $i = 1, 2, 3$ of Σ_i satisfy

$$k_i + k_j \neq 0, \quad i \neq j$$

at some point, then these are the only orthogonal complex structures. Concerning the reducible

Hermitian-symmetric space $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, our local observation fits in with the results of [13].

We also consider the flag manifold

$$F_{1,2} = \mathbf{U}(3)/(\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1))$$

with an arbitrary $\mathbf{U}(3)$-left-invariant Riemannian metric (Section 3.2). It has been already observed by the authors that with respect to a certain 1-parameter family of $\mathbf{U}(3)$-left-invariant metrics on $F_{1,2}$ the only orthogonal complex structure are the three commuting $\mathbf{U}(3)$-left-invariant complex structures, [5, Theorem 1]. This has been derived by considering $F_{1,2}$ as the twistor space over $\mathbb{C}P^2$ with its standard metric; hence there is a map $\pi : F_{1,2} \mapsto \mathbb{C}P^2$ and 1-parameter family of ($\mathbf{U}(3)$-left-invariant) Riemannian metrics $h_t, t > 0$ of $F_{1,2}$, such that for any $t > 0$, π is a Riemannian submersion from $(F_{1,2}, h_t)$ to $\mathbb{C}P^2$. It thus can be seen that any orthogonal complex structure on $(F_{1,2}, h_t)$ is either the tautological complex structure, or else it is the lift of one of the two (differing by sign) orthogonal complex structures on $\mathbb{C}P^2$. Considering now $F_{1,2}$ with its algebraic structure of a homogeneous space we extend the result for an arbitrary left-invariant metric.

Theorem 1. *Let* $F_{1,2} = \mathbf{U}(3)/(\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1))$ *be the flag manifold, endowed with a* $\mathbf{U}(3)$*-left-invariant metric. Then the only orthogonal complex structures* (*even locally defined*) *are the three* $\mathbf{U}(3)$*-left-invariant complex structures on* $F_{1,2}$.

Recall that a map from a flag manifold into a Riemannian manifold is said to be *equiharmonic* if it is harmonic with respect to *any* invariant metric of the flag manifold. As consequence of Theorem 1 we get

Corollary 1. *Let* $f : F_{1,2} \longrightarrow \mathbf{M}$ *be a stable harmonic map from the flag manifold* $F_{1,2}$ *equipped with some invariant metric into an irreducible Hermitian-symmetric space of compact type* $\mathbf{M}$, *and suppose that there is a point where the differntial of* f *has maximal rank. Then* f *is equiharmonic map which is* $\pm$*-holomorphic with respect to some of the invariant complex structures* J_1, J_2, J_3.

In Section 4 we characterize (up to a biholomorphism) $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and $F_{1,2}$ as Fano 3-folds admitting spin structure and three mutually commuting complex structures, satisfying certain natural compatible conditions (Proposition 2). Moreover, regarding to $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and $F_{1,2}$ with its (left) invariant *Kähler–Einstein* structures (see Remark 2 bellow), we prove the following result, which nicely links to Mok's characterization [26] of compact Hermitian-symmetric spaces as Kähler manifolds of non-negative bisectional curvature:

Theorem 2. *Let* (M, g, J) *be a compact Kähler–Einstein spin-manifold of real dimension* 6 *with non-negative non-identically vanishing holomorphic sectional curvature. If* (M, g) *admits another orthogonal complex structure* $J' \neq \pm J$ *which is of Kähler type, then* (M, g, J) *is biholomorphically isometric to either* $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ *or* $F_{1,2}$ *with its* (*left*) *invariant Kähler–Einstein structure, and the complex structure* J' *is invariant too.*

2. Commuting orthogonal almost complex structures and twistor spaces of Riemannian 6-manifolds

2.1. Twistor space of a Riemannian manifold of even dimension

Let (M, g) be a $2n$-dimensional oriented Riemannian manifold. We wish to study positive orthogonal (almost) complex structures on (M, g), which we view as sections of the fibre bundle $\pi_+ : Z^+M = P \times_{\mathbf{O}(2n)} (\mathbf{SO}(2n)/\mathbf{U}(n)) \mapsto M$, where $P \mapsto M$ denote the canonical principal $\mathbf{O}(2n)$-bundle. The vertical distribution $\mathcal{V} =\mathrm{Ker}(\pi_+)_*$ inherits a canonical complex structure $J^{\mathcal{V}}$ since the fibre $H_n = \mathbf{SO}(2n)/\mathbf{U}(n)$ is a Hermitian-symmetric space. Moreover, the Levi-Civita connection ∇ on M induces a splitting $TZ^+M = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of Z^+M into horizontal and vertical components, so that $\mathcal{H} \cong (\pi_+)^{-1}TM$ acquires a tautological complex structure $J^{\mathcal{H}}$ given by $J^{\mathcal{H}}_{x,j} = j$ for $x \in M$ and $j \in (\pi_+)^{-1}(x) \cong H_n$. Following [8], we define an almost-complex structure $\mathbf{J}$ on Z^+M by

$$\mathbf{J} = J^{\mathcal{H}} + J^{\mathcal{V}}.$$

It is well known that in the case when M is 4-dimensional, the almost complex structure $\mathbf{J}$ is integrable iff the positive Weyl tensor of (M, g) vanishes [8]; if $\dim M = 2n > 4$, then the integrability of $\mathbf{J}$ is equivalent to the vanishing of the Weyl tensor of (M, g) [29]. Moreover, in [32], Salamon shows that integrability of a positive orthogonal almost complex structure J of (M, g) is equivalent to the holomorphicity of J viewed as a map $J : (M, J) \mapsto (Z^+M, \mathbf{J})$. In fact one can say more. In [29], the Nijenhuis tensor $N^{\mathbf{J}}$ is calculated and it is shown that $N^{\mathbf{J}}$ vanishes at $j \in Z^+M$ iff

$$R(T^{1,0}(j), T^{1,0}(j))T^{1,0}(j) \subset T^{1,0}(j), \tag{1}$$

where $T^{(1,0)}(j) \subset T_{\pi(j)}M \otimes \mathbb{C}$ is the $(1, 0)$-space of j and R is the curvature of g.

Denote by Z^+_0M the zero-set of $N^{\mathbf{J}}$. Since any positive orthogonal complex structure J satisfies (1) we have that J lies entirely in Z^+_0M, but a section J of Z^+_0M is not necessarily integrable.

2.2. Twistor space of a Riemannian 6-manifold

Now we restrict our attention to the twistor space $Z(M, g)$ of a 6-dimensional Riemannian manifold (M, g). Since the positive and the negative twistor spaces can be identified via the map $j \mapsto -j$ on the fibre, we will consider $Z(M, g)$ as the quotient space of the principal $\mathbf{O}(6)/\mathbf{U}(3)$-bundle of all orthogonal almost complex structures of (M, g) under this action. The fiber $H_3 = \mathbf{SO}(6)/\mathbf{U}(3)$ is then isomorphic to $\mathbb{C}P^3$ and we will make use of the explicit identification given in [1] (for more details see also [33, 38, 18, 5]). Let V be a complex 4-dimensional vector space endowed with a Hermitian inner product h and a volume form $\Phi \in \Lambda^4 V$. The Hodge operator $*$ is defined on $\Lambda^2 V$ by

$$\xi \wedge *\eta = h(\xi, \eta)\Phi.$$

Thus the $\mathbb{C}$-anti-linear endomorphism $*$ induces on the 6-dimensional complex vector space $\Lambda^2 V$ a real structure. The fixed points set of $*$ forms a real 6-dimensional vector space W with a positive definite inner product g coming from h. For any $[v] \in \mathbb{P}(V)$ we then have the h-orthogonal splitting

$$\Lambda^2 V = V_v \oplus V_v^{\perp}, \tag{2}$$

where V_v denotes the vector space generated by the 2-vectors $v \wedge u$ with $h(u, v) = 0$ and $V_v^{\perp}$ is the orthogonal component of V_v in $\Lambda^2 V$. Notice that $V_v^{\perp}$ is spanned by the 2-vectors $u' \wedge u''$ with $h(u', v) = h(u'', v) = 0$. The splitting (2) of $\Lambda^2 V = W \otimes \mathbb{C}$ defines a positive g-orthogonal complex structure on the vector space W with (1,0) and (0,1)-spaces equal to V_v and $V_v^{\perp}$, respectively. This gives the identification of $\mathbb{P}(V) \cong \mathbb{C}P^3$ with the space $\mathbf{SO}(6)/\mathbf{U}(3)$ [1, Lemma 4.1]. In terms of this correspondence we have that two distinct positive orthogonal complex structures J' and J'' on (W, g) commute iff the corresponding $[v']$ and $[v'']\in \mathbb{P}(V)$ are h-orthogonal, [5, Lemma 1]. Moreover, if we fix a unitary frame $\{v_0, v_1, v_2, v_3\}$ of (V, h) with $v_0 \wedge v_1 \wedge v_2 \wedge v_3 = \Phi$, the corresponding orthogonal almost complex structures $\{J_0, J_1, J_2, J_3\}$ mutually commute, cf. [5, Corollary 1], and the complex vectors

$$Z_1 = v_0 \wedge v_1, \quad Z_2 = v_0 \wedge v_2, \quad Z_3 = v_0 \wedge v_3, \tag{3}$$

give a unitary frame of $T^{1,0}_{J_0}$; their complex-conjugated vectors are respectively

$$\overline{Z}_1 = v_2 \wedge v_3, \quad \overline{Z}_2 = -v_1 \wedge v_3, \quad \overline{Z}_3 = v_1 \wedge v_2,$$

and the corresponding $(1, 0)$-spaces of J_1, J_2, J_3 are respectively spanned by the triples

$$\{Z_1, \overline{Z}_2, \overline{Z}_3\}, \quad \{\overline{Z}_1, Z_2, \overline{Z}_3\}, \quad \{\overline{Z}_1, \overline{Z}_2, Z_3\}.$$

For an element $[v] \in \mathbb{P}(V)$ let $[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ be the homogeneous coordinates with respect to $\{v_0, v_1, v_2, v_3\}$; the corresponding positive orthogonal complex structure J is determined by its $(1, 0)$-space spanned by the vectors (see [5]):

$$\begin{aligned} Z_0^J &= -\alpha_1 Z_1 - \alpha_2 Z_2 - \alpha_3 Z_3, \quad & Z_1^J &= \alpha_0 Z_1 - \alpha_2 \overline{Z}_3 + \alpha_3 \overline{Z}_2, \\ Z_2^J &= \quad \alpha_0 Z_2 + \alpha_1 \overline{Z}_3 - \alpha_3 \overline{Z}_1, \quad & Z_3^J &= \alpha_0 Z_3 - \alpha_1 \overline{Z}_2 + \alpha_2 \overline{Z}_1. \end{aligned} \tag{4}$$

The only relation among them is $\Sigma_{j=0}^{3} \alpha_j Z_j^J = 0$.

We then decompose $\mathbb{C}P^3$ as

$$\mathbb{C}P^3 \cong \mathbb{C}^3 \cup \mathbb{C}^2 \cup \mathbb{C}P^1,$$

where:

(a) The copy of $\mathbb{C}P^1$ consists of the elements of $\mathbb{C}P^3$ with homogeneous coordinates $[0, \delta_1, \delta_2, 0]$; the corresponding $\mathbb{C}P^1$-family of positive orthogonal complex structures on (W, g) consists of all elements which commute with but differ from both J_0 and J_3. The $(1, 0)$-space of any such a structure is spanned by the complex vectors

$$\begin{aligned} Z_0^J &= \delta_1 Z_1 + \delta_2 Z_2; \\ Z_1^J &= \overline{Z}_3; \\ Z_3^J &= -\delta_2 \overline{Z}_1 + \delta_1 \overline{Z}_2. \end{aligned} \tag{5}$$

(b) The copy of $\mathbb{C}^2$ in $\mathbb{C}P^3$ is determined by the elements with homogeneous coordinates $[0, \gamma_1, \gamma_2, 1]$, i.e., the corresponding family of positive orthogonal complex structures on (W, g) consists of all elements which commute with J_0 but do not commute with J_3. The $(1, 0)$-space of any such a structure is spanned by the complex vectors

$$\begin{aligned} Z_0^J &= \gamma_1 Z_1 + \gamma_2 Z_2 + Z_3; \\ Z_2^J &= \bar{Z}_2 - \gamma_2 \bar{Z}_3; \\ Z_3^J &= -\bar{Z}_1 + \gamma_1 \bar{Z}_3. \end{aligned} \tag{6}$$

(c) The copy of $\mathbb{C}^3$ is determined by the elements of $\mathbb{C}P^3$ with homogeneous coordinates $[1, \beta_1, \beta_2, \beta_3]$, i.e., which have a $(1, 0)$-space spanned by the complex vectors

$$\begin{aligned} Z_1^J &= Z_1 + \beta_3 \bar{Z}_2 - \beta_2 \bar{Z}_3; \\ Z_2^J &= Z_2 - \beta_3 \bar{Z}_1 + \beta_1 \bar{Z}_3; \\ Z_3^J &= Z_3 + \beta_2 \bar{Z}_1 - \beta_1 \bar{Z}_2. \end{aligned} \tag{7}$$

Equivalently, the family (c) consists of all positive orthogonal almost-complex structures which do not commute with J_0.

2.3. Riemannian 6-manifolds with trivial twistor bundle

We will apply the above pointwise considerations to study global orthogonal complex structures on Riemannian 6-manifolds (M, g) whose twistor bundle $Z^+M \mapsto M$ is trivialized by four mutually commuting orthogonal almost complex structures $\{J_0, J_1, J_2, J_3\}$. Observe that $J_0 = \pm J_1 \circ J_2 \circ J_3$, hence our condition is in fact equivalent to the existence of three mutually commuting orthogonal almost complex structures. As in the 4-dimensional case, the triviality of the twistor bundle leads to some topological restrictions of the manifold. More precisely we have the following

Proposition 1. *Let M be a compact 6-manifold admitting three mutually commuting almost complex structures. Then there exist elements ω_i, $i = 1, 2, 3$ of $H^2(M, \mathbb{Z})$ satisfying the following properties*:

(i) $\omega_1 + \omega_2 + \omega_3 \equiv w_2(M) \pmod 2$;
(ii) $\omega_1^2 + \omega_2^2 + \omega_3^2 = p_1(M) \equiv \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 \pmod 2$;
(iii) $\omega_1\omega_2\omega_3 = e(M)$,

where $w_2(M)$, $p_1(M)$ and $e(M)$ are the second Stiefel class, the first Pontrjagin class and the Euler class of M, respectively, and $(\mathrm{mod}\ 2)$ *denotes the natural homomorphism $H^2(M, \mathbb{Z}) \mapsto H^2(M, \mathbb{Z}_2)$.*

Conversely, for any oriented compact 6-manifold M whose cohomology group $H^4(M, \mathbb{Z})$ has no elements of order four, existence of elements ω_i, $i = 1, 2, 3 \in H^2(M, \mathbb{Z})$ satisfying the conditions above implies the existence of three mutually commuting almost complex structures on M.

Proof. Let $\{J_1, J_2, J_3\}$ be a triple of mutually commuting almost complex structures on M and put $J_0 = J_1 \circ J_2 \circ J_3$, $Q_i = J_0 \circ J_i$, $i = 1, 2, 3$. Then J_0 is an almost complex structure on M

commuting with any almost complex structure J_i, and Q_i are J_0-invariant, mutually commuting involutions on the tangent bundle TM. We then have the splitting

$$TM = T_1 \oplus T_2 \oplus T_3, \tag{8}$$

where $T_i, i = 1, 2, 3$ are J_0-invariant 2-dimensional sub-bundles of TM, which are eigenspaces for any Q_i. The almost complex structure J_0 induces complex structures on the real vector bundles TM and T_i, $i = 1, 2, 3$. Denote by ω_i the first Chern classes of the complex line bundles T_i, respectively. We then have by (8):

$$c_1(M, J_0) = \omega_1 + \omega_2 + \omega_3,$$
$$c_2(M, J_0) = \tfrac{1}{2}(c_1^2(M, J_0) - p_1(M)) = \omega_1\omega_2 + \omega_2\omega_3 + \omega_1\omega_3,$$
$$c_3(M, J_0) = e(M) = \omega_1\omega_2\omega_3.$$

Moreover, on any oriented compact 6-manifold the Stiefel classes are determined by the Wu formula (cf. [25]); we calculate $w_2(M) = v_2$; $w_4(M) = v_2^2 = w_2(M)^2$, where, we recall, v_2 is determined as dual to

$$\mathrm{Sq}^2 : H^4(M, \mathbb{Z}_2) \mapsto H^6(M, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Now the claim follows by the well known properties: $w_2(M) = c_1(M, J_0) \pmod 2$; $w_2(M)^2 = p_1(M) \pmod 2$; $w_4(M) = c_2(M, J_0) \pmod 2$.

Let M be an oriented compact 6-manifold whose cohomology group $H^4(M, \mathbb{Z})$ has no elements of order 4 and suppose that there are elements ω_i, $i = 1, 2, 3$ of $H^2(M, \mathbb{Z})$ satisfying conditions (i), (ii) and (iii). Any ω_i determines a complex line bundle L_i over M, such that the first Chern class $c_1(L_i)$ of L_i is ω_i. Consider the 6-dimensional oriented real vector bundle $L_\mathbb{R}$ over M, underlying the complex rank 3 bundle $L = L_1 \oplus L_2 \oplus L_3$; we easily compute:

$$w_2(L_\mathbb{R}) = c_1(L) \pmod 2 = \omega_1 + \omega_2 + \omega_3 \pmod 2,$$
$$w_4(L_\mathbb{R}) = c_2(L) \pmod 2 = \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 \pmod 2,$$
$$p_1(L_\mathbb{R}) = c_1^2(L) - 2c_2(L) = \omega_1^2 + \omega_2^2 + \omega_3^2,$$
$$e(L_\mathbb{R}) = c_3(L) = \omega_1\omega_2\omega_3,$$

where $w_i(L_\mathbb{R})$, $i = 2, 4$, $p_1(L_\mathbb{R})$ and $e(L_\mathbb{R})$ are the corresponding Stiefel classes, the first Pontrjagin class and the Euler class of $L_\mathbb{R}$, respectively, and $c_i(L)$, $i = 1, 2, 3$ are the Chern classes of L. As we have already mentioned, on any oriented compact 6-manifold M, the Stiefel classes $w_4(M)$ and $w_2(M)$ are related by $w_4(M) = w_2(M)^2$; we then get from (i), (ii), (iii): $w_2(L_\mathbb{R}) = w_2(M)$; $w_4(L_\mathbb{R}) = w_4(M)$; $p_1(L_\mathbb{R}) = p_1(M)$; $e(L_\mathbb{R}) = e(M)$. According to [15, Prop.1] (see also [39]), the tangent bundle TM is then isomorphic to the real vector bundle $L_\mathbb{R}$, i.e., the splitting (8) holds true for some oriented 2-dimensional vector subbundles T_i of TM. Choosing complex structure j_i on any T_i and putting

$$J_0 = \quad j_1 + j_2 + j_3, \quad J_1 = \quad j_1 - j_2 - j_3,$$
$$J_2 = -j_1 + j_2 - j_3, \quad J_3 = -j_1 - j_2 + j_3,$$

we get four mutually commuting almost complex structures on M. □

Remark 1. Given four commuting almost complex structures $\{J_0, J_1, J_2, J_3\}$ on M it can be easily seen that there exists a Riemannian metric g compatible with any J_i, $i = 0, 1, 2, 3$. Indeed, considering the splitting (8) of the tangent bundle TM determined by $\{J_0, J_1, J_2, J_3\}$ let g_i be a Hermitian metric on T_i. Then a Riemannian metric g on M is compatible with any J_i iff the splitting (8) is g-orthogonal, i.e., iff g has the form $g = f_1g_1 + f_2g_2 + f_3g_3$ for some positive smooth functions f_1, f_2, f_3 of M.

3. Examples

3.1. Product of three Riemann surfaces

The simplest example of Riemannian 6-manifold admitting four commuting *complex* structures is the product $M = \Sigma_1 \times \Sigma_2 \times \Sigma_3$ of three oriented Riemann surfaces (Σ_i, g_i). Changing the orientation of any Σ_i we obtain in fact four commuting *Kähler* structures $\{J_0, J_1, J_2, J_3\}$, which are the products of the corresponding Kähler structures on (Σ_i, g_i). Conversely, any Riemannian 6-manifold (M, g) that admits four mutually commuting Kähler structures $\{J_0, J_1, J_2, J_3\}$ is locally isometric to the product of three Riemann surfaces. Indeed, in this case the involutions $Q_i = J_0 \circ J_i$, $i = 1, 2, 3$ are preserved by the Levi-Civita connection of (M, g), hence the holonomy group of (M, g) preserves the orthogonal splitting (8) of TM.

The next statement shows that in general $\{J_0, J_1, J_2, J_3\}$ are the only orthogonal complex structures on $\Sigma_1 \times \Sigma_2 \times \Sigma_3$:

Theorem 3. *Let $(M, g) = \Sigma_1 \times \Sigma_2 \times \Sigma_3$ be the product of oriented Riemann surfaces (Σ_i, g_i), $i = 1, 2, 3$ with Gauss curvatures k_i, respectively. Suppose that*

$$k_i + k_j \neq 0, \quad i, j \in \{1, 2, 3\},\ i \neq j \tag{9}$$

at some point of M. Then g admits exactly four positive orthogonal complex structures, which are Kähler and mutually commute.

Proof. Denote by j_i the complex structure on Σ_i and let

$$J_0 = \quad j_1 + j_2 + j_3, \quad J_1 = \quad j_1 - j_2 - j_3,$$
$$J_2 = -j_1 + j_2 - j_3, \quad J_3 = -j_1 - j_2 + j_3$$

be the four commuting Kähler structures of (M, g). Consider the open subset $\mathcal{U}$ of M, where (9) is satisfied and let $Z_0^+\mathcal{U}$ be the zero set of the Nijenhuis tensor of the twistor space of $(\mathcal{U}, g)$, see Section 2.1. Then we have

Lemma 1. *The zero set $Z_0^+\mathcal{U}$ consists of J_0, J_1, J_2 and J_3.*

Proof of Lemma 1. Consider the unitary complex $(1, 0)$-vectors Z_i, $i = 1, 2, 3$ on the complexified tangent bundle $T_i \otimes \mathbb{C}$ of each Riemannian surface (Σ_i, g_i, j_i). Then the curvature R

of the product metric g is given by

$$\begin{aligned} &R(Z_i \wedge \bar{Z}_i) = k_i Z_i \wedge \bar{Z}_i; \\ &R(Z_i \wedge Z_j) = 0;\ \ R(Z_i \wedge \bar{Z}_j) = 0, \quad i, j \in \{1, 2, 3\},\ i \neq j. \end{aligned} \tag{10}$$

Suppose that J is an orthogonal almost complex structure of $(\mathcal{U}, g)$, defined at some point $x \in M$, which satisfies (1) (i.e., J belongs to $\pi^{-1}(x) \cap Z_0^+\mathcal{U}$). If J belongs to the $\mathbb{C}P^1$-family (a) of orthogonal almost complex structures at x, then there are complex numbers δ_1 and δ_2, such that the $(1, 0)$-space of J is spanned by the complex vectors Z_0^J, Z_1^J, Z_3^J defined by (5). Thus, using (10) and the fact that J satisfies (1), we obtain $(k_1 + k_2)\delta_1^2\delta_2^2 = 0$, i.e., $\delta_1\delta_2 = 0$ since on $\mathcal{U}$ the condition (9) holds. Hence J coincides with either J_1 or J_2. Similarly, if J belongs to the $\mathbb{C}^2$-family (b), we get by (6) and (10) that the condition (1) is equivalent to $\gamma_1^2(k_2 + k_3) = \gamma_2^2(k_1 + k_3) = 0$, i.e., $\gamma_1 = \gamma_2 = 0$ because of (10). This shows that $J = J_3$. Finally, consider the case that the almost complex structure J belongs to the $\mathbb{C}^3$-family of almost complex structures at x, described in (c). Using (7) and (10) we obtain in this case that the condition (1) is equivalent to

$$\beta_1^2(k_2 + k_3) = \beta_2^2(k_1 + k_3) = \beta_3^2(k_1 + k_2) = 0,$$

hence we get from (10) $\beta_i = 0, i = 1, 2, 3$, i.e., $J = J_0$. □

Now Theorem 3 follows immediately. Indeed, if J is an *integrable* orthogonal almost complex structure of (M, g), then J belongs to $Z_0^+\mathcal{U}$ and it follows by Lemma 1 that J coincides with one of the orthogonal complex structures J_0, J_1, J_2, J_3 on any connected open subset of $\mathcal{U}$. Hence, according to [31, Remark 1.5 (2)], this holds everywhere on M. □

3.2. Flag manifold

Let $F_{1,2} = \mathbf{U}(3)/\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)$ be the complex 3-dimensional flag manifold. Consider the reductive decomposition of $\mathbf{u}(3)$

$$\mathbf{u}(3) = \mathbf{h} \oplus \mathbf{m},$$

where $\mathbf{u}(3)$ is the Lie algebra of the unitary group $\mathbf{U}(3)$ and $\mathbf{h}$ and $\mathbf{m}$ are determined by:

$$\mathbf{h} = \left\{ \begin{matrix} i\alpha & 0 & 0 \\ 0 & i\beta & 0 \\ 0 & 0 & i\gamma \end{matrix} \right\} \cong \mathbf{u}(1) \oplus \mathbf{u}(1) \oplus \mathbf{u}(1) \subset \mathbf{u}(3);$$

$$\mathbf{m} = \left\{ \begin{matrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{matrix} \right\} \subset \mathbf{u}(3).$$

Identifying any element $X \in TF_{1,2} \cong \mathbf{m}$ with the corresponding triple of complex numbers (a, b, c), we consider the $\mathbf{U}(3)$-left-invariant Riemannian metric $g_{\lambda_1,\lambda_2,\lambda_3}$ on $F_{1,2}$ defined by

$$g_{\lambda_1,\lambda_2,\lambda_3}(X, X) = \lambda_1|a|^2 + \lambda_2|b|^2 + \lambda_3|c|^2, \ \forall X \in TF_{1,2},$$

where $\lambda_1, \lambda_2, \lambda_3$ are real positive numbers. It is well known that when $\lambda_1, \lambda_2, \lambda_3$ vary, the set of the metrics $g_{\lambda_1,\lambda_2,\lambda_3}$ exhaust all $\mathbf{U}(3)$-left-invariant metrics on $F_{1,2}$.

The (left) invariant almost complex structures on $F_{1,2}$ are described by

$$\mathbf{J}_{\epsilon_1\epsilon_2\epsilon_3} : (a, b, c) \to (\epsilon_1 ia, \epsilon_2 ib, \epsilon_3 ic), \quad \epsilon_i \in \{\pm 1\}, i = 1, 2, 3.$$

The positive ones are characterized by the condition $\epsilon_1\epsilon_2\epsilon_3 = 1$, hence there are exactly four distinct positive $\mathbf{U}(3)$-left-invariant almost complex structures, $J_1 = \mathbf{J}_{-1,-1,1}$; $J_2 = \mathbf{J}_{1,1,1}$; $J_3 = \mathbf{J}_{1,-1,-1}$ and $J_0 = \mathbf{J}_{-1,1,-1}$, which are compatible with any invariant metric $g_{\lambda_1,\lambda_2,\lambda_3}$ and mutually commute. It is easily checked that J_1, J_2 and J_3 are integrable, while J_0 is *bi-invarint* with nowhere vanishing Neijenhuis tensor. Since J_1, J_2 and J_3 all satisfy (1) with respect to any invariant metric (as being integrable), so does $J_0 = J_1 \circ J_2 \circ J_3$ (see Section 2.2). More precisely, the zero set $Z_0^+(F_{1,2}, g)$ with respect to any left-invariant metric g of $F_{1,2}$ is determined by the following

Lemma 2. *Let $g = g_{\lambda_1,\lambda_2,\lambda_3}$ be a left-invariant metric of $F_{1,2}$. For any $i \in \{1, 2, 3\}$ denote by $\mathcal{P}_i$ the $\mathbb{C}P^1$-bundle over $F_{1,2}$ whose fibre at any point $x \in F_{1,2}$ consists of all positive, g-orthogonal almost complex structures at x that commute with but differ from J_0 and J_i. If we put $C_i = 3(\lambda_{i-1} + \lambda_{i+1})\lambda_i - (\lambda_{i-1} - \lambda_{i+1})^2$, $i = 1, 2, 3$, (where $\lambda_0 = \lambda_3;\ \lambda_4 = \lambda_1$), then the zero set $Z_0^+(F_{1,2}, g)$ is determined as follows*:

(i) *if for some $i \in \{1, 2, 3\}$ the positive real numbers λ_i, $i = 1, 2, 3$ satisfy $C_i = 0$ and $\lambda_{i-1} \neq \lambda_i \neq \lambda_{i+1}$, then $Z_0^+(F_{1,2}, g)$ consists of J_0, J_i and the bundle $\mathcal{P}_i$;*

(ii) *if for some $i \in \{1, 2, 3\}$ the positive real numbers λ_i, $i = 1, 2, 3$ satisfy $C_i = 0$ and $\lambda_i = \lambda_{i-1}$, then $Z_0^+(F_{1,2}, g)$ consists of J_0 and the bundles $\mathcal{P}_i$ and $\mathcal{P}_{i-1}$;*

(iii) *in any other case $Z_0(F_{1,2}, g)$ consists of J_0, J_1, J_2 and J_3.*

Proof. We will use the following well-known expression for the curvature R of g (see for example [11, Ch.7]):

$$\begin{aligned} R(X, Y, X, Y) = {} & \tfrac{3}{4} g([X, Y]_{\mathbf{m}}, [X, Y]_{\mathbf{m}}) + g([[X, Y]_{\mathbf{h}}, Y], X) \\ & + \tfrac{1}{2} g(X, [Y, [Y, X]_{\mathbf{m}}]_{\mathbf{m}}) + \tfrac{1}{2} g(Y, [X, [X, Y]_{\mathbf{m}}]_{\mathbf{m}}) \\ & + g(U(X, X), U(Y, Y)) - g(U(X, Y), U(X, Y)), \end{aligned} \tag{11}$$

where $U : \mathbf{m} \times \mathbf{m} \to \mathbf{m}$ is the tensor defined by

$$2g(U(X, Y), Z) = g([Z, X]_{\mathbf{m}}, Y) + g([Z, Y]_{\mathbf{m}}, X),$$

for any $X, Y, Z \in \mathbf{m}$ and $[\cdot, \cdot]_{\mathbf{m}}$ (resp. $[\cdot, \cdot]_{\mathbf{h}}$) denotes the projection of the commutator of two elements of $\mathbf{m}$ into $\mathbf{m}$ (resp. $\mathbf{h}$).

Consider the g-unitary frame $\{Z_1, Z_2, Z_3\}$ of $T^{1,0}_{J_0}$ determined by the elements $\lambda_1^{-1/2}(1, 0, 0)$; $\lambda_2^{-1/2}(0, 1, 0)$; $\lambda_3^{-1/2}(0, 0, 1)$ of $\mathbf{m}$. Then we have the parametrization (4) of the orthogonal almost complex structures of $(F_{1,2}, g)$, obtained with respect to the four commuting left-invariant almost complex structures $\{J_0, J_1, J_2, J_3\}$. Suppose that J is a positive orthogonal complex structure at $x \in F_{1,2}$. If J belongs to the $\mathbb{C}P^1$-family (a) (see Section 2.2), then there are complex numbers δ_1, δ_2, such that the $(1,0)$-space of J is spanned by the complex vectors Z_0^J, Z_1^J, Z_3^J, defined by (5). Observe that $Z_0^J, Z_1^J \in T_{J_3}^{0,1}$ and $Z_1^J, Z_3^J \in T_{J_0}^{0,1}$, hence

$$R(Z_0^J, Z_1^J, Z_0^J, Z_1^J) = R(Z_1^J, Z_3^J, Z_1^J, Z_3^J) = 0, \tag{12}$$

since any of J_0, J_1, J_2, J_3 belongs to $Z_0^+(F_{1,2}, g)$. Using (11) and (12) we further compute

$$\begin{aligned}
&R(Z_0^J, Z_3^J, Z_0^J, Z_3^J) = \frac{2\delta_1^2\delta_2^2}{\lambda_1\lambda_2\lambda_3}C_3;\\
&2R(Z_0^J, Z_3^J, Z_1^J, Z_3^J)\\
&\qquad = R(Z_0^J + Z_1^J, Z_3^J, Z_0^J + Z_1^J, Z_3^J) - R(Z_0^J, Z_3^J, Z_0^J, Z_3^J) = 0,\\
&2R(Z_0^J, Z_3^J, Z_0^J, Z_1^J)\\
&\qquad = R(Z_1^J + Z_3^J, Z_0^J, Z_1^J + Z_3^J, Z_0^J) - R(Z_0^J, Z_3^J, Z_0^J, Z_3^J) = 0,\\
&2R(Z_0^J, Z_1^J, Z_1^J, Z_3^J) = R(Z_0^J + Z_3^J, Z_1^J, Z_0^J + Z_3^J, Z_1^J) = 0.
\end{aligned} \tag{13}$$

It follows from (12) and (13) that J belongs to $\pi^{-1}(x) \cap Z_0^+(F_{1,2}, g)$ iff $\delta_1\delta_2 C_3 = 0$. In the case when the positive real numbers λ_i satisfy $C_3 = 0$ any element in the $\mathbb{C}P^1$-family (a) satisfies (1), i.e., $\mathcal{P}_3 \subset Z_0^+(F_{1,2}, g)$; otherwise we obtain that either $\delta_1 = 0$, i.e., $J = \pm J_2$, or $\delta_2 = 0$, i.e., $J = \pm J_1$.

If J belongs to the $\mathbb{C}^2$-family (b) at $x \in F_{1,2}$, then there are complex numbers γ_2, γ_3 such that the $(1,0)$ space of J is spanned by the complex vectors Z_0^J, Z_2^J, Z_3^J defined by (6). Using (11) and the fact that J_0, J_1, J_2, J_3 satisfy (1) we obtain

$$\begin{aligned}
R(Z_2^J, Z_3^J, Z_2^J, Z_3^J) &= 0,\\
R(Z_0^J, Z_2^J, Z_0^J, Z_2^J) &= \frac{2\gamma_2^2}{\lambda_1\lambda_2\lambda_3}C_1,\\
R(Z_0^J, Z_3^J, Z_0^J, Z_3^J) &= \frac{2\gamma_1^2}{\lambda_1\lambda_2\lambda_3}C_2,\\
2R(Z_0^J, Z_3^J, Z_2^J, Z_3^J) &= R(Z_0^J, Z_3^J, Z_0^J, Z_3^J) - R(Z_0^J - Z_2^J, Z_3^J, Z_0 - Z_2, Z_3^J)\\
&= \frac{2\gamma_1^2}{\lambda_1\lambda_2\lambda_3}C_2,\\
2R(Z_0^J, Z_2^J, Z_0^J, Z_3^J) &= R(Z_0^J, Z_2^J + Z_3^J, Z_0, Z_2^J + Z_3^J) - R(Z_0^J, Z_2^J, Z_0^J, Z_2^J)\\
&\quad - R(Z_0^J, Z_3^J, Z_0^J, Z_3^J)\\
&= \frac{2\gamma_1\gamma_2}{\lambda_1\lambda_2\lambda_3}[C_1 + C_2 - C_3],\\
2R(Z_0^J, Z_2^J, Z_2^J, Z_3^J) &= R(Z_0^J, Z_2^J, Z_0^J, Z_2^J) - R(Z_0^J - Z_3^J, Z_2^J, Z_0 - Z_3^J, Z_2^J)\\
&= \frac{2\gamma_2^2}{\lambda_1\lambda_2\lambda_3}C_1,
\end{aligned}$$

It thus follows that J belongs to $\pi^{-1}(x) \cap Z_0^+(F_{1,2}, g)$ iff

$$\gamma_1 C_2 = \gamma_2 C_1 = \gamma_1\gamma_2 C_3 = 0. \tag{14}$$

Since the equality $C_1 = C_2 = C_3 = 0$ is impossible we get from (14) $\gamma_1\gamma_2 = 0$. Moreover, in the case when the positive real numbers λ_i, $i = 1, 2, 3$ satisfy $\lambda_1 \neq \lambda_2$ and $C_1 = 0$ or $\lambda_1 \neq \lambda_2$ and $C_2 = 0$ we obtain respectively $\gamma_2 = 0$ or $\gamma_1 = 0$, i.e., $\pi^{-1}(x) \cap Z_0^+(F_{1,2}, g)$ restricted to the family (b) consists of $\mathcal{P}_1$ or $\mathcal{P}_2$, respectively; if $\lambda_1 = \lambda_2$ and $C_1 = C_2 = 0$ it consists of both $\mathcal{P}_1$ and $\mathcal{P}_2$; in any other case we get from (14) $\gamma_1 = \gamma_2 = 0$, i.e., $\pi^{-1}(x) \cap Z_0^+(F_{1,2}, g)$ consists of J_3 only.

Finally, consider the case that J belongs to the $\mathbb{C}^3$-family (c) of positive orthogonal almost complex structures at $x \in F_{1,2}$. Similarly, we get from (7) and (11) that if (1) holds for J, then for the corresponding complex numbers β_i, $i = 1, 2, 3$ the following equalities hold:

$$R(Z_1^J, Z_2^J, Z_1^J, Z_2^J) = \frac{\beta_3^2}{\lambda_1\lambda_2\lambda_3}[3\lambda_3^2 + (\lambda_1 - \lambda_2)^2] = 0;$$
$$R(Z_1^J, Z_3^J, Z_1^J, Z_3^J) = \frac{\beta_2^2}{\lambda_1\lambda_2\lambda_3}[3\lambda_2^2 + (\lambda_3 - \lambda_1)^2] = 0;$$
$$R(Z_2^J, Z_3^J, Z_2^J, Z_3^J) = \frac{\beta_1^2}{\lambda_1\lambda_2\lambda_3}[3\lambda_1^2 + (\lambda_3 - \lambda_2)^2] = 0,$$

and we get $\beta_i = 0$, $i = 1, 2, 3$, i.e., $J = J_0$.

Summarizing, the lemma follows. □

Remark 2. It can be easily deduced from (11) that $(g_{2,1,1}, J_1)$, $(g_{1,2,1}, J_2)$ and $(g_{1,1,2}, J_3)$ are Kähler–Einstein structures of non-negative (but not identically vanishing) holomorphic sectional curvature on $F_{1,2}$. It is also known that there are automorphisms of $F_{1,2}$ (coming from elements of the Weyl group of $\mathbf{SU}(3)$) which switch the three Kähler–Einstein structures. The bi-invariant metric $g_{1,1,1}$ is also Einstein but non-Kähler with respect to any J_i, $i = 1, 2, 3$ (see [7]).

Proof of Theorem 1. Assume that J is an *integrable* positive g-orthogonal almost complex structure on an open subset $\mathcal{U}$ of $F_{1,2}$, different from J_1, J_2, J_3. Since (1) is satisfied at any point of $\mathcal{U}$, according to Lemma 2, J is a section of $\mathcal{P}_i$ for some $i \in \{1, 2, 3\}$. Suppose for example that J is a section of $\mathcal{P}_1$ (the case that J is a section of $\mathcal{P}_2$ and $\mathcal{P}_3$ can be considered similarly). Any section of $\mathcal{P}_1$ has homogeneus coordinates $[0, 0, \alpha_2, \alpha_3]$ with respect to $\{J_0, J_1, J_2, J_3\}$; it thus follows by (4) that it is orthogonal with respect to the one-parameter family of left-invariant metrics $g_{\lambda_1, t\lambda_2, t\lambda_3}$, $t > 0$. We get that the complex structure J belongs to $Z_0^+(F_{1,2}, g_{\lambda_1, t\lambda_2, t\lambda_3})$ for any $t > 0$. According to Lemma 2 the equality $3t(\lambda_2 + \lambda_3)\lambda_1 - t^2(\lambda_2 - \lambda_2)^2 = 0$ holds for any $t > 0$, a contradiction. □

Proof of Corollary 1. Let $\mathcal{U}$ be the non-empty open subset of $F_{1,2}$ where the differential of f has maximal rank. As f is a harmonic map between real analytic spaces, $\mathcal{U}$ is dense in $F_{1,2}$. It follows from [14, Cor. 2] that there is a positive orthogonal complex structure J on $\mathcal{U}$ and f is J-holomorphic. According to Theorem 1, J coincides with one of the structures

$\pm J_i$, $i = 1, 2, 3$ on any connected subset of $\mathcal{U}$. But for *any* left-invariant metric g on $F_{1,2}$ the co-differential of the Kähler form of (g, J_i) is zero. Indeed, since the Kähler form of (g, J_i) is $\mathbf{U}(3)$-left-invariant, the same is true for its co-differential, hence it is of constant length and then vanishes because the Euler characteristic of $F_{1,2}$ is positive. Thus f is a harmonic map from $(\mathcal{U}, g)$ to $\mathbf{M}$ [32, Prop. 2.1], and since $\mathcal{U}$ is dense in $F_{1,2}$, we obtain that f is harmonic with respect to any left-invariant metric on $F_{1,2}$, i.e., f is equiharmonic. It is easy now to prove that f is $\pm$-holomorphic with respect to some of the complex structures J_i, $i = 1, 2, 3$: As we have already observed, on any connected subset $\mathcal{U}_0$ of $\mathcal{U}$ the function f is holomorphic with respect to some of the complex structures $\pm J_i$, $i = 1, 2, 3$, say J_1. Take the left-invariant Kähler–Einstein metric $g = g_{2,1,1}$ with respect to J_1 (see Remark 2). Then f is a harmonic map between compact Kähler manifolds, $(F_{1,2}, g, J_1)$ and $\mathbf{M}$, which is holomorphic on a non-empty open subset of $F_{1,2}$, hence on all of $F_{1,2}$ by Siu's Unique Continuation Theorem [36]. □

Remark 3. See [12] for the corresponding results concerning stable harmonic maps between Hermitian-symmetric spaces.

4. Fano 3-folds admitting commuting complex structures

We now consider the two homogeneous 6-manifolds $\mathbb{C}P^1 \times \mathbb{C}P^2 \times \mathbb{C}P^1$ and $F_{1,2}$ with some of its invariant complex structures described in the preceding section. Observing that they are both spin manifolds with positive first Chern class, we give a necessary and sufficient conditions a compact spin Fano 3-fold to be biholomorphically equivalent to either $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ or $F_{1,2}$ in terms of the existence of three commuting almost complex structures.

Proposition 2. *Suppose that a compact spin* 6-*manifold* M *admits three commuting almost complex structures* J_1, J_2, J_3 *which satisfy the following properties*:

(i) J_1 *is integrable and* (M, J_1) *is a Fano* 3-*fold, i.e.,* $c_1(M, J_1) > 0$;

(ii) J_2 *is integrable of Kähler type*;

(iii) *there exists a Kähler metric* g *on* (M, J_1), *which is compatible with* J_2.

Then (M, J_1) *is biholomorphic to either* $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ *or* $F_{1,2}$.

Proof. We claim that the second Betti number $b_2(M)$ is grater that 2. To this end, suppose that $b_2(M) = 1$. Then we have $c_1(M, J_2) = c\Omega_{J_2}$, where Ω_{J_2} is a Kähler form on (M, J_2) and c is a real constant. Suppose first that $c = 0$, i.e., $c_1(M, J_2) = 0$. Since (M, J_1) is a Fano 3-fold with $b_2(M) = 1$, we have in fact $H^2(M, \mathbb{Z}) = \mathbb{Z}$. According to Proposition 1 the existence of three commuting almost complex structures J_1, J_2, J_3 on M with $c_1(M, J_2) = 0$ implies that there are integers $a_1, a_2, a_1 + a_2 > 0$, such that

$$\begin{aligned} &c_1(M, J_1) = 2(a_1 + a_2)h;\\ &p_1(M) = 2(a_1^2 + a_2^2 + a_1a_2)h^2;\\ &e(M) = a_1a_2(a_1 + a_2)h^3, \end{aligned} \tag{15}$$

where h is the generator of $H^2(M, \mathbb{Z})$. Moreover, since the Chern numbers of any Fano 3-fold satisfy $c_1c_2 = 24$ (see for example [28] for a nice overview on the classification and some properties of Fano manifolds), we get by (15)

$$(a_1 + a_2)\big(a_1 a_2 + (a_1 + a_2)^2\big)h^3 = 12. \tag{16}$$

If $a_1 + a_2 \geqslant 2$, by the result of Kobayshi and Ochiai [22] (M, J_1) is biholomorphically equivalent to $\mathbb{C}P^3$ and then $a_1 + a_2 = 2$, $h^3 = 1$, i.e., $a_1 a_2 = 2$, a contradiction. We thus have $a_1 + a_2 = 1$ and then $1 \leqslant h^3 \leqslant 9$, cf. [28], which again contradicts with (16).

Assume now that $c \neq 0$, i.e., $c_1(M, J_2)$ is either positive or negative definite. Consider the case $c > 0$, i.e., $c_1(M, J_2) > 0$ (the case $c_1(M, J_2) < 0$ can be considered similarly). As J_1 and J_2 are mutually commuting orthogonal (almost) complex structures with respect to the Riemannian metric g, it follows that the Kähler form Ω_{J_1} of (g, J_1) is a $(1, 1)$-form with respect to J_2. From the assumption $b_2(M) = 1$ we get

$$a\Omega_{J_1} = \gamma_{J_2} + i\partial_{J_2}\bar{\partial}_{J_2} f, \tag{17}$$

where γ_{J_2} is a positive $(1, 1)$-form on (M, J_2), representing $c_1^{\mathbb{R}}(J_2)$, a is a non zero real constant, and f is a real-valued function. Let x be a point of minimum of f. Then at x the $(1, 1)$ form $i\partial_{J_2}\bar{\partial}_{J_2} f$ is semi-positive with respect to J_2. Since J_1 and J_2 commute but do not coincide, there exist non-zero tangent vectors $X', X'' \in T_x M$ such that $J_1 X' = J_2 X''$, $J_1 X'' = -J_2 X'$. We obtain from these

$$\Omega_{J_1}(J_2 X', X') = g(X', X'), \quad \Omega_{J_1}(J_2 X'', X'') = -g(X'', X''). \tag{18}$$

But we have at x

$$\gamma_{J_2}(X, J_2 X) > 0, \quad i\partial_{J_2}\bar{\partial}_{J_2} f(X, J_2 X) \geqslant 0,$$

for any non-zero tangent vector X, hence (18) contradict (17).

Thus $b_2(M) \geqslant 2$ and since M is spin, (M, J_1) is a Fano 3-fold of index 2 and Picard group of rank $\geqslant 2$. It follows from the classification of Fano 3-folds [19, 20, 27], that (M, J_1) is biholomorphic to one of the following complex 3-folds: $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, $F_{1,2}$, $\widetilde{\mathbb{C}P}^3$, where $\widetilde{\mathbb{C}P}^3 \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ is one-point blow-up of $\mathbb{C}P^3$. We are going to prove that $\widetilde{\mathbb{C}P}^3$ does not admit 3 commuting almost complex structures (but it is clear that it admits two commuting almost complex structures defined by reversing the sign of the complex structure on the fibre). We shall make use of some standard facts about Chern classes and the cohomology ring of projective bundle over complex manifold, which could be found in [17]. For any holomrphic vector bundle $p : E \to X$ there is a projectivization $\pi : \mathbb{P}(E) \to X$ and an exact sequence of sheaves:

$$\mathcal{O} \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow \pi^* E \otimes \mathcal{O}_{E(1)} \longrightarrow T_{\mathbb{P}(E)} \longrightarrow \pi^* T_X \longrightarrow 0,$$

where T_X is the holomorphic tangent bundle of X and $\mathcal{O}_E(1)$ is the bundle over $\mathbb{P}(E)$ which restricted on the fibre is $\mathcal{O}(1)$. We then have

$$c_t(T_{\mathbb{P}(E)}) = c_t(\pi^* T_x) c_t(\pi^* E \otimes \mathcal{O}_E(1)),$$

where c_t is the Chern polynomial of the corresponding bundle. It is also known that

$$c_p(E \otimes L) = \sum_{i=0}^{p} \binom{r-i}{p-i} c_i(E) c_1(L)^{p-i},$$

for the bundles E and L of ranks r and 1, respectively.

Set $\tilde{h} = c_1(\mathcal{O}_E(1))$, $h = c_1(\pi^*\mathcal{O}_{\mathbb{C}P^2}(1))$. Then the cohomology ring $H^*(\widetilde{\mathbb{C}P}^3, \mathbb{Z})$ is generated by h and $\tilde{h}$ with the relations $h^3 = \tilde{h}^3 = 0$ and $\tilde{h}^2 + h\tilde{h} = 0$ between them. Furthermore, we compute for the projectivisation $\widetilde{\mathbb{C}P}^3$ of $E = \mathcal{O} \oplus \mathcal{O}(1)$

$$c_1(\widetilde{\mathbb{C}P}^3) = c_1(\pi^* T_{\mathbb{C}P^2}) + c_1(\pi^* E \otimes \mathcal{O}_E(1)) = 4h + 2\tilde{h};$$
$$c_2(\widetilde{\mathbb{C}P}^3) = c_2(\pi^* T_{\mathbb{C}P^2}) + c_2(\pi^* E \otimes \mathcal{O}_E(1)) + c_1(T_{\mathbb{C}P^2}) c_1(\pi^* E \otimes \mathcal{O}_E(1);$$
$$c_2(\widetilde{\mathbb{C}P}^3) = 3h^2 + \tilde{h}^2 + h\tilde{h} + 3h(2\tilde{h} + h) = 6h^2 + 6h\tilde{h};$$
$$p_1(\widetilde{\mathbb{C}P}^3) = c_1^2(\widetilde{\mathbb{C}P}^3) - 2c_2(\widetilde{\mathbb{C}P}^3) = (4h + 2\tilde{h})^2 - 2(6h^2 + 6h\tilde{h}) = 4h^2.$$

Suppose that $\widetilde{\mathbb{C}P}^3$ admits three mutually commuting orthogonal almost complex structures. Then by Proposition 1 there exist $\omega_i \in H^2(\widetilde{\mathbb{C}P}^3, \mathbb{Z})$, $i = 1, 2, 3$ such that:

$$\omega_1 + \omega_2 + \omega_3 = c_1(\widetilde{\mathbb{C}P}^3) = 4h + 2\tilde{h}, \quad \omega_1^2 + \omega_2^2 + \omega_3^2 = p_1(\widetilde{\mathbb{C}P}^3) = 4h^2.$$

If $\omega_i = a_i h + b_i \tilde{h}$, $i = 1, 2, 3$, we obtain from the above formulas that

$$a_1 + a_2 + a_3 = 4, \quad a_1^2 + a_2^2 + a_3^2 = 4.$$

But the latter equalities are impossible for any integers a_1, a_2, a_3, hence $\widetilde{\mathbb{C}P}^3$ doesn't admit three commuting almost complex structures. □

Regarding now to the homogeneous spaces $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and $F_{1,2}$ with its (left) invariant *Kähler–Einstein* structures described in the preceding section (see Remark 2) we are ready to prove Theorem 2.

Proof of Theorem 2. We start with the following observation, which can be considered as a 6-dimensional analogue of [13, Theorem 5.6].

Lemma 3. *Let (M, g, J) be a Kähler–Einstein, non-Ricci-flat manifold of real dimension* 6 *with non-negative* (*non-positive*) *holomorphic sectional curvature. Then any orthogonal complex structure on (M, g) commutes with J.*

Proof of Lemma 3. Suppose that J' is a positive orthogonal complex structure on (M, g), different from J. Fix a point $x \in M$ and let $[v], [v'] \in \mathbb{P}(V) \cong \mathbb{C}P^3$ be the points of the twistor fibre $\mathbf{SO}(6)/\mathbf{U}(3) \cong \mathbb{C}P^3$ at x, corresponding to J and J' (see Section 2.2). As we have already mentioned (cf. [5, Lemma 1]), J and J' commute if and only if $h(v, v') = 0$, where h is the standard metric on the twistor fibre $\mathbb{C}P^3$. Thus, writing $v' = \alpha v/|v| + v_1$, where α is a complex number and v_1 is non-zero vector, orthogonal to v, we have to prove that $\alpha = 0$. Without lose of generality we may assume that $h(v_1, v_1) = 1$. Setting $v_0 = v/|v|$, we consider a unitary frame $\{v_0, v_1, v_2, v_3\}$ of (V, h), which defines four mutually commuting orthogonal complex structures $J = J_0, J_1, J_2, J_3$ of (T_xM, g). According to (4) we have

$$T^{1,0}_{J'} = \text{span}\{Z_1, \alpha Z_2 + \bar{Z}_3, \alpha Z_3 - \bar{Z}_2\}, \tag{19}$$

where $\{Z_1, Z_2, Z_3\}$ is the unitary frame of $T^{1,0}_J$, defined by (3). Since J' is integrable, the condition (1) is satisfied at x. Using (19) and the fact that (g, J) is Kähler we then calculate

$$\alpha^2\big(R(Z_2, \bar{Z}_2, Z_2, \bar{Z}_2) + R(Z_3, \bar{Z}_3, Z_3, \bar{Z}_3) + 2R(Z_2, \bar{Z}_2, Z_3, \bar{Z}_3)\big) = 0, \tag{20}$$

where R is the curvature of (M, g). The Einstein condition on a Kähler manifold of real dimension 6 reads as

$$R(Z_1, \bar{Z}_1, \cdot, \cdot) + R(Z_2, \bar{Z}_2, \cdot, \cdot) + R(Z_3, \bar{Z}_3, \cdot, \cdot) = -\tfrac{1}{6} isg(J\cdot, \cdot),$$

where $s \neq 0$ is the scalar curvature of (M, g). Then (20) can be rewritten as

$$\alpha^2\big[\tfrac{1}{6} s + R(Z_1, \bar{Z}_1, Z_1, \bar{Z}_1)\big] = 0,$$

and hence $\alpha = 0$ since the holomorphic sectional curvature and the scalar curvature could not have opposite signs, cf. [9, Theorem 2]. □

To prove Theorem 2 observe first that the scalar curvature s of g is a positive constant since the holomorphic sectional curvature of (g, J) is non-negative but does not identically vanish. Indeed, using the first Bianchi identity and the fact that g is Kähler, we get for the scalar curvature s:

$$\begin{aligned} s &= 2\sum_{i,j} R(Z_j, \bar{Z}_i, Z_i, \bar{Z}_j) \\ &= 2\sum_{k} R(Z_k, \bar{Z}_k, Z_k, \bar{Z}_k) + 2\sum_{i\neq j} R(Z_i, \bar{Z}_i, Z_j, \bar{Z}_j), \end{aligned}$$

where $\{Z_k\}_{k=1}^3$ is any unitary frame of $T^{1,0}_J$. On the other hand it is shown in the proof of [9, Theorem 2] that on any Kähler manifold (M, g, J) of non-negative holomorphic sectional curvature the following inequality holds:

$$2(\tfrac{1}{2}\dim_{\mathbb{R}} M - 1)\sum_{k} R(Z_k, \bar{Z}_k, Z_k, \bar{Z}_k) + 4\sum_{i\neq j} R(Z_i, \bar{Z}_i, Z_j, \bar{Z}_j) \geqslant 0.$$

It is easily seen that the inequality above is strict at any point where the holomorphic sectional curvature does not identically vanishes. On Kähler 3-folds it reduces to $s \geqslant 0$ and since the holomorphic sectional curvature does not identically vanishes, we infer that the scalar curvature s is a positive constant. Now it follows from Lemma 3 that J and J' mutually commute. As (g, J) is Kähler–Einstein of positive scalar curvature we have $c_1(M, J) > 0$, i.e., (M, J) is Fano 3-fold. Using the same arguments as in the proof of Proposition 2 we obtain that (M, J) is biholomorphic to one of the following spaces: $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, $F_{1,2}$, $\widetilde{\mathbb{C}P}^3$ where, we recall, $\widetilde{\mathbb{C}P}^3$ is one-point blow-up of $\mathbb{C}P^3$. But the automorphism group of $\widetilde{\mathbb{C}P}^3$ has non-reductive Lie algebra since its matrix representation has a zero column. By the Matsushima–Lichnerovich obstruction (see, e.g., [11, 11.D]), we know that $\widetilde{\mathbb{C}P}^3$ does not admit Kähler–Einstein metrics at all. Moreover, according to the uniqueness of Kähler–Einstein metrics modulo biholomorphisms

[10, 24, 23] we have that (g, J) must be one of the invariant Kähler–Einstein structures on $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ or $F_{1,2}$. Now the last part of the theorem follows from Theorems 3 and 1. □

Remark 4. The proof of Lemma 3 shows that the Kähler–Einstein condition can be relaxed by an appropriate pinching condition on the Ricci tensor. On the other hand the following examples are related to the necessity of the conditions of Theorem 2:

(i) $\mathbb{C}P^3$ with the Fubini–Study metric is a Kähler–Einstein spin manifold with positive holomorphic sectional curvature admitting abundance of local orthogonal complex structures, which does not admit a global one.

(ii) $\mathbb{C}P^1 \times \mathbb{C}P^2$ admits Kähler–Einstein metric of non-negative holomorphic sectional curvature and global orthogonal complex structure different from the standard one, but it is not a spin manifold.

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