EXTREMAL KÄHLER METRICS ON RULED MANIFOLDS AND STABILITY

by

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Abstract. — This article gives a detailed account and a new presentation of a part of our recent work [**3**] in the case of admissible ruled manifolds without blow-downs. It also provides additional results and pieces of information that have been omitted or only sketched in [**3**].

Contents

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Introduction

Compact complex manifolds which admit hamiltonian 2-forms of order 1 in the sense of [**1, 2**] — cf. Section 1.8 for a formal definition — have been classified in [**2**] and extensively studied in [**3**]. The main motivation in [**3**] for studying this class of Kähler manifolds is the fact that they provide a fertile testing ground for the conjectures relating extremal and CSC Kähler metrics to stability. In particular, by using recent results of X. Chen–G. Tian, here quoted as Theorem 2.1, we were able to solve in [**3**] a long pending open question since [**42**], namely the non-existence of extremal Kähler metrics in "large" Kähler classes on "pseudo-Hirzebruch surfaces", which was the last missing step towards the full resolution of the existence problem of extremal Kähler metrics on geometrically ruled complex surfaces [**5**].

The main goal of this paper is to present some salient results of our joint work [**3**]. To simplify the exposition, we here only consider the simple case of $\mathbb{P}^1$-bundles over a product of compact Kähler manifolds of constant scalar curvature, which in the terminology in [**3**] is referred to as the case *without blow-downs*. This allows us for a specific treatment, somewhat simpler than the general case worked out in [**3**], to which we refer the reader for more information and details.

For the comfort of the reader, we tried to make this paper as self-contained and easy to read as possible. With regard to [**3**], we introduce in places slightly different notation and terminology, that seem to be more adapted to the specific situations worked out in this paper. Similarly, some computations and arguments taken from [**3**] here appear in a slightly different and/or a more detailed presentation. The paper also includes new pieces of information, which were omitted or only sketched in [**3**], like Proposition 1.5 in Section 1.9, Proposition A.1 in Appendix A, a specific account of the deformation to the normal cone of the infinity section in admissible ruled manifolds, etc.

The paper is organized as follows.

In Sections 1.1 to 1.7, we set the general framework of the paper by introducing the class of *admissible ruled manifolds*, the cone of *admissible Kähler classes*, the set of *admissible momenta* and the associated set of of *admissible Kähler metrics*, and by recalling the main geometric features of these metrics (isometry groups, Ricci form, scalar curvature, etc.). In Section 1.8, we briefly explain how hamiltonian 2-forms of order 1 arise in this setting. In Section 1.9, we use a variant of the Calabi method in [**8**], also used in [**42**], to construct extremal admissible Kähler metrics in a given admissible Kähler class Ω; as in [**42**], we show that this method works successfully if and only if the *extremal polynomial* F_Ω, canonically attached to Ω, is positive on its interval of definition. Section 1.10 is devoted to the special case of admissible ruled *surfaces*, here called *Hirzebruch-like ruled surfaces*.

In Section 2.1, we review some well-known general facts concerning the space of Kähler metrics in a given Kähler class on a compact complex manifold. In Section 2.2, we recall some basic results recently obtained by X. X. Chen and G. Tian, here stated as Theorem 2.1, which play an important role in several parts of the paper. In Section 2.3, we compute the relative Mabuchi K-energy on the space of admissible Kähler metrics in any admissible Kähler class Ω and we show that Ω admits an extremal Kähler metric, which is then admissible up to automorphism, if and only if F_Ω is positive on its interval of definition (Theorem 2.2). Proposition A.1 established in Appendix A is used to complete the proof of Theorem 2.2 in the *borderline case*, when F_Ω is non-negative but has zeros, possibly irrational, in its interval of definition.

In Section 3.1, we recall the interpretation given by Donaldson and adapted by Székelyhidi to the relative case of the Futaki invariant of an S^1-action on a general polarized projective manifold. In Section 3.2, we construct the *deformation to the normal cone*, $\mathcal{D}(M)$, of the infinity section Σ_∞ of an admissible ruled manifold M. In Section 3.3, for any admissible polarization Ω on M, we turn $\mathcal{D}(M)$ into a *test configuration* in the sense of Tian [**41**] and Donaldson [**15**], by constructing a family of relative polarizations, parametrized by rational numbers in the interval of definition of the extremal polynomial F_Ω. In Section 3.4, we extend to admissible ruled manifolds a beautiful computation done by G. Székelyhidi [**39**] for ruled surfaces, and we show that, for any rational number x in $(-1, 1)$, $F_\Omega(x)$ is equal, up to a constant (negative) factor, to the relative Futaki invariant of the test configuration $\mathcal{D}(M)$ equipped with the relative polarization determined by x, see Theorem 3.1. Together with Theorem 2.2, this striking — and still mysterious — fact has the following consequence: for admissible ruled manifolds and admissible Kähler classes, the relative slope K-stability, as defined by J. Ross and R. Thomas [**34, 35**], implies the existence of extremal Kähler metrics, cf. [**3**, Theorem 2]. For a more detailed discussion on this matter, including the role of the examples of

Section 2.4 in the current refined definitions of the slope stability, the reader is referred to [**3**, Theorem 2].

Notation and convention: For any Kähler structure (g, J, ω), the riemannian metric g, the complex structure J and the Kähler form ω are linked together by $\omega = g(J\cdot, \cdot)$. The Levi-Civita connection of g, as a covariant derivative acting on any sorts of tensor fields, will be denoted by D^g, or simply D when the metric is understood. The twisted differential d^c acting on exterior forms is defined by $d^c = JdJ^{-1}$, where J acts on a p-form φ by $(J\varphi)(X_1, \ldots, X_p) = \varphi(J^{-1}X_1, \ldots, J^{-1}X_p)$; in terms of the operators ∂ and $\bar{\partial}$ we then have $d^c = i(\bar{\partial} - \partial)$ and $dd^c = 2i\partial\bar{\partial}$. Our overall convention for the curvature of a linear connection ∇ is $R^{\nabla}_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

1. Extremal metrics on admissible ruled manifolds

1.1. Admissible ruled manifolds. — Unless otherwise specified, M will denote a connected, compact, complex manifold of complex dimension $m \geq 2$, of the form

$$M = \mathbb{P}(1 \oplus L), \tag{1.1}$$

where L denotes a holomorphic line bundle over some (connected, compact) complex manifold S of complex dimension $(m-1)$. Here, 1 stands for the trivial holomorphic line bundle $S \times \mathbb{C}$ and $\mathbb{P}(1 \oplus L)$ then denotes the projective line bundle associated to the holomorphic rank 2 vector bundle $E = 1 \oplus L$: an element ξ of M over a point y of S is then a complex line through the origin in the complex 2-plane $E_y = \mathbb{C} \oplus L_y$, where E_y, L_y denote the fibres of E, L at y; if ξ is generated by the pair (z, u) in $\mathbb{C} \oplus L_y$, we write $\xi = (z : u)$. The natural (holomorphic) projection $\pi : M \to S$ admits two natural (holomorphic) sections: the *zero section* $\sigma_0 : y \mapsto \mathbb{C} \subset \mathbb{C} \oplus L_y$, and the *infinity section* $\sigma_\infty : y \mapsto L_y \subset \mathbb{C} \oplus L_y$. We denote by Σ_0, Σ_∞ the images of σ_0, σ_∞ in M, still called zero section and infinity section, both identified with S via π. Each element of $M \setminus \Sigma_\infty$ over y has a unique generator of the form $(1, u)$, with u in L_y: we thus get a natural identification of $M \setminus \Sigma_\infty$ with L and M can therefore be regarded as a compactification of (the total space of) L obtained by adding a *point at infinity* to each fiber. The open set $M_0 = M \setminus (\Sigma_0 \cup \Sigma_\infty)$ is similarly identified with the set of non-zero elements of L.

The natural $\mathbb{C}^*$-action on L extends to a holomorphic $\mathbb{C}^*$-action on M defined by: $\zeta \cdot (z : u) = (z : \zeta\, u)$. This action pointwise fixes Σ_0 and Σ_∞. The vector field on M generating the induced S^1-action is denoted by T.

We furthermore assume that $S = \prod_{i=1}^{N} S_i$ is the product of $N \geq 1$ (connected, compact) complex manifolds S_i, of complex dimensions d_i, and that L comes equipped with a (fiberwise) hermitian inner product, h, such that the curvature, R^{∇}, of the corresponding Chern connection, ∇, is of the form:

$R^\nabla = -i\sum_{i=1}^N \epsilon_i\,\omega_{S_i}$, where each ω_{S_i} is the Kähler form of a Kähler metric, g_{S_i}, on S_i (viewed as a 2-form on S. i.e. identified with $p_i^*\omega_i$, if p_i denotes the natural projection from S to S_i), and ϵ_i is equal to 1 or to -1. In particular, $\sum_{i=1}^N \epsilon_i\,[\omega_{S_i}] = 2\pi\, c_1(L^*)$, where $c_1(L^*)$ denotes the first Chern class of the dual complex line bundle L^* and $[\omega_{S_i}]$ the class of ω_{S_i} in $H^2(S,\mathbb{R})$.

Moreover, for $i = 1,\dots,N$, we assume that $R^{\nabla_i} = -i\epsilon_i\omega_{S_i}$ is the Chern curvature of a hermitian holomorphic line bundle, L_i, on S_i — so that (S_i,ω_{S_i}) is polarized by $\tilde{L}_i = L_i^{-\epsilon_i}$ — and that $L = \otimes_{i=1}^N p_i^* L_i$, equipped with the induced (fiberwise) hermitian metric.

On M_0, identified with $L\setminus\Sigma_0$ as above, define t by

$$t = \log r, \tag{1.2}$$

where $r = |\cdot|_h$ denotes the norm relative to h, viewed as a function on $L = M\setminus\Sigma_\infty$. We then have

$$d^c t(T) = 1, \qquad dd^c t = \pi^*(\sum_{i=1}^N \epsilon_i\,\omega_{S_i}), \tag{1.3}$$

where the twisted differential operator d^c, as defined above, is relative to the natural complex structure of M. The latter, as well as the complex structures of S and of each factor S_i, will be uniformly denoted by J and will be kept unchanged throughout the paper.

Definition 1.1. — Ruled manifolds of the above kind, with the additional pieces of structure described in this section, will be referred to as *admissible ruled manifolds*. Later on in this paper, we shall assume that the scalar curvature of each factor (S_i, g_{S_i}) of S is *constant*, but this assumption is not needed until Section 1.9.

1.2. Admissible Kähler classes. — We denote by e_0, resp. e_∞, the Poincaré dual of (the homology class of) Σ_0, resp. Σ_∞, in $H^2(M,\mathbb{R})$ and we set:

$$\Xi = 2\pi(e_0 + e_\infty). \tag{1.4}$$

The class $e_0 + e_\infty$ can be regarded as a projective version of the Thom class of L, whereas

$$\pi^* c_1(L) = e_0 - e_\infty, \tag{1.5}$$

where $c_1(L)$ denotes the first Chern class of L (cf. Remark 1.1 below). Any element, γ, of $H^2(M,\mathbb{R})$ can be written in a unique way as $\gamma = a\,\Xi + \pi^*\alpha$, with a in $\mathbb{R}$ and α in $H^2(S,\mathbb{R})$. Moreover, in order that γ belong to the Kähler cone of M, it certainly must satisfy the following two conditions: (i) its value on each fiber of π is positive, hence $a > 0$; (ii) $\gamma_{|\Sigma_0}$ and $\gamma_{|\Sigma_\infty}$ both belong to the Kähler cone of S, via the natural identification of Σ_0 and Σ_∞ with S. Now, $(e_0+e_\infty)_{|\Sigma_0} = e_{0|\Sigma_0} = -\frac{1}{2\pi}\sum_{i=1}^N \epsilon_i\,[\omega_{S_i}]$ and $(e_0+e_\infty)_{|\Sigma_\infty} = e_{\infty|\Sigma_\infty} = \frac{1}{2\pi}\sum_{i=1}^N \epsilon_i\,[\omega_{S_i}]$, via the natural identification of Σ_0, Σ_∞ with S (recall that $e_{0|\Sigma_0}$ is the first Chern class of the normal bundle of Σ_0 in M, identified with L on S; similarly, $e_{\infty|\Sigma_\infty}$ is the first Chern class of the normal bundle of Σ_∞ in M, identified with L^* on S). It follows that Ξ does *not* belong to the Kähler cone of M, whereas

$$\Omega_{\boldsymbol\lambda} = \sum_{i=1}^N \lambda_i\,\pi^*[\omega_{S_i}] + \Xi \tag{1.6}$$

clearly satisfies the above conditions (i)-(ii) whenever all λ_i's are real numbers greater than 1. In fact, as will be checked in the next section (cf. Remark 1.2), $\Omega_{\boldsymbol\lambda}$ *is* a Kähler class on M for any N-tuple $\boldsymbol\lambda = (\lambda_1, \dots, \lambda_N)$ of real numbers such that $\lambda_i > 1$, $i = 1, \dots, N$. Such N-tuples of real numbers will be called *admissible.*

Definition 1.2. — A *normalized admissible Kähler class* is a Kähler class of the form (1.6), where $\boldsymbol\lambda$ is an admissible N-tuple of real numbers. The *characteristic polynomial*, $p_{\Omega_{\boldsymbol\lambda}}$, of a normalized admissible Kähler class $\Omega_{\boldsymbol\lambda}$ is defined by

$$p_{\Omega_\lambda}(x) = \prod_{i=1}^N (\lambda_i + \epsilon_i\,x)^{d_i}. \tag{1.7}$$

An *admissible* Kähler class is a multiple of a normalized one by a positive real number. The *admissible Kähler cone* is the set of all admissible Kähler classes.

Remark 1.1. — Denote by $\mathcal{O}_M(-1)$ the *tautological line bundle* on M and by $\mathcal{O}_M(1)$ its complex dual: for any $\xi = (z : u)$ in M, the fiber of $\mathcal{O}_M(-1)$ at ξ is the complex line ξ itself, whereas the fiber of $\mathcal{O}_M(1)$ at ξ is $\xi^* = \mathrm{Hom}(\xi, \mathbb{C})$. The natural projection of $\mathbb{C} \oplus L$ on $\mathbb{C}$ determines a holomorphic section of $\mathcal{O}_M(1)$, whose zero locus is Σ_∞; similarly, the natural projection of $\mathbb{C} \oplus L$ on L determines a holomorphic section of $\mathcal{O}_M(1) \otimes L$, whose zero locus is Σ_0. We then have

$$e_\infty = c_1(\mathcal{O}_M(1)), \qquad e_0 = c_1(\mathcal{O}_M(1)) + c_1(\pi^*L), \tag{1.8}$$

hence

$$\Xi = 2\pi\big(2\,c_1(\mathcal{O}_M(1)) + \pi^*c_1(L)\big), \tag{1.9}$$

and

$$\Omega_{\boldsymbol\lambda} = 2\pi\,\big(2c_1(\mathcal{O}_M(1) + \sum_{i=1}^N (\lambda_i - \epsilon_i)\,c_1(\pi^*L_i^{-\epsilon_i})\big). \tag{1.10}$$

It follows that $\Omega_{\boldsymbol{\lambda}}/2\pi$ belongs to the image of $H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$ if and only if all λ_i's are (positive) integers. If so, $\Omega_{\boldsymbol{\lambda}}/2\pi = c_1(\mathcal{F}_{\boldsymbol{\lambda}})$, with

$$\mathcal{F}_{\boldsymbol{\lambda}} = \mathcal{O}_M(2) \otimes \pi^*\big(\otimes_{i=1}^N L_i^{1-\epsilon_i\lambda_i}\big). \tag{1.11}$$

1.3. Admissible momenta and Kähler metrics. — For each admissible Kähler class we construct a distinguished family of Kähler metrics called *admissible*. For convenience, we restrict our attention to *normalized* admissible Kähler classes, i.e. to Kähler classes which are of the form (1.6). The other ones are obtained by homothety.

Let $z = z(t)$ be any smooth *increasing* function of t which, as a function on M_0, smoothy extends to M, with $z_{|\Sigma_0} \equiv -1$ and $z_{|\Sigma_\infty} \equiv +1$. Equivalently, we demand that z, as a function of t, satisfies the following boundary conditions:

- $B_{-\infty}$: Near $t = -\infty$, $z(t) = \Phi_{-\infty}(e^{2t})$, where $\Phi_{-\infty}$ is smoothly defined on $[0,\varepsilon)$, for some $\varepsilon > 0$, with $\Phi_{-\infty}(0) = -1$ and $\Phi'_{-\infty}(0) > 0$.
- $B_{+\infty}$: Near $t = +\infty$, $z(t) = \Phi_{+\infty}(e^{-2t})$, where $\Phi_{+\infty}$ is smoothly defined on $[0,\varepsilon)$, for some $\varepsilon > 0$, with $\Phi_{+\infty}(0) = +1$ and $\Phi'_{+\infty}(0) < 0$.

Definition 1.3. — A smooth, increasing function $z : \mathbb{R} \to (-1,1)$, satisfying the boundary conditions $B_{-\infty}$ and $B_{+\infty}$ is called an *admissible momentum.*

For any admissible momentum z, the 2-form $\psi_z = z\sum_{i=1}^N \pi^*\epsilon_i\,\omega_{S_i} + dz\wedge d^ct$ on M_0 smoothly extends to M. Because of (1.3), ψ_z is closed. Moreover, $\psi_{z|\Sigma_0} = -\sum_{i=1}^N \epsilon_i\,\omega_{S_i}$, $\psi_{z|\Sigma_\infty} = \sum_{i=1}^N \epsilon_i\,\omega_{S_i}$, and, for any fiber $\pi^{-1}(y)$, $\int_{\pi^{-1}(y)}\psi = 4\pi$, meaning that $[\psi_z] = \Xi$ for any admissible momentum z. For any admissible Kähler class and for any admissible momentum z, we then define

$$\begin{aligned} \omega = \omega_{\boldsymbol{\lambda},z} &= \sum_{i=1}^N \lambda_i\,\pi^*\omega_{S_i} + \psi_z \\ &= \sum_{i=1}^N (\lambda_i + \epsilon_i\,z)\,\pi^*\omega_{S_i} + dz\wedge d^ct. \end{aligned} \tag{1.12}$$

Then, ω is closed, with $[\omega] = \Omega_{\boldsymbol{\lambda}}$, and is positive definite with respect to J, as $z'(t)$, the derivative of z with respect to t, is positive; it is then the Kähler form of a Kähler metric, $g = g_{\boldsymbol{\lambda},z}$, in $\Omega_{\boldsymbol{\lambda}}$. Moreover, by (1.3), $\iota_T\omega = -z'(t)dt = -dz$, meaning that z is a *momentum* of T with respect to ω.

Definition 1.4. — A Kähler metric is called *admissible* if its Kähler form is of the form (1.12) (for some admissible momentum z) or is a multiple of such metric by a positive real number.

Remark 1.2. — The above construction shows that $\Omega_{\boldsymbol{\lambda}}$ actually belongs to the Kähler cône of M, as claimed in Section 1.2. This also shows that the necessary conditions (i) and (ii) in Section 1.2 are also sufficient.

Remark 1.3. — In each admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, admissible Kähler metrics are, by their very definition, in one-to-one correspondence with the space, $\mathcal{A}$ say, of admissible momenta. Notice however that $\mathcal{A}$ is independent of $\Omega_{\boldsymbol{\lambda}}$.

Remark 1.4. — For any admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, the space of admissible Kähler metrics in $\Omega_{\boldsymbol{\lambda}}$ is preserved by the natural $\mathbb{C}^*$-action on M: each admissible Kähler metric is S^1-invariant whereas, for any real number c and any admissible Kähler metric $g_{\boldsymbol{\lambda},z}$, we have that $e^c \cdot g_{\boldsymbol{\lambda},z} = g_{\boldsymbol{\lambda},z^c}$, where z^c denotes the translated admissible momentum defined by $z^c(t) = z(t+c)$.

Proposition 1.1. — *Let $\Omega_{\boldsymbol{\lambda}_k}$ be a sequence of (normalized) admissible Kähler classes converging to a (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, meaning that $\boldsymbol{\lambda}_k$ converges to $\boldsymbol{\lambda}$ in $\mathbb{R}^N$ for the usual topology. For each k, let g_k be an admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}_k}$, determined by the admissible momentum z_k in $\mathcal{A}$. Suppose that g_k tends to a (smooth) riemannian metric g in the C^1-topology. Then, g is an admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}}$.*

Proof. — Since the g_k tend to g in the C^1-topology, the limit, ω, of the corresponding Kähler forms $\omega_k = g_k(J\cdot,\cdot)$ is closed: g is then a Kähler metric in Ω. On the other hand, ω_k is of the form (1.12) for a well-defined z_k in $\mathcal{A}$. Since the $|z_k|$ are bounded and the sequence dz_k converges to $-\iota_T\omega$, the sequence z_k converges in the C^0-topology to a smooth function z, which is the momentum of T with respect to ω. This function z still factors through t, satisfies the boundary conditions $B_{-\infty} - B_{+\infty}$ and is still increasing, since $z' = dz(T) = g(T,T)$; it then belongs to $\mathcal{A}$ and g is then the associated admissible Kähler metric in Ω. □

1.4. Admissible momentum profiles.

— It is convenient to consider an alternative parametrization of the space of admissible Kähler metrics by introducing, for any admissible momentum map $z : \mathbb{R} \to (-1,1)$, the *momentum profile* Θ defined by

$$\Theta(x) = z'(z^{-1}(x)), \tag{1.13}$$

for any x in the open interval $(-1,1)$, where, $z^{-1} : (-1,1) \to \mathbb{R}$ denote the inverse of z, cf. [**26**]. Alternatively, for any x in $(-1,1)$, $\Theta(x)$ is the square norm of T at any point of M_0 in the level set $z^{-1}(x)$ with respect to the admissible Kähler metric determined by z. In particular, Θ is positive on $(-1,1)$ and smoothly extends to the closed interval $[-1,1]$, with

$$\Theta(-1) = \Theta(1) = 0. \tag{1.14}$$

Moreover, it easily follows from the boundary conditions $B_{-\infty} and B_{+\infty}$ for z that Θ satisfies the following additional boundary conditions:

$$\Theta'(-1) = 2, \qquad \Theta'(1) = -2, \tag{1.15}$$

where Θ' denotes the derivative of Θ with respect to x.

Definition 1.5. — A positive function $\Theta : (-1,1) \to \mathbb{R}^{>0}$ is called an *admissible momentum profile* if it smoothy extends to a function $\Theta : [-1,1] \to \mathbb{R}^{\geq 0}$ and satisfies the boundary conditions (1.14) and (1.15).

Proposition 1.2. — *For any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, there is a natural 1–1 correspondence between the space of admissible momentum profiles and the space of admissible Kähler metrics in $\Omega_{\boldsymbol{\lambda}}$, up to the natural $\mathbb{C}^*$-action on M.*

Proof. — We recover z from Θ by firstly defining $t : (-1,1) \to \mathbb{R}$ by means of the differential equation $\frac{dt}{dx} = \frac{1}{\Theta(x)}$, then $z : \mathbb{R} \to (-1,1)$ as the inverse function of t (notice that $t = t(x)$ is increasing, as Θ is positive on $(-1,1)$). It is then easily checked that $z = z(t)$ defined that way is an admissible momentum, i.e. satisfies the boundary conditions $B_{-\infty}$–$B_{+\infty}$. Finally, $t = t(x)$ is only defined up to an additive constant; we already saw that the corresponding admissible Kähler metric is only defined up to the natural $\mathbb{C}^*$-action on M. □

1.5. Standard admissible metrics. — Each admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ contains a *standard* $\mathbb{C}^*$-orbit of admissible Kähler metrics, namely admissible Kähler metrics determined by the admissible momentum $z_0 = \tanh t$ or the translated momenta $z_0^c = \tanh(t+c)$. For all of them, the momentum profile, Θ_0 is given by

$$\Theta_0(x) = 1 - x^2. \tag{1.16}$$

When restricted to the affine open set $L_y \setminus \{0\}$ of each fiber $\pi^{-1}(y)$, the Kähler form $\omega_{\boldsymbol{\lambda},z}$ (corresponding to admissible momentum $z = z(t)$) is $z'(t)\, dt \wedge d^c t$, or equivalently, is equal to $dd^c\Phi(t)$, where the Kähler potential $\Phi(t)$ is a primitive of $z(t)$, defined up to an affine function of t. (Notice that the restriction of $dd^c t$ on $\pi^{-1}(y)$ vanishes.) In the standard case, when the admissible momentum is $z_0(t) = \tanh t$, we can choose as Kähler potential $\Phi_0(t) = \log(1+e^{2t}) = \log(1+r^2)$, which is the Kähler potential of the Fubini-Study of $\mathbb{P}^1$ of sectional curvature $+1$. The resulting toric Kähler manifold is then the standard unit sphere $S^2 = \{u = (x_1,x_2,x_3) \mid \sum_{i=1}^3 x_i^2 = 1\}$ in $\mathbb{R}^3$, equipped with: (i) the standard S^1-action $e^{i\theta}\cdot(x_1,x_2,x_3) = (\cos\theta\, x_1 +$

$\sin\theta\, x_2, -\sin\theta\, x_1 + \cos\theta\, x_2, x_3)$; (ii) the standard symplectic form $\omega_0 = dx_3 \wedge d\theta$; (iii) the standard complex structure $JX = u \times X$ for any X in $T_u S^2$, where $\times$ stands for the cross product in $\mathbb{R}^3$ with respect to the natural orientation; (iv) the riemannian metric g_0 induced by the standard flat metric of $\mathbb{R}^3$. The momentum of the S^1-action with respect to ω_0 is then the map $z_0 : u = (x_1, x_2, x_3) \mapsto x_3$.

For a general admissible Kähler metric in a normalized admissible Kähler class, the induced toric Kähler structure on the fibres of π are again isomorphic to S^2, equipped with the same S^1-action and the same complex structure J, and with symplectic form $\omega = f\,\omega_0$ and the metric $g = f\,g_0$, where $f = f(x_3)$ denotes an S^1-invariant invariant positive function on S^2, submitted to the only constraint that $\int_{S^2} f\,\omega_0 = \int_{S^2} \omega_0$ or, equivalently, $\int_{-1}^{1} f(x)\,dx = 2$; the corresponding admissible momentum is then

$$z(t) = -1 + \int_{-1}^{\tanh t} f(x)\,dx. \tag{1.17}$$

1.6. Symmetries of admissible Kähler metrics. — In general, for any (connected) compact complex manifold (M, J), we denote by $\mathrm{H}(M, J)$ the identity component of the group of complex automorphisms of (M, J) and by $\mathfrak{h} = \mathfrak{h}(M, J)$ its Lie algebra, which we regard as the Lie algebra of real vector fields X such that $\mathcal{L}_X J = 0$, where $\mathcal{L}_X$ denotes the Lie derivative along X; X is then called a (real) holomorphic vector field. Equivalently, X is the real part of a complex vector field of type $(1,0)$, $X^{1,0}$, which is a holomorphic section of the holomorphic tangent bundle $T^{1,0}M$.

For any riemannian metric g which is Kähler with respect to J, a (real) vector field X is holomorphic if and only if $D^- X^\flat = 0$ — where $X^\flat$ denotes the riemannian dual 1-form of X and $D^- X^\flat$ denotes the J-anti-invariant part of $DX^\flat$ — and X can then be written in a unique way as

$$X = X_H + \mathrm{grad}_g f_g^X + J\mathrm{grad}_g h_g^X, \tag{1.18}$$

where X_H is the dual of a g-harmonic (real) 1-form and f_g^X, h_g^X are real functions normalized by $\int_M f_g^X\, v_g = \int_M h_g^X\, v_g = 0$; f_g^X, called the (real) *potential* of X, is determined by $\mathcal{L}_X \omega = dd^c f_g^X$, where $\omega = g(J\cdot, \cdot)$ is the Kähler form of the pair (g, J), cf. *e.g.* [**27**].

A (real) vector field X is called a Killing vector field with respect to g if $\mathcal{L}_X g = 0$. The Lie algebra, denoted by $\mathfrak{k}$, of Killing vector fields is the Lie algebra of the identity component, $\mathrm{K}(M, g)$, of the group of isometries of (M, g). It is well-known(1) that $\mathrm{K}(M, g)$ is a (compact) subgroup of $\mathrm{H}(M, J)$.

(1) The easy argument goes as follows: for any γ in $\mathrm{K}(M, g)$, $\gamma \cdot \omega$ is g-harmonic, as γ is an isometry, and it belongs to the de Rham class $[\omega]$, as γ is homotopic to the identity; since M is compact, this implies that $\gamma \cdot \omega = \omega$, hence also $\gamma \cdot J = 0$.

In view of the above, $\mathfrak{k}$ then coincides with the space of those (real) holomorphic vector fields whose (real) potential is identically zero.

The space, $\mathfrak{h}_0$, of (real) holomorphic vector fields such that $X_H \equiv 0$ in the decomposition (1.18) is the Lie algebra of a closed subgroup, $\mathrm{H}_0(M,J)$, of $\mathrm{H}(M,J)$, namely the kernel of the *Albanese map* from $\mathrm{H}(M,J)$ to the *Albanese torus* of (M,J): $\mathfrak{h}_0$ is then the space of (real) vector fields of the form $X = \mathrm{grad}_g f + J\mathrm{grad}_g h$, with $D^-(df + d^c h) = 0$. It can be alternatively described as the space of (real) holomorphic vector fields whose zero set is not empty [**29**]. The space $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{h}_0$ is the Lie algebra of *hamiltonian Killing* vector fields, i.e. the space of Killing vector fields of the form $X = J\mathrm{grad} h_g^X = \mathrm{grad}_\omega h_g^X$; it is the Lie algebra of a closed subgroup of $\mathrm{K}(M,g)$ denoted $\mathrm{K}_0(M,g)$.

We denote by P_g the space of *Killing potential with respect to* g, i.e. the space of a real functions, h, on M such that $X = J\mathrm{grad}_g h$ is a hamiltonian Killing vector field (notice that constants are included in P_g). This space is the kernel(2) of the *Lichnerowicz fourth order differential operator* $(D^-d)^* D^- d$.

The group $\mathrm{H}_0(M,J)$ and its subgroup $\mathrm{K}_0(M,g)$ will be referred to as the *reduced automorphism group* of (M,J) and the *reduced isometry group* of (M,g) respectively. We then have (cf. [**3**, Proposition 2]):

Proposition 1.3. — (i) *For any admissible ruled manifold* $M = \mathbb{P}(1 \oplus L)$, $\mathrm{H}_0(M,J)$ *projects surjectively to* $\mathrm{H}_0(S,J) = \prod_{i=1}^N \mathrm{H}_0(S_i,J)$, *with kernel the semi-direct product* $\mathbb{C}^* \ltimes H^0(S,L^\pm)$, *where* $H^0(S,L^\pm)$ *stands for the space of holomorphic sections of* L *or* $L^* = L^{-1}$ *according as* $H^0(S,L^*)$ *or* $H^0(S,L)$ *is reduced to zero*(3).

(ii) *For any admissible Kähler metric* g *on* M, $\mathrm{K}_0(M,g)$ *projects surjectively to* $\mathrm{K}_0(S,g_S) = \prod \mathrm{K}_0(S_i,g_{S_i})$, *with kernel* S^1, *which is contained in the center of* $\mathrm{K}_0(M,J)$. *In particular,* $\mathrm{K}_0(M,g)$ *is independent of the chosen admissible Kähler metric.*

Proof. — For any X in $\mathfrak{h}(M,J)$ and for any y in S, the projection of $X_{|\pi^{-1}(y)}$ to T_yS can be viewed as a holomorphic map from the fiber $\pi^{-1}(y)$ to $T_y^{1,0}S$, which is then constant: each X in $\mathfrak{h}$ is then projectable and we thus get a Lie algebra homomorphism from $\mathfrak{h}(M,J)$ to $\mathfrak{h}(S,J)$. This implies that any element of $\mathrm{H}(M,J)$ maps fiber to fiber, hence that the above Lie algebra homomorphism is induced by a homomorphism from $\mathrm{H}(M,J)$ to $\mathrm{H}(S,J)$. Moreover, if

(2)Since M is compact, f belongs to the kernel of $(D^-d)^*D^-d$ if and only if the Hessian Ddf is J-invariant, which amounts to saying that the hamiltonian vector field $J\mathrm{grad}_g f$ is Killing.
(3) For any non-trivial holomorphic line bundle over a connected compact complex manifold M, either $H^0(M,L)$ or $H^0(M,L^*)$ is reduced to $\{0\}$: if σ belongs to $H^0(M,L)$ and α belongs to $H^0(M,L^*)$, the holomorphic function $\langle\sigma,\alpha\rangle$ is constant, as M is compact, hence identically zero, as L is non-trivial; since M is connected, it follows that either σ or α is identically zero.

X belongs to $\mathfrak{h}_0(M,J)$, its projection on S belongs to $\mathfrak{h}_0(S,J)$, as each zero of X is mapped to a zero of its projection. We denote by τ the resulting homomorphism from $\mathfrak{h}_0(M,J)$ to $\mathfrak{h}_0(S,J)$ and by $\tilde{\tau}$ the corresponding Lie group homomorphism from $\mathrm{H}_0(M,J)$ to $\mathrm{H}_0(S,J)$. We show that τ is surjective by constructing a right inverse. Any element V of $\mathfrak{h}_0(S,J)$ splits as $V = \sum_{i=1}^N V_i$, with V_i in $\mathfrak{h}_0(S_i,J)$; we can then assume that $V = \mathrm{grad}_{g_{S_i}} f + J\mathrm{grad}_{g_{S_i}} h$ belongs to $\mathfrak{h}_0(S_i,J)$ for some i. Define $\hat{V}$ by

$$\hat{V} = \tilde{V} + \epsilon_i(\pi^* h)\, T - \epsilon_i\, (\pi^* f)\, JT \tag{1.19}$$

on M, where $\tilde{V}$ denotes the horizontal lift of V on M_0 with respect to the Chern connection of L. In general, for any vector field X on any almost-complex manifold (M,J), the Lie derivative of J along X is given by $\mathcal{L}_X J = [X, J\cdot] - J[X,\cdot]$; in particular, for any function f on M, we have:

$$\mathcal{L}_{fX} J = f\, \mathcal{L}_X J + d^c f \otimes X + df \otimes JX. \tag{1.20}$$

We thus get:

$$\begin{aligned}\mathcal{L}_{\hat{V}} J = \mathcal{L}_{\tilde{V}} J &+ \epsilon_i\, df \otimes T + \epsilon_i\, d^c h \otimes T\\ &- \epsilon_i\, d^c f \otimes JT + \epsilon_i\, dh \otimes JT.\end{aligned} \tag{1.21}$$

In particular, $(\mathcal{L}_{\hat{V}} J)(T) = 0$, as $\tilde{V}$ commutes with T and JT for *any* vector field V on S. For any vector field Z on S, the horizontal component of $(\mathcal{L}_{\tilde{V}} J)(\tilde{Z}) = [\tilde{V}, \tilde{JZ}] - J[\tilde{V}, \tilde{Z}]$ is zero, as V is (real) holomorphic, whereas its vertical component is equal to $-\epsilon_i\, \omega_i(V, JZ)\, T + \epsilon_i\, \omega_i(V,Z)\, JT$, hence to

$$-\epsilon_i\, df(Z) - \epsilon_i\, d^c(Z)\, T + \epsilon_i\, d^c f(Z) - \epsilon_i\, dh(Z)\, JT.$$

By substituting in (1.21), we get $\mathcal{L}_{\hat{V}} J = 0$. The map $\hat{\tau} : V \mapsto \hat{V}$ is then a linear map — in fact a Lie algebra homomorphism (easy verification) — from $\mathfrak{h}_0(S,J)$ to $\mathfrak{h}_0(M,J)$, hence is right inverse of τ. The kernel of τ is the Lie algebra of those holomorphic vector fields on M which are tangent to the fibers of π, hence restrict to holomorphic vector fields on the projective lines $\mathbb{P}(\mathbb{C} \oplus L_y)$, for all y on S: $\ker \tau$ is then identified with the space $H^0(S, \mathrm{End}_0(1 \oplus L))$ of holomorphic sections of the holomorphic vector bundle $\mathrm{End}_0(E)$ of trace-free endomorphisms of $E = 1 \oplus L$, which is isomorphic to $\mathbb{C} \oplus H^0(S, L^{\pm})$, cf. footnote 3 of page 11. The kernel of $\tilde{\tau}$ in $\mathrm{H}_0(M,J)$ is therefore identified with $\mathbb{C}^* \ltimes H^0(S, L^{\pm})$[4]. This proves (i). For any admissible metric $g = g_{\boldsymbol{\lambda}, z}$, (1.19) can be re-written as

$$\hat{V} = \mathrm{grad}_g((\lambda_i + \epsilon_i\, z)\, \pi^* f) + J\, \mathrm{grad}_g((\lambda_i + \epsilon_i\, z)\, \pi^* h) \tag{1.22}$$

[4] An element α of $H^0(S, L^*)$ acts on $M = \mathbb{P}(1 \oplus L)$ as follows: for any element $\xi = (z : u)$ of M over y in S, $\alpha \cdot \xi = (z + \langle \alpha(y), u\rangle : u)$; similarly, any σ of $H^0(S,L)$ acts on M by: $\sigma \cdot \xi = (z : u + z\, \sigma(y))$. In the former case, $\mathbb{C}^*$ acts on $H^0(S,L^*)$ by $\zeta \cdot \alpha = \zeta^{-1}\, \alpha$, in the latter case $\mathbb{C}^*$ acts on $H^0(S,L)$ by $\zeta \cdot \sigma = \zeta\, \sigma$.

In particular, $\hat{V}$ is Killing with respect to g if and only if V is Killing with respect to g_{S_i}. Moreover, all admissible Kähler metrics are invariant under the natural S^1-action; since S^1 is a maximal subgroup of $\mathbb{C}^* \ltimes H^0(S.L^{\pm})$, we get (ii). □

In the sequel, the common reduced isometry group $\mathrm{K}_0(M,g)$ for all admissible Kähler metrics will be simply denoted by G. The Lie algebra, $\mathfrak{g}$, of G splits as

$$\mathfrak{g} = \mathbb{R}\, T \oplus \oplus_{i=1}^N \mathfrak{k}_0(S_i, g_{S_i}), \tag{1.23}$$

which is a Lie algebra direct sum; in particular, T belongs to the center of $\mathfrak{g}$. For any $X = a\,T + \sum_{i=1}^N X_i$ in $\mathfrak{g}$ and for any admissible metric $g = g_{\boldsymbol{\lambda},z}$ in the (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, a Killing potential of X with respect of g — cf. Section 1.6 — is $h_g^X = a\,z + \sum_{i=1}(\lambda_i + \epsilon_i\, z)\, \pi^* h_i$, where h_i is a Killing potential of X_i with respect to g_{S_i}.

1.7. Ricci form and scalar curvature. — Throughout this section we fix a (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$. For any admissible momentum z, $p_{\Omega_{\boldsymbol{\lambda}}}(z)$ then denotes the function on M obtained by substituting $z = x$ in the characteristic polynomial; $p'_{\Omega_{\boldsymbol{\lambda}}}(z)$, $p''_{\Omega_{\boldsymbol{\lambda}}}(z), \ldots$, etc. are defined similarly, by substituting $z = x$ in the derivatives of $p_{\Omega_{\boldsymbol{\lambda}}}$. We then have (cf. [**1**, Section 5.1]):

Lemma 1.1. — *For any admissible metric $g_{\boldsymbol{\lambda},z}$ in $\Omega_{\boldsymbol{\lambda}}$, the Ricci form, ρ, and the scalar curvature, s, of $g_{\boldsymbol{\lambda},z}$, on M_0, are given by*

$$\rho = \sum_{i=1}^N \pi^* \rho_i - \frac{1}{2} dd^c \log\, (p_{\Omega_{\boldsymbol{\lambda}}}\, \Theta)(z), \tag{1.24}$$

and

$$s = \sum_{i=1}^N \frac{\pi^* s_i}{(\lambda_i + \epsilon_i\, z)} - \frac{(p_{\Omega_{\boldsymbol{\lambda}}}\, \Theta)''(z)}{p_{\Omega_{\boldsymbol{\lambda}}}(z)}, \tag{1.25}$$

where ρ_i and s_i denote the Ricci form and the scalar curvature of the Kähler structure (g_{S_i}, ω_{S_i}) on S_i.

Proof. — In general, the Ricci form of a Kähler structure (g,ω) of complex dimension m is defined by $\rho(\cdot,\cdot) = r(J\cdot,\cdot)$, where r denotes the Ricci tensor of g, and has the following local expression on the domain of any system of holomorphic coordinates

$$\rho =_{\mathrm{loc}} -\frac{1}{2} dd^c \log \frac{v_g}{v_0}, \tag{1.26}$$

where $v_g = \frac{\omega^m}{m!}$ denotes the volume form of g and v_0 stands for the volume form of the standard flat Kähler metric determined by the chosen holomorphic coordinates. (If these are denoted $z_1, \dots, z_m$, we then have $v_0 = \prod_{j=1}^m \frac{i}{2} dz_j \wedge d\bar{z}_j$, but the rhs of (1.26) is independent of this choice.)

For any admissible Kähler metric g, whose Kähler form is given by (1.12), we clearly have

$$v_g = p_{\Omega_{\boldsymbol{\lambda}}}(z) \prod_{i=1}^N v_{g_{S_i}} \wedge dz \wedge d^c t = p_{\Omega_{\boldsymbol{\lambda}}}(z)\, \Theta(z) \prod_{i=1}^N v_{g_{S_i}} \wedge dt \wedge d^c t. \tag{1.27}$$

To compute v_0, we use holomorphic coordinates on each factor S_i, viewed as holomorphic coordinates on M, and complete them to a system of holomorphic coordinates on an appropriate open subset of M_0, by choosing any local non-vanishing holomorphic section σ of L and adding the associated holomorphic coordinate λ determined by $u = \lambda(u)\, \sigma(\pi(u))$ for any u in L (viewed as an element of M_0). We then have $\frac{i}{2} d\lambda \wedge d\bar{\lambda} = |\lambda|^2\, dt \wedge d^c t$ up to terms which only involve the differential of holomorphic coordinates coming from the base S, hence contribute nothing to v_0. We thus get

$$v_0 = |\lambda|^2 \prod_{i=1}^N v_{i,0} \wedge dt \wedge d^c t, \tag{1.28}$$

where $v_{i,0}$ denotes the volume form of the flat Kähler metric determined by the chosen local holomorphic coordinates on S_i. By comparing (1.27) and (1.28) and by using (1.26), we get (1.24). The scalar curvature s is deduced from the Ricci form ρ via the general identity:

$$\rho \wedge *\omega = \rho \wedge \frac{\omega^{m-1}}{(m-1)!} = \frac{s}{2}\, v_g. \tag{1.29}$$

From (1.12), we infer[5]

$$\begin{aligned} \frac{\omega^{m-1}}{(m-1)!} &= p_{\Omega_{\boldsymbol{\lambda}}}(z) \prod_{i=1}^N \pi^* v_{g_{S_i}} \\ &+ p_{\Omega_{\boldsymbol{\lambda}}}(z) \sum_{i=1}^N \frac{1}{(\lambda_i + \epsilon_i\, z)} \frac{\pi^* \omega_i^{d_i - 1}}{(d_i - 1)!} \wedge \prod_{k \neq i} \pi^* v_{g_{S_k}} \wedge dz \wedge d^c t. \end{aligned} \tag{1.31}$$

---

[5] In this and the above computation we use the general combinatorial identity

$$\frac{(\sum_{i=1}^d a_i)^k}{k!} = \sum_{\substack{k_1, \dots, k_d \\ \sum k_i = k}} \prod_{i=1}^d \frac{a_i^{k_i}}{k_i!}. \tag{1.30}$$

The contribution of $\pi^*\rho_i$ in $\rho\wedge\frac{\omega^{m-1}}{(m-1)!}$ only involves the second term in the rhs of (1.31); by using (1.29) for each factor S_i, this contribution is found to be equal to $\frac{1}{2}(\sum_{i=1}^N \frac{\pi^* s_i}{(\lambda_i+\epsilon_i)})\, v_g$. On the other hand, $d^c \log\Theta(z) = \frac{\Theta'(z)}{\Theta(z)}\, d^c z = \Theta'(z)\, d^c t$ and $d^c \log p_{\Omega_{\boldsymbol\lambda}}(z) = \frac{p'_{\Omega_{\boldsymbol\lambda}}(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}\, d^c z = \frac{p'_{\Omega_{\boldsymbol\lambda}}(z)\,\Theta(z)}{p_{\Omega_{\boldsymbol\lambda}}}\, d^c t$, so that $d^c \log\left(p_{\Omega_{\boldsymbol\lambda}}(z)\Theta(z)\right) = \frac{(p_{\Omega_{\boldsymbol\lambda}}\,\Theta)'(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}\, d^c t$; it follows that:

(1.32)
$$-\frac{1}{2} dd^c \log\left(p_{\Omega_{\boldsymbol\lambda}}(z)\Theta(z)\right) = -\frac{1}{2}\,\frac{(p_{\Omega_{\boldsymbol\lambda}}\,\Theta)''(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}\, dz\wedge d^c t + \frac{1}{2}\,\frac{(p_{\Omega_{\boldsymbol\lambda}}\,\Theta)'(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}\left(\frac{p'_{\Omega_{\boldsymbol\lambda}}(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}\, dz\wedge d^c t - \sum_{i=1}^N \epsilon_i\,\omega_{S_i}\right).$$

In the wedge product with $\frac{\omega^{m-1}}{(m-1)!}$, $dz\wedge d^c t$ contributes via the first term in the rhs of (1.31) only, whereas $\sum_{i=1}^N \epsilon_i\,\omega_{S_i}$ contributes via the second term only, giving $\sum_{i=1}^N \frac{d_i\epsilon_i}{(\lambda_i+\epsilon_i\, z)}\, v_g = \frac{p'_{\Omega_{\boldsymbol\lambda}}(z)}{p_{\Omega_{\boldsymbol\lambda}}(z)}\, v_g$; the second term of (1.32) then contributes to zero. □

1.8. Hamiltonian 2-forms. — In general, a *hamiltonian* 2*-form* on a (connected) Kähler manifold (M, g, J, ω) of complex dimension m is a J-invariant real 2-form ϕ such that

$$D_X\phi = \frac{1}{2}(d\,\mathrm{tr}\phi\wedge JX^\flat - d^c\,\mathrm{tr}\phi\wedge X^\flat) \tag{1.33}$$

for any vector field X, where $X^\flat$ denotes the dual 1-form of X with respect to g and $\mathrm{tr}\varphi = (\phi, \omega)$ denotes the *trace* of ϕ with respect to g, defined as follows: If ϕ is viewed as a skew-hermitian $\mathbb{C}$-linear endomorphism of (TM, J) via the metric g, so that $\phi(X, Y) = g(\phi(X), Y)$, and if $\lambda_1 \leq \ldots \leq \lambda_m$ denote the (real) eigenvalues of the corresponding hermitian operator $-J\circ\phi$, then $\mathrm{tr}\phi := \sum_{i=1}^m \lambda_i$ (for simplicity, the λ_i's will be referred to as the eigenfunctions of ϕ). Hamiltonian 2-forms in Kähler geometry have nice properties, extensively studied in [**1, 2, 3, 4**]. In particular, for any hamiltonian 2-form ϕ, the elementary symmetric functions of its eigenfunctions $\sigma_1 = \mathrm{tr}\phi = \lambda_1+\ldots+\lambda_m, \sigma_2 = \sum_{i<j}\lambda_i\lambda_j, \ldots, \sigma_m = \lambda_1\ldots\lambda_m$ are Poisson commuting Killing potentials. Moreover, if $K_r = J\,\mathrm{grad}_g\sigma_r$, $r = 1, \ldots, m$, denote the corresponding hamiltonian vector fields, there exists an integer $0 \leq \ell \leq m$, called the *order* of ϕ, and an open dense subset M_0 of M such that $K_1, \ldots, K_\ell$ are linearly independent, whereas K_r linearly depends of $K_1, \ldots, K_\ell$ for any $r > \ell$. If $\ell = 1$, the case of main interest in this paper, $K = K_1 = J\mathrm{grad}_g\mathrm{tr}\phi$ is called the *hamiltonian Killing vector field* of ϕ.

Proposition 1.4. — *Let M be an admissible ruled manifold and let $\Omega_{\boldsymbol{\lambda}}$ be a normalized admissible Kähler class on M. Then, any admissible Kähler metric $g = g_{\boldsymbol{\lambda},z}$ in $\Omega_{\boldsymbol{\lambda}}$ admits a hamiltonian 2-form of order 1, whose hamiltonian Killing vector field is T, namely the 2-form φ defined by*

$$\phi = -\sum_{i=1}^{N} \epsilon_i \lambda_i (\lambda_i + \epsilon_i z)\, \pi^* \omega_{S_i} + z\, dz \wedge d^c t. \tag{1.34}$$

Proof. — We fist observe that the eigenfunctions of the J-invariant 2-form φ defined by (1.34) with respect to g are the admissible momentum z, of multiplicity 1, and the constant functions $\xi_i = -\epsilon_i \lambda_i$, each of multiplicity d_i. In particular,

$$\mathrm{tr}\phi = z - \sum_{i=1}^{N} d_i\, \epsilon_i \lambda_i. \tag{1.35}$$

The fact that φ is hamiltonian with respect to g is a straightforward consequence of the following two lemmas, whose easy verification is left to the reader:

Lemma 1.2. — *The covariant derivative of T with respect to the Levi-Civita connection of g is given by*

$$\begin{aligned} D_T T &= \frac{1}{2}\Theta'(z)\, JT, \qquad D_{JT} T = -\frac{1}{2}\Theta'(z)\, T, \\ D_{\tilde{X}} T &= \frac{1}{2}\Theta(z) \sum_{i=1}^{N} \frac{\epsilon_i\, J\tilde{X}_i}{(\lambda_i + \epsilon_i z)} \end{aligned} \tag{1.36}$$

for any vector field $X = \sum_{i=1}^{N} X_i$ on S, where X_i sits in TS_i, and $\tilde{X} = \sum_{i=1}^{N} \tilde{X}_i$ denotes its horizontal lift on M with respect to the Chern connection ∇.

Lemma 1.3. — *With the same notation, for $i = 1, \ldots, N$, the covariant derivative of $\pi^*\omega_{S_i}$ is given by:*

$$\begin{aligned} D_T(\pi^*\omega_{S_i}) &= 0, \qquad D_{JT}(\pi^*\omega_{S_i}) = \Theta(z) \sum_{i=1}^{N} \frac{\epsilon_i \pi^* \omega_{S_i}}{(\lambda_i + \epsilon_i z)}, \\ D_{\tilde{X}}(\pi^*\omega_{S_i}) &= \frac{1}{2} \sum_{i=1}^{N} \frac{\epsilon_i}{(\lambda_i + \epsilon_i z)} \left(d^c z \wedge \pi^*(X_i^\flat) - dz \wedge \pi^*(JX_i^\flat) \right), \end{aligned} \tag{1.37}$$

where $X_i^\flat$ stands for the dual 1-form of X_i with respect to g_{S_i}.

□

1.9. Extremal admissible Kähler class. — In general, a Kähler structure (g, J, ω) is called *extremal* if the scalar curvature $s = s_g$ is a Killing potential with respect to g, i.e. if the hamiltonian vector field $K = \mathrm{grad}_\omega s = J\mathrm{grad}_g s$, is Killing or, equivalently, (real) holomorphic, cf. Section 1.6 and Section 2.1.

Proposition 1.5. — *Let g be an admissible Kähler metric in a (normalized) admissible Kähler class $\Omega_{\boldsymbol\lambda}$, determined by an admissible momentum z. Then, g is extremal if and only if its scalar curvature s is an affine function of z. In this case the scalar curvatures of (S_i, g_{S_i}) are constant.*

Proof. — For any $i = 1, \ldots, N$, the dual vector field of $d^c\pi^* s_i$ with respect to the chosen admissible Kähler metric on M is $\frac{1}{(\lambda_i+\epsilon_i\, z)}\,\tilde K_i$, where K_i denotes the dual vector field of $d^c s_i$ on S_i with respect to g_{S_i}, and $\tilde K_i$ denotes the horizontal lift of K_i on M_0. Notice that for any vector field, X, on S, the horizontal lift $\tilde X$ commutes with T and JT; we thus have $[\tilde K_i, T] = [\tilde K_i, JT] = 0$ for all i. On the other hand, for any admissible Kähler metric, T is the symplectic gradient of z. We thus infer from (1.25) the following expression of K:

$$K = \sum_{i=1}^N \frac{1}{(\lambda_i+\epsilon_i\, z)^2}\,(\tilde K_i - \epsilon_i\,(\pi^* s_i)\,T) - \left(\frac{(p_{\Omega_{\boldsymbol\lambda}}\,\Theta)''}{p_{\Omega_{\boldsymbol\lambda}}}\right)'(z)\,T. \tag{1.38}$$

By using (1.20), we infer:

(1.39)

$$\begin{aligned}\mathcal{L}_K J = &\sum_{i=1}^N \frac{1}{(\lambda_i+\epsilon_i\, z)^2}\,\mathcal{L}_{(\tilde K_i - \epsilon_i\,(\pi^* s_i)\,T)} J\\ &+\sum_{i=1}^N \left(\frac{\epsilon_i}{(\lambda_i+\epsilon_i\, z)^2}\right)'(z)\,(d^c z\otimes(\tilde K_i - \epsilon_i\,(\pi^* s_i)\,T) + dz\otimes J(\tilde K_i - \epsilon_i\,(\pi^* s_i)\,T))\\ &-\left(\frac{(p_{\Omega_{\boldsymbol\lambda}}\,\Theta)''}{p_{\Omega_{\boldsymbol\lambda}}}\right)''(z)\,(d^c z\otimes T + dz\otimes JT).\end{aligned}$$

Since the $\tilde K_i$'s commute with T and JT, we have that $(\mathcal{L}_{(\tilde K_i - \epsilon_i\,(\pi^* s_i)\,T)} J)(T) = 0$, whereas $d^c z(T) = \Theta(z)$, $dz(T) = 0$; we thus get

(1.40)

$$(\mathcal{L}_K J)(T) = \Theta(z)\,\big((\sum_{i=1}^N \big(\frac{\epsilon_i}{(\lambda_i+\epsilon_i\, z)^2}\big)'(\tilde K_i - \epsilon_i\,(\pi^* s_i)\,T) - \big(\frac{(p_{\Omega_{\boldsymbol\lambda}}\,\Theta)''}{p_{\Omega_{\boldsymbol\lambda}}}\big)''\,T\big).$$

Assume that the chosen admissible Kähler metric is extremal; then $(\mathcal{L}_K J)(T)$ is identically zero. Since T and the $\tilde K_i$'s sit in separate spaces, we infer that the $\tilde K_i$'s, hence the K_i's are all identically zero; the scalar curvatures s_i are then constant, so that s is a function of z. Moreover, the coefficient of T in the rhs of (1.40), which is identically zero, is then equal to $\frac{d^2s}{dz^2}$, cf. (1.25); it follows that s is an affine function of z. Conversely, if s is an affine function of z, then K is a constant multiple of T, hence a Killing vector field, meaning that g is extremal. □

In view of Proposition 1.5, we henceforth assume without further comment that the s_i's are constant.

This assumption has in particular the following consequence, cf. [**3**, Proposition 5]:

Proposition 1.6. — *The common reduced isometry group G of all admissible Kähler metrics — cf.* Proposition 1.3 — *is a* maximal *compact subgroup of the reduced automorphism group* $\mathrm{H}_0(M, J)$.

Proof. — It is a well-known fact that for any compact Kähler manifold (M, J) of constant scalar curvature the reduced isometry group $\mathrm{K}_0(M, J)$ is a maximal compact subgroup of the reduced automorphism group $\mathrm{H}_0(M, J)$. Proposition 1.6 is then a direct consequence of Proposition 1.3. □

For any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, we infer from (1.25) and Proposition 1.5 that an admissible Kähler metric $g = g_{\boldsymbol{\lambda},z}$ of momentum profile Θ is extremal if and only if

$$(p_{\Omega_{\boldsymbol{\lambda}}}\,\Theta)''(x) = R(x), \tag{1.41}$$

by setting

$$R(x) = p_{\Omega_{\boldsymbol{\lambda}}}(x) \sum_{i=1}^{N} \frac{s_i}{(\lambda_i + \epsilon_i\, x)} - p_{\Omega_{\boldsymbol{\lambda}}}(x)(\alpha x + \beta), \tag{1.42}$$

for some (unknown) real constants α, β. All functions appearing in (1.41)–(1.42) are defined on the open interval $(-1, 1)$. Because of the boundary conditions (1.14)–(1.15) for Θ, the polynomial R is subjected to the following two constraints:

$$\int_{-1}^{1} R(x)\, dx = -2\, p_{\Omega_{\boldsymbol{\lambda}}}(-1) - 2\, p_{\Omega_{\boldsymbol{\lambda}}}(1), \tag{1.43}$$

$$\int_{-1}^{1} R(x)\, x\, dx = 2\, p_{\Omega_{\boldsymbol{\lambda}}}(-1) - 2\, p_{\Omega_{\boldsymbol{\lambda}}}(1). \tag{1.44}$$

These constraints in turn determine α, β, hence the polynomial R in terms of the (constant) scalar curvatures s_i, and the characteristic polynomial $p_{\Omega_{\boldsymbol{\lambda}}}(x)$. In particular, R is entirely determined by the chosen admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, as are the constants α, β.

In view of the extremality equation (1.41), we *define* $F = F(x)$ — a polynomial of degree at most $(m+2)$ — by

$$F''(x) = R(x) \tag{1.45}$$

and

$$F(-1) = F(1) = 0, \tag{1.46}$$

cf. [**3**, Proposition 8]. The constraints (1.43)–(1.44) then insure that F also satisfies:

$$F'(-1) = 2\, p_{\Omega_{\boldsymbol\lambda}}(-1), \qquad F'(1) = -2\, p_{\Omega_{\boldsymbol\lambda}}(1). \tag{1.47}$$

Like $R(x)$, the polynomial $F(x)$ determined that way only depends of the admissible Kähler $\Omega_{\boldsymbol\lambda}$.

Definition 1.6. — For any (normalized) admissible Kähler class $\Omega_{\boldsymbol\lambda}$ on M, the polynomial F of degree at most $(m+2)$ determined by (1.45)–(1.46) is called the *extremal polynomial* of $\Omega_{\boldsymbol\lambda}$, henceforth denoted by $F_{\Omega_{\boldsymbol\lambda}}$.

From the above discussion, we readily infer:

$$F_{\Omega_{\boldsymbol\lambda}}(x) = 2p_{\Omega_{\boldsymbol\lambda}}(-1)(1+x) + \int_{-1}^{x} R(s)(x-s)\, ds. \tag{1.48}$$

Remark 1.5. — It readily follows from (1.42) and from the above definition of the extremal polynomial $F_{\Omega_{\boldsymbol\lambda}}$ that for each $i = 1, \ldots, N$, the scalar curvature s_i can be expressed by

$$s_i = \frac{F''_{\Omega_{\boldsymbol\lambda}}(-\epsilon_i\lambda_i)}{\prod_{k\neq i}(\lambda_k - \epsilon_k\epsilon_i\lambda_i)} \tag{1.49}$$

provided that $\epsilon_i\lambda_i \neq \epsilon_k\lambda_k$ for $k \neq i$.

Proposition 1.7. — *A* (*normalized*) *admissible Kähler class* $\Omega_{\boldsymbol\lambda}$ *on* M *admits an extremal admissible Kähler metric,* $g = g_{\boldsymbol\lambda, z}$*, for some admissible momentum* z*, if and only if its extremal polynomial* $F_{\Omega_{\boldsymbol\lambda}}$ *is positive on the open interval* $(-1, 1)$*. The momentum profile of* g *is then given by*

$$\Theta(x) = \frac{F_{\Omega_{\boldsymbol\lambda}}(x)}{p_{\Omega_{\boldsymbol\lambda}}(x)}. \tag{1.50}$$

In particular, g *is then uniquely defined up to the natural* $\mathbb{C}^*$*-action on* M*. Moreover, the scalar curvature* s *of* g *is given by*

$$s = \alpha\, z + \beta, \tag{1.51}$$

where α, β *are the real constants determined by* (1.42)–(1.43)–(1.44)*. In particular,* s *is constant if and only if the leading coefficient of* $F_{\Omega_{\boldsymbol\lambda}}$ *is equal to zero; it is identically zero if and only if the leading and the sub-leading coefficients of* $F_{\Omega_{\boldsymbol\lambda}}$ *are both equal to zero.*

Proof. — In view of the above discussion, g is extremal if and only if its momentum profile is given by (1.50). From (1.46)–(1.47), we deduce that the function Θ *defined* by (1.50) is smoothy defined on the closed interval $[-1,1]$ and satisfies the boundary conditions (1.14)–(1.15). It is then an admissible momentum profile if and only if it is positive on $(-1,1)$. Since $p_{\Omega_{\boldsymbol{\lambda}}}(x)$ is positive on $[-1,1]$, this is equivalent to $F_{\Omega_{\boldsymbol{\lambda}}}$ being positive on $(-1,1)$. In view of Proposition 1.2, Θ is then the momentum profile of an extremal admissible Kähler metric, uniquely defined up to the natural $\mathbb{C}^*$-action. For a general admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}}$, the scalar curvature is given by (1.25), or equivalently,

$$s = \alpha\, z + \beta + \frac{R(z) - (p_{\Omega_{\boldsymbol{\lambda}}}\,\Theta)''(z)}{p_{\Omega_{\boldsymbol{\lambda}}}(z)}, \tag{1.52}$$

where the constants α, β are determined by (1.42)–(1.43)–(1.44). If g is extremal, this reduces to (1.51), because of (1.50) and (1.45). Moreover, from (1.42) and (1.45), we readily infer that the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is of the form $F_{\Omega_{\boldsymbol{\lambda}}}(x) = \sum_{j=0}^{m+2} a_j x^{m+2-j}$, where the leading and the sub-leading coefficients are given by

$$a_0 = \pm\,\frac{\alpha}{(m+1)(m+2)}, \qquad a_1 = \pm\,\frac{\beta + (\sum_{k=1}^{N} d_k\lambda_k\epsilon_k)\,\alpha}{m(m+1)}, \tag{1.53}$$

with $\pm = -\prod_{i=1}^{N}\epsilon_i^{d_i}$. The last statement of Proposition 1.7 follows immediately. □

In view of of (1.53), the constants α, β will be referred to as the *renormalized leading coefficients* of the extremal polynomial.

Definition 1.7. — An admissible Kähler class Ω is said to be *far from the boundary* if Ω is a positive multiple of a normalized admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, with $\lambda_i \gg 1$, $i = 1, \dots, N$.

Lemma 1.4. — *The extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ of a normalized admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ far from the boundary has the following asymptotic behavior:*

$$F_{\Omega_\lambda}(x) = \left(\prod_{i=1}^{N}\lambda_i^{d_i}\right)(1-x^2) + o(\boldsymbol{\lambda}), \tag{1.54}$$

meaning that each coefficients of the polynomial $\frac{F_{\Omega_\lambda}(x)}{\prod_{i=1}^{N}\lambda_i^{d_i}} - (1-x^2)$ *tends to* 0 *when all λ_i's tend to $+\infty$.*

Proof. — By dividing both sides of (1.43)–(1.44) by $\prod_{i=1}^{N}\lambda_i^{d_i}$ and observing that $\left(\prod_{i=1}^{N}\lambda_i^{d_i}\right)^{-1} p_{\Omega_{\boldsymbol{\lambda}}}(x)$ tends to the constant polynomial 1 on $[-1,1]$ when the λ_i's tend to $+\infty$, we get the following limits for $\alpha = \alpha(\lambda_1, \dots, \lambda_N)$ and $\beta = \beta(\lambda_1, \dots, \lambda_N)$:

$$\lim_{\lambda_1 \to +\infty, \dots, \lambda_N \to +\infty} \alpha = 0, \qquad \lim_{\lambda_1 \to +\infty, \dots, \lambda_N \to +\infty} \beta = 2. \tag{1.55}$$

This, in turn, implies that the polynomial R in (1.42) tends to the constant polynomial -2; since $R = F''_{\Omega_{\boldsymbol{\lambda}}}$ and $F_{\Omega_\lambda}(-1) = F_{\Omega_\lambda}(1) = 0$ for all λ_i's, we infer that F_{Ω_λ} tends to the polynomial $1 - x^2$ when all λ_i's tend to 0. □

Proposition 1.8. — *Each admissible Kähler class far enough from the boundary admits an extremal admissible Kähler metric.*

Proof. — We can assume that Ω is a normalized admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$. It follows from (1.54) that, when the λ_i's go to infinity, all roots of the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ other than ± 1 go to infinity. In particular, $F_{\Omega_{\boldsymbol{\lambda}}}$ has no root in the open interval $(-1, 1)$ when Ω_λ is far enough from the boundary; because of the boundary conditions (1.46)–(1.47) and the fact that $p_{\Omega_{\boldsymbol{\lambda}}}(-1) = \prod_{i=1}^N (\lambda_i - \epsilon_i)^{d_i}$ and $p_{\Omega_{\boldsymbol{\lambda}}}(1) = \prod_{i=1}^N (\lambda_i + \epsilon_i)^{d_i}$ are both positive, $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on $(-1, 1)$. Proposition 1.8 then follows from Proposition 1.7. □

A further consequence of Proposition 1.7 is the following result ([**3**, Proposition 11]):

Proposition 1.9. — *In the case when all s_i are non-negative, any admissible Kähler class admits an admissible extremal Kähler metric.*

Proof. — By Proposition 1.7, it is sufficient to check that $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on $(-1, 1)$ for any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$. Assume, for a contradiction, that $F_{\Omega_{\boldsymbol{\lambda}}}$ has zeros on $(-1, 1)$. Because of the boundary conditions (1.46)-(1.47), where $p_{\Omega_{\boldsymbol{\lambda}}}(-1)$ and $p_{\Omega_{\boldsymbol{\lambda}}}(1)$ are both positive, $F_{\Omega_{\boldsymbol{\lambda}}}$ must have at least two maxima and two inflection point on $(-1, 1)$. Denote respectively by $x_m < x_M$ the smallest and greatest point of maxima in $(-1, 1)$. Note also that $F''_{\Omega_{\boldsymbol{\lambda}}} = R(x)$ has at least two zeros in $(-1, 1)$.

By (1.42), $R(x)$ can be re-written as $R(x) = \left(\prod_{a=1}^N (\lambda_a + \epsilon_a x)^{d_a - 1}\right) q(x)$, where q is the polynomial defined by

$$q(x) = \sum_{a=1}^N s_a \prod_{b \neq a} (\lambda_b + \epsilon_b x) - (\alpha x + \beta) \prod_{a=1}^N (\lambda_a + \epsilon_a x). \tag{1.56}$$

In this expressions and in the sequel of the argument, we (temporarily) change our overall notation in the following manner: N denotes the number of *distinct* $\epsilon_i \lambda_i$ — that is to say the number of distinct constant values of the hamiltonian

2-form ϕ, cf. Section 1.8 — and the latter are labeled by $a, b = 1, \dots, N$ in such a way that

$$\epsilon_K \lambda_K < \dots < \epsilon_1 \lambda_1 < -1 < 1 < \epsilon_N \lambda_N < \dots < \epsilon_{K+1} \lambda_{K+1} \tag{1.57}$$

where K is the number of *negative* ϵ_a's. For each label a, we put $d_a = \sum_{i\,|\epsilon_i\lambda_i=\epsilon_a\lambda_a} d_i$ so that $p_{\Omega_{\boldsymbol\lambda}}(x) = \prod_{a=1}^N (\lambda_a + \epsilon_a x)^{d_a}$ and $s_a = \sum_{i\,|\epsilon_i\lambda_i=\epsilon_a\lambda_a} s_i$.

With this notation, the roots of $R(x)$ are counted as follows: (1) the N real numbers $-\epsilon_a\lambda_a$ — each with multiplicity $d_a - 1$ — which all sit outside $[-1, 1]$, and (2) the roots of q. With our assumption, q has at least two roots, r_1, r_2 say, in $(-1, 1)$, in fact in the subinterval (x_m, x_M). Moreover, $F''_{\Omega_{\boldsymbol\lambda}}(x_m)$ and $F''_{\Omega_{\boldsymbol{x}_M}}$ are both non-positive; since $\prod_{a=1}^N (\lambda_a + \epsilon_a x)^{d_a-1}$ is positive on $(-1, 1)$, we then have $q(x_m) \le 0$ and $q(x_M) \le 0$.

Denote by n_-, resp. n_+, the number of real roots of q in the interval $(-\infty, -1]$, resp. in the interval $[1, +\infty)$ (counted with multiplicity). From (1.56), we infer

$$q(-\epsilon_a\lambda_a) = s_a \prod_{b\neq a} (\lambda_b - \epsilon_b\epsilon_a\lambda_a). \tag{1.58}$$

Since all s_i's, hence all s_a's in the new notation, are non-negative, we infer that $q(-\epsilon_a\lambda_a)\, q(-\epsilon_b\lambda_b) \le 0$ for any pair a, b, such that $a, b \le K$ or $a, b > K$ and $|a - b| = 1$. There is then at least one real root of q between any two consecutive $-\epsilon_a\lambda_a$, $-\epsilon_b\lambda_b$, with $a, b \le K$ or $a, b > K$. It follows that

$$n_+ + 1 \ge K, \qquad n_- + 1 \ge N - K, \tag{1.59}$$

hence

$$n_+ + n_- + 2 \ge N. \tag{1.60}$$

On the other hand,

$$n_+ + n_- + 2 \le N + 1, \tag{1.61}$$

as the degree of q is at most equal to $N + 1$ and q has at least $n_+ + n_- + 2$ real roots: the 2 roots r_1, r_2 in $(-1, 1)$ and $n_+ + n_-$ roots outside this interval. From (1.60) and (1.61), we infer that $n_+ + 1 = K$ or $n_- + 1 = N - K$.

First assume that $n_+ + 1 = K$; there is then exactly one root of q between any two consecutive $-\epsilon_a\lambda_a, -\epsilon_b\lambda_b$, with $i, j \le K$ and no roots in $[1, +\infty)$. In particular, there is no root in the interval $[1, -\epsilon_1\lambda_1)$. From (1.58) we easily infer $q(-\epsilon_1\lambda_1) \ge 0$, whereas $q(x_M) \le 0$; then, there exists a root, r_3 say, of q in the interval $[x_M, 1)$, hence distinct from r_1, r_2; we thus get at least *three* roots of q in $(-1, 1)$ and (1.61) can then be replaced by $n_+ + n_- + 2 \le N$; this, together with (1.60), implies $n_+ + n_- + 2 = N$, hence $n_- + 1 = N - K$; as above, we infer that there is no root of q in the interval $(-\epsilon_N\lambda_N, -1]$; by (1.58) again, $q(-\epsilon_N\lambda_N) \ge 0$, whereas $q(x_m) \le 0$; there then exists a root of

q, r_4 say, in the interval $(-1, x_m]$, hence distinct from r_1, r_2 and r_3; we thus obtain (at least) *four* roots, r_1, r_2, r_3, r_4, of q in $(-1, 1)$. It follows that (1.61) can be improved by $n_+ + n_- + 2 \leq N - 1$, which contradicts (1.60). Same reasoning and same conclusion apply if we assume $n_- + 1 = N - K$. □

Remark 1.6. — Proposition 1.8 is a part of [**3**, Proposition 9]. Proposition 1.9 is [**3**, Proposition 10]; similar results have previously appeared in the literature, in particular in [**25**] and [**21**], cf. [**3**] for more details.

1.10. Hirzebruch-like surfaces. — In this section, we consider the particular case when $N = 1$ and the base $S = S_1$ is a compact Riemann surface of genus $\boldsymbol{g}$. The resulting complex surface $M = \mathbb{P}(1 \oplus L)$ will be called a *Hirzebruch-like surface* of genus $\boldsymbol{g}$: it is a genuine *Hirzebruch surface* [**23**] when $\boldsymbol{g} = 0$, a *pseudo-Hirzebruch surface* in the sense of [**42**] if $\boldsymbol{g} > 1$. We assume that the *degree* $\deg(L) = \langle c_1(L), [S] \rangle$ is *negative* — meaning that $\epsilon_1 = 1$ — equal to $-\ell$, and that g_S is of constant scalar curvature $s = 2\kappa$. It then follows from the Gauss-Bonnet formula that

$$\kappa = \frac{2(1 - \boldsymbol{g})}{\ell}. \tag{1.62}$$

With the above assumption, for any real number $\lambda > 1$, the characteristic polynomial of the (normalized) admissible Kähler Ω_λ is simply

$$p_{\Omega_\lambda}(x) = \lambda + x. \tag{1.63}$$

In view of (1.5), $\Omega_{\boldsymbol{\lambda}}$ can also be written:

$$\Omega_\lambda = 2\pi \left(-(\lambda - 1)\, e_0 + (\lambda + 1)\, e_\infty\right) \tag{1.64}$$

for $\lambda > 1$. In the notation of Section 1.9, we have

$$R(x) = -\alpha\, x^2 - (\lambda\, \alpha + \beta)\, x + 2\kappa - \lambda\, \beta. \tag{1.65}$$

The constraints (1.43)–(1.44) then read:

$$\begin{aligned} \int_{-1}^{1} R(x)\, dx &= -\frac{2\,\alpha}{3} + 4\,\kappa - 2\lambda\,\beta = -4\lambda, \\ \int_{-1}^{1} R(x)\, x\, dx &= -\frac{2\lambda\alpha}{3} - \frac{2\beta}{3} = -4, \end{aligned} \tag{1.66}$$

so that

$$\alpha = \frac{12\lambda - 6\kappa}{3\lambda^2 - 1}, \qquad \beta = \frac{6\lambda^2 + 6\lambda\,\kappa - 6}{3\lambda^2 - 1}. \tag{1.67}$$

The extremal polynomial is then $F_{\Omega_{\boldsymbol{\lambda}}} = (1 - x^2)\, Q(x)$, with

$$Q(x) = A(x^2 - 1) + x + \lambda, \tag{1.68}$$

by setting

$$A = A(\lambda) = \frac{\lambda - \kappa/2}{3\lambda^2 - 1} \tag{1.69}$$

(because of (1.62), A is positive; moreover, $\lim_{\lambda \to +\infty} A = 0$). By Proposition 1.7, $\Omega_{\boldsymbol{\lambda}}$ admits an admissible extremal Kähler metric if and only if $Q(x)$ is positive on the open interval $(-1, 1)$. Notice that $Q(-1) = \lambda - 1$ and $Q(1) = \lambda + 1$ are both positive, whereas $Q'(-1) = 1 - 2A = \frac{(\lambda-1)(3\lambda+1)+\kappa}{3\lambda^2-1}$ and $Q'(1) = 1 + 2A > 0$. If $\kappa \geq 0$, i.e. if the genus $\boldsymbol{g}$ of S is 0 or 1, then $Q'(-1) > 0$ and $Q(x)$ is then positive on $(-1, 1)$ for any $\lambda > 1$. If $\kappa < 0$, i.e. $\boldsymbol{g} > 1$, $Q'(-1)$ is positive for large values of λ — hence $Q(x)$ is positive on $(-1, 1)$ — but it takes negative values when λ is small, namely for any λ such that $(\lambda - 1)(3\lambda + 1) < -\kappa$. For these values of λ, Q achieves its minimum at $x_0 = -\frac{1}{2A}$; this belongs to the open interval $(-1, 0)$, as $Q'(-1) = 1 - 2A < 0$, and $Q(x_0) = -\frac{D(\lambda)}{4(3\lambda^2-1)(\lambda-\kappa/2)}$, where

$$\begin{aligned} D(\lambda) &= -3\lambda^4 + 6\kappa\lambda^3 + 2\lambda^2 - 6\kappa\lambda + 1 + \kappa^2 \\ &= (\lambda^2 - 1)(-3\lambda^2 + 6\kappa\,\lambda - 1) + \kappa^2. \end{aligned} \tag{1.70}$$

It is easyto check that, for any negative value of κ, the rhs of (1.70) decreases from $\Delta(1) = \kappa^2 > 0$ up to $-\infty$, when λ runs from 1 to $+\infty$; it follows that the equation $D(\lambda) = 0$ has a unique root greater than 1, called λ_0. From this and from Proposition 1.7 we infer:

Theorem 1.1. — *Let M be a Hirzebruch-like surface of genus $\boldsymbol{g}$. Then, each Kähler class Ω is admissible, hence a positive multiple of a normalized admissible Kähler class Ω_λ for some $\lambda > 1$.*

Denote by λ_0 the unique root greater than 1 of the equation $D(\lambda) = 0$, where $D(\lambda)$ is defined by (1.70). Then:

(i) *If $\boldsymbol{g} \leq 1$ or if $\boldsymbol{g} > 1$ and $\lambda > \lambda_0$, then Ω_λ admits an extremal admissible metric, unique up to the natural action of $\mathbb{C}^*$.*

(ii) *If $\boldsymbol{g} > 1$ and $\lambda \leq \lambda_0$, then Ω_λ admits no extremal admissible Kähler metric.*

Remark 1.7. — The case when $\boldsymbol{g} = 0$ in Theorem 1.1, and, more generally, the case when S is a complex projective space of any dimensions, are due to E. Calabi [**8**] and constitute the first examples of (compact) extremal Kähler manifolds of non-constant scalar curvature (cf. also [**37**] for an alternative approach). As mentioned earlier, our general approach can be viewed as a generalization of Calabi's method. The case when $\boldsymbol{g} = 1$ was worked out by A. Hwang in [**25**] and D. Guan in [**21**]. The case when $\boldsymbol{g} > 1$ is due to the fourth author [**42**] and constitute the first known family of examples of (compact) Kähler manifolds where the extremal Kähler cone is non-empty but does not fill the Kähler cone. Notice that in the latter case, Theorem 1.1 does not imply the non-existence of — non-admissible — extremal Kähler metric if $\lambda \geq \lambda_0$ (more on this point in [**42**]). This question will be settled in Section 2.3 (an alternative treatment can be found in [**40**].)

2. Relative K-energy and extremal metrics

2.1. The space of Kähler metrics. — In this section, we briefly review some general facts concerning the space $\mathcal{M}_\Omega$ of Kähler metrics on a compact complex manifold (M, J) of (complex) dimension m, in a fixed Kähler class Ω. The presentation and the notations are taken from [**19**].

An element of $\mathcal{M}_\Omega$ will be indifferently designated by a Kähler riemannian metric g or by its Kähler form $\omega = g(J\cdot, \cdot)$, with $[\omega] = \Omega$, or by the pair (g, ω). As a consequence of the dd^c-lemma in Kähler geometry, cf. *e.g.*, [**20**], $\mathcal{M}_\Omega$ is essentially a space of (real) functions. More precisely, for any fixed reference element ω_0 in $\mathcal{M}_\Omega$, we have that

$$\mathcal{M}_\Omega = \{\varphi \,|\, \omega := \omega_0 + dd^c\varphi > 0\}, \tag{2.1}$$

where φ, the *relative Kähler potential* of ω relative to ω_0, is well-defined up to a (real) additive constant (here, $\omega > 0$ means that $g = \omega(\cdot, J\cdot)$ is a riemannian metric). The relative potential can be normalized, cf. [**14**], in such a way that, for any g in $\mathcal{M}_\Omega$, the tangent space $T_g\mathcal{M}_\Omega$ be identified with the space of real functions f on M such that $\int_M f\, v_g = 0$. The L^2-norm on this space then gives $\mathcal{M}_\Omega$ a structure of riemannian Fréchet manifold, first introduced and studied by T. Mabuchi [**32**].

The Mabuchi metric on $\mathcal{M}_\Omega$ admits a Levi-Civita connection, denoted by $\mathcal{D}$. For any real function f on M, let $\hat{f}$ be the *constant vector field* on $\mathcal{M}_\Omega$ defined by $g \mapsto f - \bar{f}$, where $\bar{f} = \frac{\int_M f v_g}{V_\Omega}$ denotes the mean value of f. The covariant derivative $\mathcal{D}$ is entirely determined by the $\mathcal{D}\hat{f}$'s, which are given by

$$\mathcal{D}_{f_1}\hat{f}_2 = -(df_1, df_2)_g + \frac{\int_M (df_1, df_2)\, v_g}{V_\Omega} \tag{2.2}$$

for any g in $\mathcal{M}_\Omega$ and any f_1 in $T_g\mathcal{M}_\Omega$. In particular, a curve $\omega_t = \omega_0 + dd^c\varphi_t$, $t \in [0, 1]$, in $\mathcal{M}_\Omega$ is a geodesic if and only if

$$\ddot{\varphi}_t - (d\dot{\varphi}_t, d\dot{\varphi}_t)_{g_t} = 0. \tag{2.3}$$

As observed by S. Semmes [**36**], the geodesic equation (2.3) can be re-written as a degenerate homogeneous Monge-Ampère equation my considering φ_t as a function, Φ say, defined on the product $\hat{M} := M \times \Sigma$, where Σ here stands for the cylinder $[0, 1] \times S^1$, equipped with the complex structure determined by $J\partial/\partial t = \partial/\partial s$, where s denotes the natural parameter of the additional circle factor S^1. By still denote by ω the pull-back of ω on $\hat{M}$, the geodesic equation can be rewritten as

$$(\omega + dd^c\Phi)^{m+1} = 0 \tag{2.4}$$

for S^1-invariant functions Φ defined on $M \times \Sigma$ such that $\Phi(\cdot, t)$ is a relative Kähler potential on M with respect to ω_0.

Remark 2.1. — The Monge-Ampère equation (2.4) makes sense when Σ is replaced by any Riemann surface with boundary. Let Φ be a (smooth) solution of (2.4), such that $\Phi(\cdot, \tau)$ is a relative Kähler potential on M with respect to ω_0 for any τ in Σ. Choose a local holomorphic coordinate $z = t + is$ on Σ: Φ then appears as a family of relative Kähler potentials parametrized by s, t, $\varphi = \varphi(t, s)$, and the Monge-Ampère equation (2.4) is then equivalent to

$$\ddot{\varphi}_{tt} + \ddot{\varphi}_{ss} - |d\dot{\varphi}_t - d^c\dot{\varphi}_s|^2_{g_{t,s}} = 0, \tag{2.5}$$

where $g_{s,t}$ denotes the Kähler metric of relative Kähler potential $\varphi(s, t)$, cf. [**14**].

The Monge-Ampère equation (2.4) makes sense in particular when Σ is the (closed) unit disk D in $\mathbb{C}$. In this case, it has a nice interpretation in terms of holomorphic disks [**31**], [**36**], [**13**], which plays a crucial rôle in the theory, in particular in the proof given by Chen-Tian of Theorem 2.1 below.

The group $\mathrm{H}(M, J)$ — cf. Section 1.6 — acts on $\mathcal{M}_\Omega$ and preserves its riemannian structure. For any (real) vector field X in its Lie algebra $\mathfrak{h}$ and any (g, ω) in $\mathcal{M}_\Omega$, the induced vector field $\hat{X}$ on $\mathcal{M}_\Omega$ is $g \mapsto f_g^X$, where f_g^X denotes the real potential of X with respect to g, as defined in Section 1.6.

The scalar curvature determines a vector field, $\hat{s}$, on $\mathcal{M}_\Omega$ via the assignment $g \mapsto (s_g - \bar{s})$, with $\bar{s} = \frac{\int_M s_g\, v_g}{V_\Omega}$ (notice that $\int_M s_g\, v_g = 2\pi\, (c_1(M) \cup \frac{\Omega^{m-1}}{(m-1)!})[M] =: S_\Omega$ is independent of g in $\mathcal{M}_\Omega$). The dual 1-form, σ, is $\sigma(g) = s_g\, v_g$, via the duality relation $\langle \sigma, f \rangle = \int_M s_g f v_g$, for any f in $T_g\mathcal{M}_\Omega$. Both $\hat{s}$ and σ are left invariant by $\mathrm{H}(M, J)$. The covariant derivative of σ is given by

$$\mathcal{D}_f \sigma = -2\, (D^- d\,)^* D^- df\, v_g, \tag{2.6}$$

for any g in $\mathcal{M}_\Omega$ and any f in $T_g\mathcal{M}_\Omega$, cf. *e.g.* [**19**, Chapter 4] and Section 1.6 for the notation. Recall, cf. Section 1.6, that the kernel of the operator $(D^- d\,)^* D^- d$ is the space P_g of Killing potentials for g. It then follows from (2.6) that the critical point of the *Calabi functional* $\mathcal{C}(g) = \int_M (s_g - \bar{s})^2\, v_g = \sigma_g(\hat{s})$ on $\mathcal{M}_\Omega$ are those metrics g in $\mathcal{M}_\Omega$ whose scalar curvature is a Killing potential.

Since $(D^- d)^* D^- d$ is self-adjoint, a further direct consequence of (2.6) is that the 1-form σ is *closed*. Since σ is $\mathrm{H}(M, J)$-invariant, by using the Cartan formula $0 = \mathcal{L}_{\hat{X}}\sigma = \iota_{\hat{X}} d\sigma + d(\iota_{\hat{\sigma}}\sigma)$, we infer that $\sigma(\hat{X})$ is *constant* on $\mathcal{M}_\Omega$ for any X in $\mathfrak{h}$, cf. [**7**]. We thus obtain an $\mathbb{R}$-linear form $\mathcal{F}_\Omega : \mathfrak{h} \to \mathbb{R}$, defined by

$$\mathcal{F}_\Omega(X) = \sigma(\hat{X}) = \int_M f_g^X s_g \, v_g. \tag{2.7}$$

By the above discussion, the rhs of (2.7) is independent of the choice of the metric g in $\mathcal{M}_\Omega$. This linear form has been first introduced by A. Futaki in [**17**] for Fano manifolds, then extended to general Kähler manifolds by E. Calabi in [**9**]. It will be referred to as the *Futaki invariant* or the *Futaki character*(6) of Ω.

We also consider the *Futaki-Mabuchi bilinear form*, B_Ω, defined on $\mathfrak{h}_0$, the Lie algebra of the reduced group of automorphisms $\mathrm{H}_0(M, J)$, cf. Section 1.6, by

$$B_\Omega(X, Y) = \int_M f_g^X f_g^Y \, v_g - \int_M h_g^X h_g^Y \, v_g, \tag{2.8}$$

for any $X = \mathrm{grad}_g f^X + J\mathrm{grad}_g h^X$, $Y = \mathrm{grad}_g f^Y + J\mathrm{grad}_g h^Y$ in $\mathfrak{h}_0$. It can be checked that the rhs of (2.8) is independent of the metric g in $\mathcal{M}_\Omega$, cf. [**18**]. Notice that $B_\Omega(JX, JY) = -B_\Omega(X, Y)$, for any X, Y in $\mathfrak{h}_0$ and that B_Ω is negative definite on the space, $\mathfrak{k}_0$, of hamiltonian Killing vector fields, and positive definite on $J\mathfrak{k}_0 \subset \mathfrak{h}_0$. For any two elements X, Y in $\mathfrak{h}_0$, with $B_\Omega(Y, Y) \neq 0$, we define the *relative Futaki invariant of X with respect to Y* by

$$\mathcal{F}_\Omega(X \mathrm{mod}\, Y) = \mathcal{F}_\Omega(X) - \frac{B_\Omega(X, Y)}{B_\Omega(Y, Y)} \mathcal{F}_\Omega(Y). \tag{2.9}$$

The *Mabuch K-energy*, $\mathcal{E}$, is defined on $\mathcal{M}_\Omega$ by

$$\sigma = -d\mathcal{E}, \tag{2.10}$$

i.e.

$$d\mathcal{E}_g(f) = -\int_M f \, s_g \, v_g, \tag{2.11}$$

for any g in $\mathcal{M}_\Omega$ and any f in $T_g\mathcal{M}_\Omega$. Since σ is closed and $\mathcal{M}_\Omega$ is contractible, $\mathcal{E}$ exists and is well-defined up to an additive constant; we denote by $\mathcal{E}_{\omega_0}$ the unique determination of $\mathcal{E}$ which vanishes at the chosen base element ω_0 on $\mathcal{M}_\Omega$. It follows from (2.6) that $\mathcal{E}$ is $\mathcal{D}$-convex on $\mathcal{M}_\Omega$, meaning that its Hessian $\mathcal{D}d\mathcal{E}$ is non-negative; moreover, for any g in $\mathcal{M}_\Omega$, its kernel in $T_g\mathcal{M}_\Omega$ is the space of Killing potentials of mean value zero.

(6)It easily follows from its definition that $\mathcal{F}_\Omega$ is a character of the Lie algebra $\mathfrak{h}$, i.e. vanishes on the derived ideal $[\mathfrak{h}, \mathfrak{h}]$.

Because of (2.10), the critical points of $\mathcal{E}$ are the zeros of σ, hence the metrics of constant scalar curvature in $\mathcal{M}_\Omega$. To generalize the setting to include extremal metrics of non-constant scalar curvature — the case of main interest in this paper — it is convenient to substitute a relative version introduced by D. Guan in [**22**] and S. Simanca in [**38**]. This is done as follows.

Let G be a maximal compact subgroup of $\mathrm{H}_0(M,J)$ and denote by $\mathcal{M}_\Omega^G$ the space of G-invariant Kähler metrics in Ω. $\mathcal{M}_\Omega^G$ is a totally geodesic submanifold of $\mathcal{M}_\Omega$. In virtue of a celebrated theorem of Calabi [**9**], any extremal Kähler metric in $\mathcal{M}_\Omega$ — if any — belongs to the $\mathrm{H}_0(M,J)$-orbit of an element of $\mathcal{M}_\Omega^G$. Since G is maximal in $\mathrm{H}_0(M,J)$, its Lie algebra, $\mathfrak{g}$, is the Lie algebra of *all* hamiltonian Killing vector fields for each element, g, of $\mathcal{M}_\Omega^G$. Notice that, while the latter is independent of g, the space, P_g, of Killing potentials with respect to g does depend of g.

For any g in $\mathcal{M}_\Omega^G$, of scalar curvature s_g, the *Killing part*, $\Pi_g^G(s_g)$, of s_g is defined as the L^2-projection of s relative to g in P_g. The *reduced scalar curvature* of g, denoted by s_g^G, is defined by

$$s_g^G = s_g - \Pi_g^G(s_g). \tag{2.12}$$

Then, g is extremal if and only if its reduced scalar curvature s_g^G is identically zero.

The vector field $Z_\Omega^G = \mathrm{grad}(\Pi_g^G(s))$ — called the *extremal vector field* of the pair (Ω, G) — is independent of g in $\mathcal{M}_\Omega^G$ and can be alternatively determined by

$$\mathcal{F}_\Omega(JX) = B_\Omega(JX, Z_\Omega^G), \tag{2.13}$$

for any X in $\mathfrak{g}$. Notice that Z_Ω^G belongs to the center $\mathfrak{z}$ of $\mathfrak{g}$. Its lift, $\hat{Z}_\Omega^G$, on $\mathcal{M}_\Omega^G$ is the vector field $g \mapsto \Pi_g^G(s_g)$. It turns out that $\hat{Z}_\Omega^G$ is $\mathcal{D}$-parallel, and so is its dual 1-form ζ_Ω^G, cf. [**19**]. We now consider the 1-form on $\mathcal{M}_\Omega^G$ defined by

$$\sigma^G = \sigma_{|\mathcal{M}_\Omega^G} - \zeta_\Omega^G. \tag{2.14}$$

Since ζ_Ω^G is $\mathcal{D}$-parallel, we infer from (2.6)

$$\mathcal{D}_f \sigma^G = -2\,(D^- d)^* D^- df, \tag{2.15}$$

for any f in $T_g\mathcal{M}_\Omega^G$. In particular, σ^G is closed.

Denote by $\mathrm{H}_G(M,J)$ the normalizer of G in $\mathrm{H}_0(M,G)$ and by $\mathfrak{h}_G$ the Lie algebra of $\mathrm{H}_G(M,J)$. The group $\mathrm{H}_G(M,J)$ acts on $\mathcal{M}_\Omega^G$ and we define as above the *relative Futaki character* $\mathcal{F}_\Omega^G : \mathfrak{h}_G \to \mathbb{R}$ by

$$\mathcal{F}_\Omega^G(X) = \sigma^G(\hat{X}) = \int_M f_g^X\, s_g^G v_g. \tag{2.16}$$

As before, $\Omega : \mathfrak{h}_G \to \mathbb{R}$ is independent of g in $\mathcal{M}_\Omega^G$.

The *relative K-energy* $\mathcal{E}^G$ is defined by

$$\sigma^G = -d\mathcal{E}^G, \tag{2.17}$$

i.e. by

$$d\mathcal{E}^G_g(f) = -\int_M f\, s^G_g\, v_g, \tag{2.18}$$

for any g in $\mathcal{M}^G_\Omega$ and any f in $T_g\mathcal{M}^G_\Omega$. Since σ^G is closed and $\mathcal{M}^G_\Omega$ is contractible, $\mathcal{E}^G$ is well-defined up to an additive real constant. As before, we denote by $\mathcal{E}^G_{\omega_0}$ the determination of $\mathcal{E}^G$ which is zero at the chosen base point ω_0. By (2.17), the critical points of $\mathcal{E}^G$ are the zeros of σ^G, hence the extremal metrics in $\mathcal{M}^G_\Omega$. Moreover, since $\mathcal{D}\sigma^G = \mathcal{D}\sigma_{|\mathcal{M}^G_\Omega}$, $\mathcal{E}^G$ is $\mathcal{D}$-convex and, at each g in $\mathcal{M}^G_\Omega$, the kernel of the Hessian $\mathcal{D}d\mathcal{E}^G$ is the space of G-invariant Killing potentials relative to g.

2.2. The Chen-Tian Theorem. — The K-energy $\mathcal{E}$ and the relative K-energy $\mathcal{E}^G$ defined in Section 2.1 play an important role in the theory of extremal Kähler metrics, due in particular to the following observation.

***Proposition 2.1* (S. Donaldson [14]).** — *Let ω_0, ω be any two elements of $\mathcal{M}^G_\Omega$. Assume that ω_0 is extremal. Assume, moreover, that there exists a geodesic $\omega_t = \omega_0 + dd^c\varphi_t$, $t \in [0,1]$, between ω_0 and $\omega = \omega_1$. Then*

$$\mathcal{E}^G(\omega) \geq \mathcal{E}^G(\omega_0) \tag{2.19}$$

and equality holds if and only if ω is extremal. If so, ω belongs to the $\mathrm{H}_0(M,J)$*-orbit of ω_0.*

Proof. — (Sketch) To simplify notation, let's write $f(t)$ for $\mathcal{E}^G(\omega_t)$; we can assume $f(0) = 0$. By (2.17), we have that $f'(t) = -\sigma^G(T)$, where T denotes the tangent vector field along the geodesic ω_t, given by the assignment $t \mapsto \dot\varphi_t \in T_{\omega_t}\mathcal{M}^G_\Omega$. In particular, $f'(0) = 0$, since ω_0 is extremal. By using (2.15), we get:

$$\begin{aligned} f''(t) &= -(\mathcal{D}_T\sigma^G)(T) - \sigma^G(\mathcal{D}_T T) \\ &= -(\mathcal{D}_T\sigma^G)(T) = 2\int_M ((D^-d)D^-d\dot\varphi_t, \dot\varphi_t)\, v_{g_t} \\ &= 2\int_M |D^-d\dot\varphi_t|^2\, v_{g_t} \end{aligned} \tag{2.20}$$

The last term is non-negative and is zero if and only if $\dot\varphi_t$ is a Killing potential with respect to g_t for each t in $[0,1]$, cf. Section 1.6. Proposition 2.1 follows easily. □

This argument has been extended by X. X. Chen and G. Tian in the following way (cf. also Remark 2.1 for the notation):

***Proposition 2.2* (X. Chen–G. Tian [13]).** — *Let ω_0 be a fixed element of $\mathcal{M}^G_\Omega$ and let Φ be a smooth G-invariant solution of the Monge-Ampère equation (2.4) defined on $M \times \Sigma$ for any Riemann surface with boundary Σ. Suppose that, for any τ in Σ, $\Phi(\cdot, p)$ is the relative Kähler potential of an element $\omega^{(\tau)} = \omega_0 + dd^c\Phi(\cdot, \tau)$ in $\mathcal{M}^G_\Omega$, so that the relative energy $\mathcal{E}^G(\tau) := \mathcal{E}^G(\omega^{(\tau)})$ can be regarded as a function defined on Σ. Let $z = t+is$ be a local holomorphic coordinate on Σ. Then, with the notation of* Remark 2.1, *$\mathcal{E}^G(\tau)$ satisfies the following equality*

$$\frac{d^2\mathcal{E}^G}{dt^2} + \frac{d^2\mathcal{E}^G}{ds^2} = 2\int_M |D^-(d\dot\varphi_t - d^c\dot\varphi_s)|^2_{\omega^{(\tau)}}\, v_{g^{(\tau)}}. \tag{2.21}$$

In particular, $\frac{d^2\mathcal{E}^G}{dt^2} + \frac{d^2\mathcal{E}^G}{ds^2} \geq 0$, with equality if and only if $Z := \mathrm{grad}_{g_{t,s}}\dot\varphi_t - J\mathrm{grad}_{g_{t,s}}\dot\varphi_s$ is a (real) holomorphic vector field on M for any τ in Σ.

Proof. — From (2.18), we infer $\frac{d\mathcal{E}^G}{dt} = -\int_M s^G_{g^{(\tau)}}\dot\varphi_t\, v_{g^{(\tau)}}$. It is easily deduced from (2.15) that, in general, the first variation of the reduced scalar curvature at g in $\mathcal{M}^G_\Omega$ in the direction of f is given by

$$\dot{s^G}(f) = -2(D^-d)^*D^-df + (ds^G_g, f), \tag{2.22}$$

whereas the first variation of the volume form is given by $\dot{v_g}(f) = -\Delta_g f\, v_g$. The second derivative of $\mathcal{E}^G$ with respect to t is then given by

$$\frac{d^2\mathcal{E}^G}{dt^2} = 2\int_M |D^-d\dot\varphi_t|^2\, v_g - \int_M \left(\ddot\varphi_{tt} - (d\dot\varphi_t, d\dot\varphi_t)_{g^{(\tau)}}\right) s^G_{g^{(\tau)}} v_{g^{(\tau)}}. \tag{2.23}$$

We get a similar formula by replacing t by s, hence, by using (2.5):

$$\begin{aligned}\frac{d^2\mathcal{E}^G}{dt^2} + \frac{d^2\mathcal{E}^G}{ds^2} &= 2\int_M |D^-(d\dot\varphi_t - d^c\dot\varphi_s)|^2\, v_{g_\tau}\\ &+ 2\int_M \left(2(D^-d\dot\varphi_t, D^-d^c\dot\varphi_s) + (d\dot\varphi_t, d^c\dot\varphi_s)\, s^G_{g_\tau}\right) v_{g_\tau}\end{aligned} \tag{2.24}$$

where the second term in the rhs is actually zero(7). The last assertion of Proposition 2.2 follows easily (see Section 1.6). □

(7) This is an easy consequence of the following general formula (see Section 1.6 for the notation):

$$(D^-d)^*D^-d^cf = -\frac{1}{2}\mathcal{L}_K f, \tag{2.25}$$

for any function f on a Kähler manifold of scalar curvature s_g, with $K := J\mathrm{grad}_g s_g$. Here, (2.25) is applied to $f = \dot\varphi_s$. Moreover, since $\dot\varphi_s$ is G-invariant, K can be replaced by $K^G := J\mathrm{grad}_g s^G_g$.

The argument in Proposition 2.1 only holds for metrics which are linked to extremal metrics by a geodesic. On the other hand, the existence issue for geodesics in $\mathcal{M}_\Omega$ has remained an open question, principally because of the lack of regularity for solutions of the Monge-Ampère equation (2.4). In [**13**], X. X. Chen and G. Tian established a (weak) regularity theorem for solutions of (2.4), improving a previous regularity result by X. X. Chen [**10**] which asserts the existence of solutions in the class $C^{1,1}$. From this, and by using the above Proposition 2.2, they were able to deduce the following fudamental results:

Theorem 2.1 **(X. X. Chen–G. Tian [11], [12], [13])**

(i) *All extremal metrics in $\mathcal{M}_\Omega$, if any, belong to a unique* $\mathrm{H}_0(M,J)$*-orbit.*

(ii) *Let ω_0 be an extremal metric in $\mathcal{M}_\Omega$. Without loss of generality, assume that ω_0 belongs to $\mathcal{M}_\Omega^G$. Then,*

$$\mathcal{E}^G(\omega) \geq \mathcal{E}^G(\omega_0), \tag{2.26}$$

with equality if and only if ω is extremal.

2.3. The relative energy of admissible metrics. — Denote by $\mathcal{M}_{\Omega_{\boldsymbol{\lambda}}}^{\mathrm{adm}}$ the space of admissible Kähler metrics in a given (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$. Then, $\mathcal{M}_{\Omega_{\boldsymbol{\lambda}}}^{\mathrm{adm}} \subset \mathcal{M}_{\Omega_{\boldsymbol{\lambda}}}^G$, where G is the maximal compact subgroup of $\mathrm{H}_0(M,J)$ given by Propositions 1.3–1.6, and the reduced scalar curvature is given by the following proposition (cf. [**3**, Proposition 6]):

Proposition 2.3. — *For any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ and for any admissible Kähler metric $g = g_{\boldsymbol{\lambda},z}$ in $\Omega_{\boldsymbol{\lambda}}$, of scalar curvature s_g, the Killing part of s_g is given by*

$$\Pi_g^G(s_g) = \alpha\, z + \beta, \tag{2.27}$$

where α, β denote the renormalized leading coefficients of the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$, defined by (1.53)*, whereas the reduced scalar curvature has the following expression:*

$$s_g^G = \frac{(F_{\Omega_{\boldsymbol{\lambda}}} - p_{\Omega_{\boldsymbol{\lambda}}}\,\Theta)''(z)}{p_{\Omega_{\boldsymbol{\lambda}}}(z)}. \tag{2.28}$$

Proof. — For any admissible Kähler metric in a (normalized) Kähler class, it follows from (1.23) that the space P_g of Killing potentials relative to g splits as

$$P_g = \mathbb{R} \oplus \mathbb{R}\, z \oplus \left(\oplus_{i=1}^N P_{g_{S_i}}^0\right), \tag{2.29}$$

where: $\mathbb{R}$ denotes the space of constant functions; $\mathbb{R}\, z$ the space generated by z; $P_{g_{S_i}}^0$ denotes the space of Killing potentials of mean value zero on (S_i, g_{S_i}). By (1.52), the scalar curvature s is a function of z only; by (1.27), s is then

L^2-orthogonal to all Killing potentials in $\oplus_{i=1}^N P^0_{g_{S_i}}$. In order to prove (2.27), it is sufficient to check that $\frac{R(x)-(p_{\Omega_{\boldsymbol\lambda}}\Theta)''(x)}{p_{\Omega_{\boldsymbol\lambda}}(x)}$ is orthogonal to 1 and to z. In view of (1.27), this amounts to checking that $\int_{-1}^1 (R(x) - (p_{\Omega_{\boldsymbol\lambda}}\Theta)''(x))\,dx = 0$ and $\int_{-1}^1 (R(x) - (p_{\Omega_{\boldsymbol\lambda}}\Theta)''(x))\,x\,dx = 0$; in view of the boundary conditions (1.14)–(1.15) for Θ, these two conditions are equivalent to (1.43)–(1.44); since $R = F''_{\Omega_{\boldsymbol\lambda}}$, (2.28) follows from (2.27) and (1.52). □

Corollary 2.1. — *For any admissible Kähler class $\Omega_{\boldsymbol\lambda}$, denote by $Z^G_{\Omega_{\boldsymbol\lambda}}$ the extremal vector field relative to the pair $(G, \Omega_{\boldsymbol\lambda})$, see* Section 2.1. *Then*

$$JZ^G_{\Omega_{\boldsymbol\lambda}} = \alpha\, T. \tag{2.30}$$

Proof. — By definition, $Z^G_{\Omega_{\boldsymbol\lambda}} = \mathrm{grad}_g(\Pi^G_g(s_g)$, for any g in $\mathcal{M}^G_{\Omega_{\boldsymbol\lambda}}$, hence for $g_{\boldsymbol\lambda,z}$. Since, $-JT = \mathrm{grad}_{g_{\boldsymbol\lambda,z}} z$, (2.30) readily follows from (2.27). □

Corollary 2.2. — *For any admissible Kähler class $\Omega_{\boldsymbol\lambda}$, we have*

(2.31)
$$\mathcal{F}_{\Omega_{\boldsymbol\lambda}}(-JT) = \frac{2\pi V(S)}{\int_{-1}^1 p_{\Omega_{\boldsymbol\lambda}}(s)ds} \times \alpha\,\Big(\int_{-1}^1 s^2 p_{\Omega_{\boldsymbol\lambda}}(s)ds \int_{-1}^1 p_{\Omega_{\boldsymbol\lambda}}(s)ds - \int_{-1}^1 s\, p_{\Omega_{\boldsymbol\lambda}}(s)ds \int_{-1}^1 s\, p_{\Omega_{\boldsymbol\lambda}}(s)ds\Big)$$

and

(2.32)
$$B_{\Omega_{\boldsymbol\lambda}}(-JT,-JT) = \frac{2\pi V(S)}{\int_{-1}^1 p_{\Omega_{\boldsymbol\lambda}}(s)ds} \times \Big(\int_{-1}^1 s^2 p_{\Omega_{\boldsymbol\lambda}}(s)ds \int_{-1}^1 p_{\Omega_{\boldsymbol\lambda}}(s)ds - \int_{-1}^1 s\, p_{\Omega_{\boldsymbol\lambda}}(s)ds \int_{-1}^1 s\, p_{\Omega_{\boldsymbol\lambda}}(s)ds\Big),$$

where $V(S) = \prod V(S_i, g_{S_i})$ denotes the volume of S. In particular,

$$\mathcal{F}_{\Omega_{\boldsymbol\lambda}}(-JT) = \alpha\, B_{\Omega_{\boldsymbol\lambda}}(-JT,-JT). \tag{2.33}$$

Proof. — Since T is a hamiltonian Killing vector field of momentum z, $-JT$ belongs to $J\mathfrak{g}$ and its real holomorphic potential is $z - \bar z$, where $\bar z = \frac{\int_M z v_g}{\int_M v_g}$ is the mean value of z. Since $z - \bar z$ belongs to P_g, in (2.7) only the Killing part $\Pi^G_g(s_g) = \alpha z + \beta$ contributes: we then get $\mathcal{F}_\Omega(-JT) = \alpha \int_M (z - \bar z) z v_g$ and $B_{\Omega_{\boldsymbol\lambda}}(-JT,-JT) = \int_M (z-\bar z)^2\, v_g$. By using the expression (1.27) of v_g, we readily get (2.31) and (2.32); (2.33) follows readily; alternatively, (2.33) follows from (2.32) and Corollary 2.1, via (2.13). □

Choose a reference element in $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$, e.g. the standard admissible metric ω_0 corresponding to the admissible momentum $z_0(t) = \tanh t$, cf. Section 1.5. Any other element ω of $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ can be written $\omega = \omega_0 + dd^c\phi$, where $\phi = \phi(t)$, called the *relative potential* of ω, is uniquely determined by ω up to an additive constant. Notice that

$$z = z_0 + \frac{d\phi}{dt}. \tag{2.34}$$

For any curve ω_s in $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ we set $\dot{\omega} = \frac{d\omega_s}{ds}_{|s=0}$ and we denote similarly the first variations of all objects determined by ω; we thus have: $\dot{\omega} = dd^c\dot{\phi}$, $\dot{z} = \frac{d\dot{\phi}}{dt}$, etc. By identifying $\dot{\omega}$ with $\dot{\phi}$ we identify each tangent space $T_g\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ of $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ with the space of all smooth real functions of t mod constant functions.

Although it is a hard task to get an explicit expression of the relative energy $\mathcal{E}^G(g)$ for a general element of $\mathcal{M}^G_{\Omega_{\boldsymbol{\lambda}}}$, it turns out that the restriction of $\mathcal{E}^G$ to $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ admits a simple explicit expression in terms of the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$, given by the following proposition (cf. [**3**, Proposition 7]):

Proposition 2.4. — *For any admissible metric g in $\Omega_{\boldsymbol{\lambda}}$, of momentum profile Θ, we have*

$$\mathcal{E}^G(g) = C\int_{-1}^{1}\left(\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\Theta(x)} + p_{\Omega_{\boldsymbol{\lambda}}}(x)\log\Theta(x)\right)dx \mod \mathbb{R}, \tag{2.35}$$

with $C = 2\pi\prod_{i=1}^N V_i$, where V_i denotes the volume of (S_i, g_{S_i}).

Proof. — The restriction of $\mathcal{E}^G$ to $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ is determined by

$$d\mathcal{E}^G_g(\dot{\phi}) = -\int_M s^G_g\,\dot{\phi}\,v_g \tag{2.36}$$

for any $g = g_{\boldsymbol{\lambda},z}$ in $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ and for $\dot{\phi} = \dot{\phi}(t)$, any function of t mod $\mathbb{R}$, where, we recall, s^G_g denotes the reduced scalar curvature of g with respect to G. By using (2.28) and (1.27), we get

$$(d\mathcal{E}^G)_g(\dot{\phi}) = -C\int_{-1}^{1}\left(F_{\Omega_{\boldsymbol{\lambda}}} - p_{\Omega_{\boldsymbol{\lambda}}}\,\Theta\right)''(x)f(x)\,dx, \tag{2.37}$$

where C is as above and by setting

$$f(x) = \dot{\phi}(z^{-1}(x)). \tag{2.38}$$

By integrating by part twice and by observing that at each step the intregrated terms vanish because of (1.14)–(1.15)–(1.46)–(1.47), we get

$$(d\mathcal{E}^G)_g(\dot{\phi}) = -C\int_{-1}^{1}\left(F_{\Omega_{\boldsymbol{\lambda}}} - p_{\Omega_{\boldsymbol{\lambda}}}\,\Theta\right)(x)f''(x)\,dx. \tag{2.39}$$

From (2.38) we get $f'(x) = \frac{\dot{z}(z^{-1}(x))}{\Theta(x)}$, hence $f''(x) = \frac{1}{\Theta^2(x)}\left(\frac{d\dot{z}}{dt}(z^{-1}(x)) - \Theta'(x)(\dot{z}(z^{-1}(x))\right)$. On the other hand, from (1.13), we get $\dot{\Theta}(x) = \frac{d\dot{z}}{dt}(z^{-1}(x)) - \Theta'(x)\dot{z}(z^{-1}(x))$. We thus end up with

$$f''(x) = \frac{\dot{\Theta}(x)}{\Theta^2(x)}. \tag{2.40}$$

By substituting in (2.39), we eventually obtain

$$(d\mathcal{E}^G)_g(\dot{\phi}) = C \int_{-1}^{1} \left(-\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\Theta^2(x)} + \frac{p_{\Omega_{\boldsymbol{\lambda}}}(x)}{\Theta(x)}\right) \dot{\Theta}(x)\, dx, \tag{2.41}$$

for any $\dot{\phi} = \dot{\phi}(t)$ in $T_g\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$, where the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ and the characteristic polynomial $p_{\Omega_{\boldsymbol{\lambda}}}$ are both independent of g in $\mathcal{M}_{\Omega_{\boldsymbol{\lambda}}}$. The rhs of (2.41) is then the first derivative in $\mathcal{M}^{\mathrm{adm}}_{\Omega_{\boldsymbol{\lambda}}}$ of the rhs of (2.35). □

Proposition 2.5. — *Let $\Omega_{\boldsymbol{\lambda}}$ be any admissible Kähler class on M.*

(i) *Assume that $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on $(-1,1)$ and denote by g_0 an admissible extremal Kähler metric in $\Omega_{\boldsymbol{\lambda}}$, of momentum profile $\Theta_0 = \frac{F_{\Omega_{\boldsymbol{\lambda}}}}{p_{\Omega_{\boldsymbol{\lambda}}}}$* (cf. Proposition 1.7). *Then, for any admissible Kähler metric in $\Omega_{\boldsymbol{\lambda}}$, we have*

$$\mathcal{E}^G(g) \geq \mathcal{E}^G(g_0), \tag{2.42}$$

with equality if and only if g is extremal, hence equal to g_0 up to the natural $\mathbb{R}^{>0}$-action on M.

(ii) *Assume that $F_{\Omega_{\boldsymbol{\lambda}}}$ is negative on a non-empty open subinterval I of $(-1,1)$. Then, for any admissible Kähler metric g in $\Omega_{\boldsymbol{\lambda}}$, there exists a half-line g_s of admissible Kähler metrics in $\Omega_{\boldsymbol{\lambda}}$, with s in $[0,+\infty)$ and $g_0 = g$, such that $\mathcal{E}^G(g_s)$ tends to $-\infty$ when s tends to $+\infty$.*

Proof. — (i) From (2.35) we infer

$$\mathcal{E}^G(g_0) = C \int_{-1}^{1} (1 + \log \frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{p_{\Omega_{\boldsymbol{\lambda}}}}(x))\, p_{\Omega_{\boldsymbol{\lambda}}}(x)\, dx, \tag{2.43}$$

whereas

$$\mathcal{E}^G(g) = C \int_{-1}^{1} (\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{p_{\Omega_{\boldsymbol{\lambda}}}(x)\,\Theta(x)} + \log \Theta(x))\, p_{\Omega_{\boldsymbol{\lambda}}}(x)\, dx. \tag{2.44}$$

We thus get

$$\mathcal{E}^G(g) - \mathcal{E}^G(g_0) = C \int_{-1}^{1} (A(x) - 1 - \log A(x))\, p_{\Omega_{\boldsymbol{\lambda}}}(x)\, dx \tag{2.45}$$

by setting

$$A(x) = \frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{p_{\Omega_{\boldsymbol{\lambda}}}(x)\,\Theta(x)}. \tag{2.46}$$

Now, $A(x)$ is positive for any x in $(-1,1)$ by hypothesis and, by Proposition 1.7, is identically equal to 1 if and only if g is extremal. It is easy to check that the function $\phi(t) := t - 1 - \log t$ defined on $(0,+\infty)$ is convex, tends to $+\infty$ when t tends to 0 or to $+\infty$, and reaches its unique minimum 0 at $t = 1$. It follows that the rhs of (2.45) is positive except when $A = A(x)$ is identically equal to 1, i.e. when g is extremal.

(ii) Let Θ be the momentum profile of any admissible Kähler metric g in $\Omega_{\boldsymbol{\lambda}}$. Let φ be a non-negative, non-constant smooth function on $(-1,1)$ which is compactly supported in the interval I, and set

$$\Theta_s(x) = \frac{\Theta(x)}{1 + s\,\varphi(x)\Theta(x)}, \tag{2.47}$$

for any non-negative real number s. By Proposition 1.2, Θ_s is the momentum profile of an admissible Kähler metric, g_s, in $\Omega_{\boldsymbol{\lambda}}$ for any $s \geq 0$, with $g_0 = g$. Moreover

$$\begin{aligned} \mathcal{E}^G(g_s) =& \mathcal{E}^G(g) + C \int_{-1}^{1} s\,\varphi(x) F_{\Omega_{\boldsymbol{\lambda}}}(x)\,dx \\ & - C \int_{-1}^{1} \log\left(1 + s\,\varphi(x)\Theta(x)\right) dx \end{aligned} \tag{2.48}$$

where, $\int_{-1}^{1} s\,\varphi(x) F_{\Omega_{\boldsymbol{\lambda}}}(x))\,dx = \int_I \varphi(x) F_{\Omega_{\boldsymbol{\lambda}}}(x))\,dx$ is a *negative* multiple of s. It follows that the rhs of (2.48) tends to $-\infty$ when s tends to $+\infty$. □

Remark 2.2. — The expression (2.35) of the (relative) energy of admissible metrics, as well as the argument in Proposition 2.5, are quite reminiscent to Donaldson's paper [**15**] for toric manifolds.

Now we are ready to state and prove the main result of [**3**]:

Theorem 2.2. — *Let $M = \mathbb{P}(1 \oplus L)$ be any admissible ruled manifold and let $\Omega_{\boldsymbol{\lambda}}$ be any (normalized) admissible Kähler class on M. Then, $\Omega_{\boldsymbol{\lambda}}$ contains an extremal Kähler metric — which is then admissible up to the action of $\mathrm{H}_0(M,J)$ — if and only if the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is (strictly) positive on $(-1,1)$.*

Proof. — By Proposition 1.7, if $F_{\Omega_{\boldsymbol{\lambda}}}$ is positive on $(-1,1)$, $\Omega_{\boldsymbol{\lambda}}$ contains an admissible extremal Kähler metrics. By Proposition 2.5, if $F_{\Omega_{\boldsymbol{\lambda}}}$ is negative on some open subinterval of $(-1,1)$, the relative K-energy $\mathcal{E}^G$ is not bounded from below: by Theorem 2.1 (ii), $\Omega_{\boldsymbol{\lambda}}$ contains no extremal Kähler metric.

It remains to consider the limiting case, when $F_{\Omega_{\boldsymbol{\lambda}}}$ is non-negative but has (repeated) zeros on $(-1,1)$. Suppose that $F_{\Omega_{\boldsymbol{\lambda}}}$ is of this form and assume, for a contradiction, that $\Omega = \Omega_{\boldsymbol{\lambda}}$ contains an extremal Kähler metric, (g,ω) say. In view of the already mentioned Calabi theorem, we can assume that the pair (g,ω) is G-invariant (cf. Proposition 1.6). By LeBrun-Simanca's openness theorem [**29, 30**], any (normalized) admissible Kähler class $\Omega_{\boldsymbol{\lambda}'}$, with $\boldsymbol{\lambda}'$ close to $\boldsymbol{\lambda}$ in $\mathbb{R}^N$, contains an extremal Kähler metric. More precisely, LeBrun-Simanca's theorem asserts the existence of a sequence of extremal Kähler metrics $(\tilde{g}_k, \tilde{\omega}_k)$, with $[\tilde{\omega}_k] = \Omega_k$, which converges to (g, ω) in the Fréchet topology and the $(\tilde{g}_k, \tilde{\omega}_k)$ can be again chosen G-invariant.

Two cases then may *a priori* occur: (i) either, $F_{\Omega_{\boldsymbol{\lambda}'}}$ has repeated roots on $(-1, 1)$ for all $\boldsymbol{\lambda}'$ in some open neighborhood of $\boldsymbol{\lambda}$ in $\mathbb{R}^N$, or else: (ii) there exists a sequence of (normalized) admissible Kähler classes $\Omega_k = \Omega_{\boldsymbol{\lambda}_k}$ converging to Ω — meaning that $\boldsymbol{\lambda}_k$ converges to $\boldsymbol{\lambda}$ in the usual sense — such that F_{Ω_k} is positive on $(-1, 1)$ for each k.

Case (i) would imply that the discriminant of $F_{\Omega_{\boldsymbol{\lambda}}}$ is zero as a polynomial with coefficients in the field $R(\lambda_1, \ldots, \lambda_N)$ of rational fractions in $\{\lambda_1, \ldots, \lambda_N\}$: this would contradict Proposition A.1 in Appendix A (by substituting $\lambda_i = \epsilon_i \lambda$ in $F_{\Omega_{\boldsymbol{\lambda}}}$, regarded as a polynomial with coefficients in $R(\lambda_1, \ldots, \lambda_N)$, up to a factor $\prod_{i=1}^N \epsilon_i^{d_i}$, we get the extremal polynomial of an admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$, as a polynomial with coefficient in $R(\lambda)$, on an admissible ruled manifold with $N = 1$, $d = \sum_{i=1} d_i$ and $s = \sum_{i=1}^N s_i$). Case (i) is thus discarded.

Now assume, again for a contradiction, that Case (ii) occurs. LeBrun-Simanca openness theorem actually guarantees the existence of a sequence, $(\tilde{g}_k, \tilde{\omega}_k)$, of G-invariant extremal Kähler metrics, with $[\tilde{\omega}_k] = \Omega_k$ for each k, which converges to (g, ω) in the Fréchet topology. On the other hand, since F_{Ω_k} is positive on $(-1, 1)$, Proposition 1.7 guarantees the existence of an *admissible* extremal Kähler metric, (g_k, ω_k) say, in each Ω_k, unique up to the natural $\mathbb{C}^*$-action, with $\omega_k = \sum_{i=1}^N ((\boldsymbol{\lambda}_k)_i + \epsilon_i z_k) \pi^* \omega_i + dz_k \wedge d^c t$, cf. Section 1.3.

By Theorem 2.1, for any k the extremal Kähler metrics (g_k, ω_k) and $(\tilde{g}_k, \tilde{\omega}_k)$ in Ω_k are linked together by $\tilde{g}_k = \Psi_k \cdot g_k$, for some Ψ_k in $\mathrm{H}_0(M, J)$. Moreover, from the invariance of the extremal vector field $Z^G_{\Omega_k}$ of each pair (Ω_k, G) — see Sections 2.1 and 2.3 — we get $Z^G_{\Omega_k} = \mathrm{grad}_{g_k} s_{g_k} = \mathrm{grad}_{\tilde{g}_k} s_{\tilde{g}_k} = \Psi_k \cdot \mathrm{grad}_{g_k} s_{g_k}$, meaning that $Z^G_{\Omega_k}$, hence also T by Corollary 2.1, are preserved by Ψ_k for any k. We infer that the Ψ_k's all belong to the subgroup of elements of $\mathrm{H}_0(M, J)$ which commute with $\mathbb{C}^*$, hence, by Proposition 1.3, to the extension of $\mathrm{H}_0(S, J)$ by $\mathbb{C}^*$. Moreover, since the (g_k, ω_k) are only defined up to the natural $\mathbb{C}^*$-action, we can actually arrange that the Ψ_k's all belong to a lift of $\mathrm{H}_0(S, J)$ in $\mathrm{H}_0(M, J)$, meaning that each Ψ_k is induced by a linear lift on L of an element, Φ_k say, of $\mathrm{H}_0(S, J)$. Each $\tilde{\omega}_k$ is then of the form $\tilde{\omega}_k = \sum_{i=1}^N ((\boldsymbol{\lambda}_k)_i + \epsilon_i \Psi_k \cdot z_k) \pi^* (\Phi_k \cdot \omega_i) + d(\Psi_k \cdot z_k) \wedge d^c (\Psi_k \cdot t)$, hence the Kähler form of an (extremal) admissible Kähler metrics on the admissible ruled manifold obtained by simply substituting the hermitian inner product $\tilde{h}_k = \Psi_k \cdot h$ on L. Since any two hermitian inner products on L are conformal, $\tilde{h}_k$ can be alternatively written as $\tilde{h}_k = e^{2F_k}\, h$ for some well-defined (real) smooth function F_k on S and we then have $\tilde{t}_k = \Psi_k \cdot t = t + \pi^* F_k$. Since $\Psi_k \cdot T = T$, we also have that $\tilde{z}_k = \Psi_k \cdot z_k$ is a momentum of T with respect to $\tilde{\omega}_k$.

By assumption, the sequence $\tilde{\omega}_k$ converges to ω in the Fréchet topology: it follows that $\tilde{z}_k$ converges to a momentum of T with respect to ω; similarly, since $\iota_{JT}\tilde{\omega}_k = -\tilde{g}_k(T,T)\, d^c\tilde{t}_k = -\tilde{g}_k(T,T)\, d^c(t + \pi^* F_k)$, the sequence F_k converges to a smooth function F on S, meaning that the sequence $\tilde{h}_k$ converges to the hermitian inner product $\tilde{h} = e^{2F}\, h$, whereas each $\Psi_k \cdot \omega_i$ converges to $\tilde{\omega}_i$, which is the curvature form of $L^{-\epsilon_i}$ equipped with the hermitian inner product induced by $\tilde{h}$.

It follows that ω is the Kähler form of an extremal admissible Kähler metric on M with respect to $(L, \tilde{h})$. Since the extremal polynomial F_Ω of Ω only depends of the N-tuple $\boldsymbol{\lambda}$ and of the ϵ_i's, F_Ω should then be positive on $(-1, 1)$ by Proposition 1.7 again. Case (ii) is then discarded as well. □

2.4. A borderline case example. — In this section, we present a family of examples of (normalized) admissible Kähler classes on an admissible rules manifold $M = \mathbb{P}(1 \oplus L) \to S$ whose extremal polynomials are non-negative but have a repeated root, which can be chosen irrational, on $(-1, 1)$.

The simplest examples are obtained by considering (complex) four-dimensional admissible ruled manifolds for which $S = \prod_{i=1}^3 S_i$, where each S_i is a Riemann surfaces of genus $\boldsymbol{g}_i$ greater than one. For $i = 1, 2, 3$, the (constant) scalar curvatures s_i of S_i is then negative; more precisely, by the Gauss-Bonnet formula,

$$s_i = \frac{4(1 - \boldsymbol{g}_i)}{k_i}, \tag{2.49}$$

where k_i denotes the degree of the polarizing line bundle $\tilde{L}_i = L_i^{-\epsilon_i}$ (cf. Section 1.1 and formula (1.62) in Section 1.10). In particular, each s_i can be made equal to any negative rational number by an appropriate choice of the genus $\boldsymbol{g}_i$ and of the degree k_i.

Our aim is to construct a family of (normalized) admissible Kähler classes $\Omega_{\boldsymbol{\lambda}}$ on M, for an appropriate choice of the scalar curvatures s_i — hence of the line bundles L_i on S_i by (2.49) — in such a way that the extremal polynomials be of the form

$$F_{\Omega_{\boldsymbol{\lambda}}}(x) = C(1 - x^2)(x^2 + rx - 1)^2, \tag{2.50}$$

for some positive constants C and r. The polynomial in the rhs of (2.50) satisfies the first boundary condition (1.46) for extremal polynomials and is non-negative on $(-1, 1)$. It has two repeated roots: a positive one, $r_+ = \frac{-r+\sqrt{r^2+4}}{2}$, in the open interval $(0, 1)$; a negative one, $r_- = \frac{-r-\sqrt{r^2+4}}{2}$, in $(-\infty, -1)$. The first and second derivatives of $F_{\Omega_{\boldsymbol{\lambda}}}$ are given by:

$$F'_{\Omega_{\boldsymbol{\lambda}}}(x) = C(-6x^5 - 10rx^4 + 4(3 - r^2)x^3 + 12rx^2 + 2(r^2 - 3)x - 2r) \tag{2.51}$$

and

$$F''_{\Omega_{\boldsymbol{\lambda}}}(x) = C(-30x^4 - 40rx^3 + 12(3 - r^2)x^2 + 24rx + 2(r^2 - 3)). \tag{2.52}$$

In particular, $F'_{\Omega_{\boldsymbol{\lambda}}}(-1) = 2Cr^2$ and $F'_{\Omega_{\boldsymbol{\lambda}}} = -2Cr^2$. It follows that F_{Ω_λ} satisfies the second boundary condition (1.47) for extremal polynomials if and only if

$$p_{\Omega_{\boldsymbol{\lambda}}}(-1) = p_{\Omega_{\boldsymbol{\lambda}}}(1) = Cr^2, \tag{2.53}$$

where $p_{\Omega_{\boldsymbol{\lambda}}}(x) = \prod_{i=1}^{3}(\lambda_i + \epsilon_i x)$ denotes the characteristic polynomial [(8)] of $\Omega_{\boldsymbol{\lambda}}$, cf. (1.7). If we write $p_{\Omega_{\boldsymbol{\lambda}}}(x) = \sum_{j=0}^{3} p_j x^{3-j}$, with $p_0 = \epsilon_1\epsilon_2\epsilon_3$, $p_1 = \sum_{ijk} \epsilon_i\epsilon_j\lambda_k$, $p_2 = \sum_{ijk} \epsilon_i\lambda_j\lambda_k$, $p_3 = \lambda_1\lambda_2\lambda_3$ (summation over the circular permutation of $(1, 2, 3)$), (2.53) is equivalent to the two conditions:

$$p_0 + p_2 = 0, \tag{2.54}$$

$$p_1 + p_3 = Cr^2. \tag{2.55}$$

The condition (2.54) cannot be satisfied if all ϵ_i are equal to 1 or -1: We then assume

$$\epsilon_1 = \epsilon_2 = 1, \qquad \epsilon_3 = -1, \tag{2.56}$$

and (2.54) then reads:

$$\lambda_3 = \frac{1 + \lambda_1\lambda_2}{\lambda_1 + \lambda_2}. \tag{2.57}$$

Notice that $\frac{1+\lambda_1\lambda_2}{\lambda_1+\lambda_2} = 1 + \frac{(\lambda_1-1)(\lambda_2-1)}{\lambda_1+\lambda_2} > 1$. The condition (2.55) determines the constant C as follows:

$$C = \frac{p_1 + p_3}{r^2} = \frac{\lambda_1\lambda_2\lambda_3 + \lambda_3 - \lambda_1 - \lambda_2}{r^2}. \tag{2.58}$$

Notice, by using (2.57), that $\lambda_1\lambda_2\lambda_3 + \lambda_3 - \lambda_1 - \lambda_2 = \frac{(1+\lambda_1\lambda_2)^2-(\lambda_1+\lambda_2)^2}{\lambda_1+\lambda_2} = \frac{(\lambda_1^2-1)(\lambda_2^2-1)}{\lambda_1+\lambda_2} > 0$. Also notice that

$$\lambda_1 - \lambda_3 = \frac{\lambda_1^2 - 1}{\lambda_1 + \lambda_2} > 0, \qquad \lambda_2 - \lambda_3 = \frac{\lambda_2^2 - 1}{\lambda_1 + \lambda_2} > 0. \tag{2.59}$$

Now, for any positive real number r and for any admissible triple $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$ satisfying (2.57), the polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ defined by (2.50), where C is defined by (2.58), is actually the extremal polynomial of the (normalized) admissible

(8) As long as the s_i and the ϵ_i — hence the S_i and the polarizing line bundles $L_i^{-\epsilon_i}$ over S_i — have not been fixed, $\Omega_{\boldsymbol{\lambda}}$ is only a "virtual" admissible Kähler class encoded by an admissible triple $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$.

Kähler class Ω_λ if and only if $F''_{\Omega_{\boldsymbol\lambda}}(x) = R(x)$, where, in general, $R(x)$ is defined by (1.42) in Section 1.9. In the present situation, this condition is then:

(2.60)
$$F''_{\Omega_{\boldsymbol\lambda}}(x) = s_1(\lambda_2 + x)(\lambda_3 - x) + s_2(\lambda_3 - x)(\lambda_1 + x) + s_3(\lambda_1 + x)(\lambda_2 + x) - (\alpha x + \beta)(\lambda_1 + x)(\lambda_2 + x)(\lambda_3 - x),$$

where α, β are real constants. In view of (2.60), we now assume that λ_1 and λ_2 are distinct, hence $\lambda_1 > \lambda_2$ say. This implies that the s_i's are uniquely determined by

(2.61)
$$s_1 = \frac{F''_{\Omega_{\boldsymbol\lambda}}(-\lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 + \lambda_1)}, \quad s_2 = \frac{F''_{\Omega_{\boldsymbol\lambda}}(-\lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_2)}, \quad s_3 = \frac{F''_{\Omega_{\boldsymbol\lambda}}(\lambda_3)}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)},$$

a special case of the general formula (1.49). By using (2.52), this can be re-written as

$$(2.62) \quad s_1 = \frac{2C}{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_3)}\big((6\lambda_1^2 - 1)\, r^2 - 4\lambda_1(5\lambda_1^2 - 3)\, r + 15\lambda_1^4 - 18\lambda_1^2 + 3)\big),$$

(2.63)
$$s_2 = \frac{2C}{(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3)}\big(- (6\lambda_2^2 - 1)\, r^2 + 4\lambda_2(5\lambda_2^2 - 3)\, r - 15\lambda_2^4 + 18\lambda_2^2 - 3)\big),$$

(2.64)
$$s_3 = \frac{2C}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}\big(- (6\lambda_3^2 - 1)\, r^2 - 4\lambda_3(5\lambda_3^2 - 3)\, r - 15\lambda_3^4 + 18\lambda_3^2 - 3)\big).$$

Conversely, if s_1, s_2, s_3 are given these values, then $F''_{\Omega_{\boldsymbol\lambda}}(x)$ is of the form (2.60) — as $F''_{\Omega_{\boldsymbol\lambda}}(x) - s_1(\lambda_2 + x)(\lambda_3 - x) + s_2(\lambda_3 - x)(\lambda_1 + x) + s_3(\lambda_1 + x)(\lambda_2 + x)$ is then divisible by $(\lambda_1 + x)(\lambda_2 + x)(\lambda_3 - x)$ — so that $F_{\Omega_{\boldsymbol\lambda}}$ is indeed an extremal polynomial provided however that the real numbers s_i defined by (2.62)-(2.63)-(2.64) can be realized as the scalar curvatures of Riemann surfaces S_i of genus greater than 1, polarized by a holomorphic line bundle $L_i^{-\epsilon_i}$. According to (2.49), this can be done whenever s_i are (arbitrary) negative rational numbers. This forces us to assume that λ_1, λ_2 — hence also λ_3 by (2.57) — are rational, as well as the parameter r.

By (2.64), s_3 is negative for any $r > 0$ and any admissible triple $\{\lambda_1, \lambda_2, \lambda_3\}$. By (2.63)-(2.64), s_1 is negative if and only if

$$(2.65) \qquad \psi_-(\lambda_1) < r < \psi_+(\lambda_1),$$

and s_2 is negative if and only if

$$(2.66) \qquad r < \psi_-(\lambda_2) \quad \text{or} \quad r > \psi_+(\lambda_2),$$

by setting

$$\psi_{\pm}(\lambda) = \frac{2\lambda(5\lambda^2-3) \pm \sqrt{10\lambda^6+3\lambda^4+3}}{6\lambda^2-1}. \tag{2.67}$$

It is easy to check that ψ_- is increasing from $\psi_-(1)=0$ to $+\infty$ and that ψ_+ is increasing from $\psi_+(1)=8/5$ to ∞ when λ runs from 1 to $+\infty$. We readily infer: For any (rational) admissible triple $\{\lambda_1,\lambda_2,\lambda_3\}$ satisfying (2.57) and $\lambda_1>\lambda_2$, the (rational) numbers s_1,s_2,s_3 given by (2.62)-(2.63)-(2.64) are all negative if and only if

$$\psi_-(\lambda_1) < r < \psi_+(\lambda_1) \tag{2.68}$$

if $\psi_-(\lambda_1) \geq \psi_+(\lambda_2)$, or

$$\psi_+(\lambda_2) < r < \psi_+(\lambda_1), \tag{2.69}$$

if $\psi_-(\lambda_1) \leq \psi_+(\lambda_2)$. The above discussion can be summarized by the following statement ([**3**, Example 1]):

Proposition 2.6. — *For any admissible triple $\boldsymbol{\lambda}=\{\lambda_1,\lambda_2,\lambda_3\}$ of rational numbers satisfying* (2.57) *and $\lambda_1>\lambda_2$, denote by $I_{\boldsymbol{\lambda}}$ the open interval in* $(8/5,+\infty)$ *defined by* (2.68)-(2.69)*. Then, for any rational number r in $I_{\boldsymbol{\lambda}}$, there exists a* (*complex*) *four-dimensional ruled manifold $M=\mathbb{P}(1\oplus L)\to S=\prod_{i=1}^3 S_i$, where each S_i is a Riemann surface of hyperbolic type, such that $\Omega_{\boldsymbol{\lambda}}$ is a* (*normalized*) *admissible Kähler class on M whose extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is of the form* (2.50)*, with C defined by* (2.58)*.*

Remark 2.3. — In view of the current conjectures concerning the link between the existence of extremal Kähler metrics and stability questions considered in the next chapter, the case of particular interest in Proposition 2.6 is when r is chosen so that the repeated root $r_+=\frac{-1+\sqrt{r^2+4}}{2}$ of $F_{\Omega_{\boldsymbol{\lambda}}}$ in $(0,1)$ is *irrational.* If r is written as $r=p/q$, for two (relatively prime) positive integers, this happens if and only if the integer p^2+4q^2 is *not* a square, hence for "most" rational numbers in $I_{\boldsymbol{\lambda}}$.

3. Extremal metrics and stability

3.1. The Futaki character on polarized manifolds. — In this section, $M=(M,J,g,\omega)$ denotes a general compact Kähler manifold of complex dimension m, polarized by a hermitian holomorphic line bundle L, meaning that $R^{\nabla}=i\omega$, i.e. that the Kähler form ω is the curvature form of the Chern connection ∇ of L. In particular, $\Omega=[\omega]=2\pi\, c_1(L)$. We denote by π the projection of L on M. As usual, L is viewed as a complex manifold of complex dimension $m+1$.

We consider an S^1-action on M which preserves the whole Kähler structure. Denote by X the generator of this action, *i.e.* the (real) vector field X defined by $X(x) = \frac{d}{dt}_{|t=0} e^{it} \cdot x$, for any x in M. We assume that the action is hamiltonian, *i.e.* that $X = \mathrm{grad}_\omega f^X = J\mathrm{grad}_g f^X$, for some real function f^X well-defined up to an additive constant.

For any choice of f^X, X lifts to a vector field $\hat{X}$ on L, preserving the natural complex structure of L, defined by $\hat{X} = \tilde{X} - (\pi^* f^X)\, T$, where $\tilde{X}$ denotes the horizontal lift of X on L determined by ∇ and T the generator of the standard S^1-action on L (= the usual multiplication by S^1 on each fiber). Moreover, for an appropriate choice of f^X, $\hat{X}$ is the generator of a holomorphic S^1-action on L which lifts the given S^1-action on M, cf. e.g. [**19**, Proposition 7.5.1]. Such a *distinguished momentum* is well-defined up to an additive integer. We henceforth assume that $\hat{X}$ is the generator of a lifted S^1-action on L, corresponding to the distinguished momentum f^X. Notice that the lifted action on L determines a lifted S^1-action on all tensor powers L^k of L.

The lifted action induces a $\mathbb{C}$-linear S^1-action on the space, $\Gamma(L)$, of smooth sections of L, defined by

$$(\zeta \cdot s)(x) = \zeta \cdot (s\,(\zeta^{-1} \cdot x)), \tag{3.1}$$

for any s in $\Gamma(L)$, any ζ in S^1 and any x in M. According to the general definition of the Lie derivative, we then define:

$$\mathcal{L}_X s = -\frac{d}{dt}_{|t=0} e^{it} \cdot s, \tag{3.2}$$

for any s in $\Gamma(L)$ and any x in M. In terms of covariant derivative, this can be rewritten as

$$\mathcal{L}_X s = \nabla_X s + i f^X\, s. \tag{3.3}$$

The Lie derivative $\mathcal{L}_X$ preserves the subspace $H^0(M, L)$ of holomorphic sections of L and thus induces a $\mathbb{C}$-linear, skew-symmetric action on $H^0(M, L)$ and, more generally, on $H^0(M, L^k)$ for any positive integer k.

Definition 3.1. — The *infinitesimal weight* of the lifted S^1-action on L is the trace of the hermitian operator $-i\mathcal{L}_X$ on $H^0(M, L)$.

Example 3.1. — Let $(V, \langle\cdot,\cdot\rangle)$ be any hermitian $(m+1)$-dimensional complex vector space and denote by $\mathbb{P}(V)$ the corresponding complex projective space, equipped with the induced Fubini-Study Kähler metric of holomorphic sectional curvature equal to 2: the Kähler form ω is then the curvature form $-iR^\nabla$ of the Chern connection of the dual tautological line bundle $\mathcal{O}(1)$, equipped with the induced hermitian inner product, cf. Section 1.1. Any hermitian endomorphism A of V with integer eigenvalues $a_0, a_1, \dots, a_m$ determines an S^1-action on $\mathbb{P}(V)$ by: $e^{it} \cdot x = e^{itA}(x)$, for any x in $\mathbb{P}(V)$. This action preserves the whole Kähler metric. The generator of this action is the (real) Hamiltonian Killing vector field X^A defined by $X^A(x) : u \in x \mapsto iA(u)$ mod x (we here the natural identification $T_x\mathbb{P}(V) = \mathrm{Hom}(x, V/x)$). This action has a natural, tautological, lift on the tautological bundle $\mathcal{O}(-1)$, namely $e^{it} \cdot u = e^{itA}(u)$, for any x in $\mathbb{P}(V)$ and any u in the complex line x. The dual S^1-action on $\mathcal{O}(1)$ is then $(e^{it} \cdot \alpha)(u) = \alpha(e^{-itA}(u))$, for any α in $\mathcal{O}(1)_x = x^*$. This is a lift of the above S^1-action on $\mathcal{O}(1)$, corresponding to the distinguished momentum defined by $f^{X^A}(x) = \langle Au, u\rangle$, for any unit generator u of x. The space $H^0(\mathbb{P}(V), \mathcal{O}(1))$ is naturally identified with the dual space V^*: each element α of V^* can be viewed as a holomorphic section of $\mathcal{O}(1)$ by setting $\alpha(x) = \alpha_{|x}$. From the above discussion, we readily infer $\mathcal{L}_{X^A}\alpha = \alpha \circ A$. In particular, the infinitesimal weight of X^A is the trace of A, *i.e.* $\sum_{i=0}^m a_i$.

It is a far reaching observation by S. Donaldson [**15**] that $\mathcal{F}_\Omega(-JX)$ can be computed by using the asymptotic expansions of the infinitesimal weights, $w_k(X)$, of the lifted S^1-action on L^k, when k tends to infinity. More precisely, denote by d_k the (complex) dimension of $H^0(M, L^k)$; then

$$\frac{w_k(X)}{k\, d_k} = \frac{\int_M f^X\, v_g}{V_\Omega} + \frac{1}{4}\frac{\mathcal{F}_\Omega(-JX)}{V_\Omega}\, k^{-1} + O(k^{-2}), \tag{3.4}$$

where f^X denotes the distinguished momentum of X determined by the chosen lifted S^1-action on L and V_Ω the volume of (M, Ω).

If Y is the generator of another hamiltonian S^1-action on M, preserving the whole Kähler structure, the *combined infinitesimal weight* $w(X, Y)$ on L is defined as the trace of the product operator $(-i\mathcal{L}_X) \circ (-i\mathcal{L}_Y)$ on $H^0(M, L)$. Denote by $w_k(X, Y)$ the combined infinitesimal weight on $H^0(M, L^k)$. We then have

$$\frac{w_k(X,Y)}{k^2 d_k} - \frac{w_k(X)}{k d_k}\,\frac{w_k(Y)}{k d_k} = \frac{B_\Omega(-JX, -JY)}{V_\Omega} + O(k^{-1}). \tag{3.5}$$

The key point is that formulae (3.4)-(3.5) can be used to *define* $\mathcal{F}_\Omega(-JX)$ and $B_\Omega(-JX, -JY)$ in the case when M is singular and these objects cannot be defined directly in geometric terms. Such situations occur in particular when considering *test configurations* introduced by G. Tian [**41**] and S. Donaldson [**15**] to check the stability of polarized projective manifolds.

3.2. Deformation to the normal cone. — In general, for any closed subscheme Σ of a complex variety M, the *deformation to the normal cone* of Σ in M is a classical construction in algebraic geometry, by which the embedding of Σ in M is connected to its embedding in its normal cone $C = C_\Sigma M$ as the zero section.

This is done by considering the blow-up — call it $\mathcal{D}(M)$ — of the product $M \times \mathbb{P}^1$ along $\Sigma \times (1:0)$, where $(1:0)$ is the *point at infinity* of the standard complex projective line $\mathbb{P}$, and the induced projection $p : \mathcal{D}(M) \to \mathbb{P}^1$. Denote by $q : \mathcal{D}(M) \to M \times \mathbb{P}^1$ the blow-down mapping: the exceptional divisor $q^{-1}(\Sigma \times (1:0))$ is then the projectivized normal cone $\mathbb{P}(C \oplus 1)$ of $\Sigma \times (1:0)$ in $M \times \mathbb{P}^1$. For each $(\lambda : \mu) \neq (1:0)$ in $\mathbb{P}^1$, the fiber $p^{-1}((\lambda : \mu))$ is naturally identified with M, whereas the *central fiber* $p^{-1}((1:0))$ splits into two pieces:

(i) the exceptional divisor $\mathbb{P}(1 \oplus C)$, and

(ii) the blow-up $\hat{M}$ of M along Σ.

Notice that the two pieces $\hat{M}$ and $\mathbb{P}(1 \oplus C)$ of the central fiber intersect at the divisor at infinity $\mathbb{P}(C)$ in $\mathbb{P}(1 \oplus C)$, which is also the exceptional divisor of the blow-up of M along Σ.

Since the blow-up of $\Sigma \times \mathbb{P}^1$ along $\Sigma \times ((1:0))$ is $\Sigma \times \mathbb{P}^1$ again, $\Sigma \times \mathbb{P}^1$ is naturally embedded over $\mathbb{P}^1$ in $\mathcal{D}(M)$: For any $(\lambda : \mu) \neq (1:0)$ in $\mathbb{P}^1$, the induced embedding $\Sigma \hookrightarrow p^{-1}((\lambda : \mu)) \cong M$ is isomorphic to the initial embedding $\Sigma \hookrightarrow M$, whereas, over $(1:0)$, Σ is embedded in $p^{-1}(1:0)) = \mathbb{P}(1 \oplus C) \cup \hat{M}$ as the zero section in the normal cone $C \subset \mathbb{P}(1 \oplus C)$ (cf. [**16**, Chapter 5] for details).

In this paper, we consider this construction in the case when $M = \mathbb{P}(1 \oplus L)$ is an admissible ruled manifold and $\Sigma = \Sigma_\infty$ is the infinity section[9]. Since Σ_∞ is smooth, its normal cone $\mathcal{C}$ is simply the normal bundle $TM_{|\Sigma_\infty}/T\Sigma_\infty \cong (\pi^* L^*)_{|\Sigma_\infty}$. With the above notation, the central fiber $p^{-1}((1:0))$ is the union of

(i) $\hat{M}$, identified with M, as Σ_∞ is a divisor of M, and

(ii) the exceptional divisor $\mathbb{P}(C \oplus 1)$, identified with $\mathbb{P}(L^* \oplus 1)$ via the natural identification $\Sigma_\infty = S$.

Via the natural isomorphism $\mathbb{P}(L^* \oplus 1) = \mathbb{P}(1 \oplus L)$ obtained by tensoring $L^* \oplus 1$ by L, $\mathbb{P}(C \oplus 1)$ is naturally identified with M again and its intersection with $\hat{M} = M$ in $\mathcal{D}(M)$ is then the zero section Σ_0.

As observed in Remark 1.1 of Section 1.2, Σ_∞ is the zero divisor of the holomorphic section of $\mathcal{O}_M(1)$, s say, determined by the natural projection of $1 \oplus L$ to the trivial bundle $1 = S \times \mathbb{C}$. This allows for the following alternative description of $\mathcal{D}(M)$, which is a particular case of the general *MacPherson's graph construction* [**33**]. Let $\mathbb{P}(1 \oplus \mathcal{O}_M(-1))$ denote the natural compactification of $\mathcal{O}_M(-1)$ over M and consider the embedding $M \times (\mathbb{P}^1 \setminus (1:0)) \hookrightarrow \mathbb{P}(1 \oplus \mathcal{O}_M(-1)) \times \mathbb{P}^1$ defined by

$$\big(\xi = (z:u), (\lambda:\mu)\big) \to \big((\lambda z : \mu\,(z,u)), (\lambda:\mu)\big) \in \mathbb{P}(\mathbb{C} \oplus \xi) \times \mathbb{P}^1, \tag{3.6}$$

[9]The choice of Σ_∞ instead of the zero section Σ_0 is inessential.

for any $\xi = (z : u) \in \mathbb{P}(\mathbb{C} \oplus L_y)$ in M — cf. Section 1.1 for the notation — and for any $(\lambda : \mu) \neq (1 : 0)$ in $\mathbb{P}^1$. In (3.6), λz has to be regarded as $\lambda\, s(\xi)((z, u))$. Then, $\mathcal{D}(M)$ is alternatively defined as the closure of the image of $M \times (\mathbb{P}^1 \setminus (1 : 0))$ in $\mathbb{P}(1 \oplus \mathcal{O}_M(-1)) \times \mathbb{P}^1$ by the embedding (3.6), hence as the (closed) complex submanifold of $\mathbb{P}(1 \oplus \mathcal{O}_M(-1)) \times \mathbb{P}^1$ whose elements are of the form $((\alpha : (\beta, u)), (\lambda : \mu))$, for any pair (α, β) of complex numbers such that $\lambda\beta - \mu\alpha = 0$, cf. Example 5.1.2 and Example 18.1.6 (d) in [**16**].

We denote by $\tilde{\pi} : \mathcal{D}(M) \to S$ the natural projection induced by $\pi : M \to S$; for any y in S, we set $\mathcal{D}(M)_y = \tilde{\pi}^{-1}(y)$.

In order to get a more concrete grasp on $\mathcal{D}(M)_y$, we write $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2)$, where $\mathbb{C}_1$ and $\mathbb{C}_2$ stand for two copies of $\mathbb{C}$, we rewrite $M_y = \pi^{-1}(y) = \mathbb{P}(\mathbb{C}_2 \oplus L_y)$ and we introduce the complex projective plane $\mathbb{P}^2_y = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2 \oplus L_y)$: $\mathcal{D}(M)_y$ can then be viewed as a (compact) complex submanifold of the product $M_y \times \mathbb{P}^1 \times \mathbb{P}^2_y$, namely the space of $\big((z : u), (\lambda : \mu), (\alpha : \beta : v)\big)$ in $M_y \times \mathbb{P}^1 \times \mathbb{P}^2_y$ such that (α, β) belongs to the complex line $(\lambda : \mu)$ (in $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2)$) and (β, v) belongs to the complex line $(z : u)$ (in $M_y = \mathbb{P}(\mathbb{C}_2 \oplus L_y)$), that is to say the 2-dimensional (compact, smooth) complex submanifold of $M_y \times \mathbb{P}^1 \times \mathbb{P}^2_y$ defined by the equations:

$$\mu\alpha - \lambda\beta = 0, \qquad zv - \beta u = 0. \tag{3.7}$$

For any y in S, denote by $p_{1,y} : \mathcal{D}(M)_y \to M_y$, $p_{2,y} : \mathcal{D}(M)_y \to \mathbb{P}^1$, $p_{3,y} : \mathcal{D}(M)_y \to \mathbb{P}^2_y$ the induced projections and by $C_{1,y}, C_{2,y}, C_{3,y}$ the (complex) curves in $\mathcal{D}(M)_y$ defined by

$$C_{1,y} = \{\big((z : u), (1 : 0), (1 : 0 : 0)\big), \quad (z : u) \in M_y = \mathbb{P}(\mathbb{C}_2 \oplus L_y)\}, \tag{3.8}$$

$$C_{2,y} = \{\big((0 : u), (\lambda : \mu), (0 : 0 : u)\big), \quad (\lambda : \mu) \in \mathbb{P}^1 = \mathbb{P}(\mathbb{C}_1 \oplus \mathbb{C}_2)\}, \tag{3.9}$$

$$C_{3,y} = \{\big((0 : u), (1 : 0), (\alpha : 0 : v)\big), \quad (\alpha : v) \in \mathbb{P}(\mathbb{C}_1 \oplus L_y)\}. \tag{3.10}$$

The curves $C_{1,y}$ and $C_{2,y}$ are tautologically identified with M_y and $\mathbb{P}^1$ respectively, whereas $C_{3,y}$ will be identified with M_y via the the natural identification $\mathbb{C}_1 = \mathbb{C}_2$, i.e. via the map $(\alpha : v) \in \mathbb{P}(\mathbb{C}_1 \oplus L_y) = \mathbb{P}(\mathbb{C}_2 \oplus L_y) \mapsto \big((0 : u), (1 : 0), (\alpha : 0 : v)\big)$. The curves $C_{1,y}$ and $C_{2,y}$ are disjoint; the intersection $C_{1,y} \cap C_{3,y}$ is $\sigma_\infty(y)$ in $C_{1,y} = M_y$ and $\sigma_0(y)$ in $C_{3,y} = M_y$; the intersection $C_{2,y} \cap C_{3,y}$ is $(1 : 0)$ in $C_{2,y} = \mathbb{P}^1$ and $\sigma_\infty(y)$ in $C_{3,y} = M_y$.

Each fiber $\mathcal{D}(M)_y$ of $\tilde{\pi} : \mathcal{D}(M) \to S$ is a blow-up of $\mathbb{P}^2_y$ at two points, via the map $p_{3,y}$, which contracts the curves $C_{1,y}$ and $C_{2,y}$ to the points $[\mathbb{C}_1]$ and $[L_y]$ of $\mathbb{P}^2_y$ respectively, *and* a blow-up of $M_y \times \mathbb{P}^1$ at one point, via the map $(p_{1,y}, p_{2,y})$, which contracts the curve $C_{3,y}$ to the point $([L_y] = \sigma_\infty(y), (1 : 0))$ of $M_y \times \mathbb{P}^1$.

Denote by $q : \mathcal{D}(M) \to M \times \mathbb{P}^1$, resp. $p : \mathcal{D}(M) \to \mathbb{P}^1$, the map whose restriction to each $\mathcal{D}(M)_y$ is $(p_{1,y}, p_{2,y})$, resp. $p_{2,y}$. Then, q realizes $\mathcal{D}(M)$ as a blow-up of $M \times \mathbb{P}^1$ along $\Sigma_\infty \times (1:0)$, hence as the deformation to the normal cone of Σ_∞, according to the general construction described at the beginning of this section, and p is the induced projection on $\mathbb{P}^1$. Accordingly, each $(p_{1,y}, p_{2,y})$, resp. $p_{2,y}$, will be renamed q_y, resp. p_y.

For any y in S and for any $(\lambda : \mu) \neq (1:0)$, $p_y^{-1}((\lambda:\mu))$ is isomorphic to M_y, via the embedding $M_y \hookrightarrow \mathcal{D}(M)_y$ defined by:

$$(z:u) \mapsto \big((z:u), (\lambda:\mu), (\lambda z : \mu z : \mu\, u)\big). \tag{3.11}$$

This family of embeddings parametrized by $\mathbb{P}^1 \setminus (1:0)$ can be viewed as a unique embedding of $M \times (\mathbb{P}^1 \setminus (1:0))$ in $\mathcal{D}(M)_y$. The restriction of this embedding to $\sigma_\infty(y) \times (\mathbb{P}^1 \setminus (1:0))$ then extends to an embedding of $\sigma_\infty(y) \times \mathbb{P}^1$ in $\mathcal{D}(M)_y)$, given by

$$((0:u), (\lambda:\mu)) \mapsto \big((0:u), (\lambda:\mu), (0:0:u)\big), \tag{3.12}$$

whose image is $C_{2,y}$.

The central fiber $p_y^{-1}((1:0))$ is $C_{1,y} \cup C_{3,y}$ over each y in S. By setting $C_1 = \cup_{y \in S} C_{1,y}$, $C_2 = \cup_{y \in S} C_{2,y}$ and $C_3 = \cup_{y\in S} C_{3,y}$, we then get

$$p^{-1}((1:0)) = C_1 \cup C_3, \tag{3.13}$$

where C_1 and C_3 are both identified with M as explained above. The intersection $C_1 \cap C_3$ is then identified with Σ_0 in $C_1 \cong M$ and with Σ_∞ in $C_3 \cong M$.

3.3. The space $\mathcal{D}(M)$ as a test configuration: Polarizations. — For any y in S, denote by $\Lambda_{1,y}, \Lambda_{2,y}, \Lambda_{3,y}$ the holomorphic line bundles on $\mathcal{D}(M)_y$ defined by $p_{1,y}^*(\mathcal{O}_{M_y}(1))$, $p_{2,y}^*(\mathcal{O}_{\mathbb{P}^1}(1))$, $p_{3,y}^*(\mathcal{O}_{\mathbb{P}^2_y}(1))$ respectively. Each $\Lambda_{1,y}, \Lambda_{2,y}, \Lambda_{3,y}$ admits a distinguished holomorphic section whose zero divisor is $C_{2,y} + C_{3,y}$, $C_{1,y} + C_{3,y}$, $C_{1,y} + C_{2,y} + C_{3,y}$ respectively. If $C_{1,y}, C_{2,y}, C_{3,y}$ are regarded a elements of $H^2(\mathcal{D}(M)_y, \mathbb{Z})$, by Poincaré duality, we then have

$$\begin{aligned} C_{1,y} &= c_1(\Lambda_{1,y}^{-1} \otimes \Lambda_{3,y}),\\ C_{2,y} &= c_1(\Lambda_{2,y}^{-1} \otimes \Lambda_{3,y}),\\ C_{3,y} &= c_1(\Lambda_{1,y} \otimes \Lambda_{2,y} \otimes \Lambda_{3,y}^{-1}), \end{aligned} \tag{3.14}$$

where $c_1(\cdot)$ stands for the (first) Chern class.

We now choose an admissible polarization on M, i.e. an admissible Kähler class $\Omega_{\boldsymbol{\lambda}}$ on M in the image of $H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{R})$. By Remark 1.1, this means that the λ_i's are integers and that $\Omega/2\pi = c_1(\mathcal{F}_{\boldsymbol{\lambda}})$, where $\mathcal{F}_{\boldsymbol{\lambda}}$ is given by (1.11).

In order to turn $\mathcal{D}(M)$ into a test configuration compatible with this polarization, we need a hermitian holomorphic line bundle, $\mathcal{L}$, on $\mathcal{D}(M)$, whose restriction to $p^{-1}((\lambda:\mu))$ is the chosen polarization of $M = p^{-1}((\lambda:\mu))$ if $(\lambda:\mu) \neq (1:0)$ and which induces, in some sense, a polarization on the central fiber $p^{-1}((1:0))$ (however, $\mathcal{L}$ is not required to be a polarization on the whole space $\mathcal{D}(M)$).

For each $\mathcal{D}(M)_y$, this will be done by twisting the pull-back of $(\mathcal{F}_{\boldsymbol{\lambda}})_{|M_y}$ on $\mathcal{D}(M)_y$ by an appropriate multiple $-aC_{3,y}$ of the exceptional divisor, i.e. by tensoring the pull-back of $(\mathcal{F}_{\boldsymbol{\lambda}})_{|M_y}$ by $\Lambda_{1,y}^{-a} \otimes \Lambda_{2,y}^{-a} \otimes \Lambda_{3,y}^{a}$ for some positive rational number a (strictly speaking, a should be chosen an integer but, for our purposes, it will be sufficient that ka be an integer for k a positive integer growing to infinity). By using (1.11), we thus get:

$$\mathcal{L}_{|\mathcal{D}(M)_y} = \Lambda_{1,y}^{2-a} \otimes \Lambda_{2,y}^{-a} \otimes \Lambda_{3,y}^{a} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})_y. \tag{3.15}$$

We now show that the restriction of $\mathcal{L}$ to each fiber $p^{-1}((\lambda:\mu))$, is ample whenever $0 < a < 2$.

We first consider the case when $(\lambda:\mu) \neq (1;0)$. From (3.11) we infer that the restriction of $\Lambda_{3,y}$ to $p_y^{-1}((\lambda:\mu))$ is naturally identified with the restriction of $\Lambda_{1,y} \otimes \Lambda_{2,y}$, so that:

$$\mathcal{L}_{|p_y^{-1}((\lambda:\mu))} = \Lambda_{1,y}^{2} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})_y = (\mathcal{F}_{\boldsymbol{\lambda}})_{|M_y}, \tag{3.16}$$

for any a.

We now consider the central fiber $p^{-1}((1:0))$, which is $C_{1,y} \cup C_{3,y}$ in each $\mathcal{D}(M)_y$. On $C_{1,y}$, we have $\Lambda_{2,y} = \Lambda_{3,y} = \mathbb{C}_1^*$, so that:

$$\begin{aligned} \mathcal{L}_{|C_{1,y}} &= \Lambda_{1,y}^{2-a} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})_y \\ &= (\mathcal{F}_{\boldsymbol{\lambda}}^{(1-\frac{a}{2})})_{|M_y} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})_y^{\frac{a}{2}}, \end{aligned} \tag{3.17}$$

whereas, on $C_{3,y}$, we have $\Lambda_{1,y} = L_y^*$, $\Lambda_{2,y} = \mathbb{C}_1^*$, $\Lambda_{3,y} = \Lambda_{1,y}$, so that:

$$\begin{aligned} \mathcal{L}_{|C_{3,y}} &= \Lambda_{1,y}^{a} \otimes \mathbb{C}_1^{a} \otimes L_y^{a-2} \otimes (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})_y \\ &= (\mathcal{F}_{\boldsymbol{\lambda}}^{\frac{a}{2}})_{|M_y} \otimes (\bigotimes_{i=1}^{N} L_i^{-1-\epsilon_i\lambda_i})_y^{1-\frac{a}{2}}. \end{aligned} \tag{3.18}$$

By setting: $\Omega^{(0)} = 2\pi c_1(\mathcal{L}_{|C_1})$ and $\Omega^{(\infty)} = 2\pi c_1(\mathcal{L}_{|C_3})$, both regarded as defined on M, we thus get

$$\Omega^{(0)} = (1 - a/2)\big(\Xi + \sum_{i=1}^{N} \frac{\lambda_i - a/2\,\epsilon_i}{1 - a/2}\,\pi^*[\omega_{S_i}]\big), \tag{3.19}$$

and

$$\Omega^{(\infty)} = a/2\,\big(\Xi + \sum_{i=1}^{N} \frac{\lambda_i + (1 - a/2)\,\epsilon_i}{a/2}\,\pi^*[\omega_{S_i}]\big). \tag{3.20}$$

These evidently belong to the (admissible) Kähler cone of M if and only if $0 < a < 2$. Moreover, via the common identification $\Sigma_0 = \Sigma_\infty = S$, the restriction of $\Omega^{(0)}$ to Σ_∞ coincides with the restriction of $\Omega^{(\infty)}$ to Σ_0, as it must be. More precisely, by setting

$$a = 1 - x, \tag{3.21}$$

we have

$$\Omega^{(0)}_{|\Sigma_\infty} = \Omega^{(\infty)}_{|\Sigma_0} = \sum_{i=1}^{N} (\lambda_i + x\epsilon_i)\,[\omega_{S_i}], \tag{3.22}$$

which is the class of the Kähler form of the Kähler reduction of M, equipped with the admissible Kähler metric (1.12) in $\Omega_{\boldsymbol{\lambda}}$, for the level set $z = x$. We infer that the pair $(\Omega^{(0)}, \Omega^{(\infty)})$ determines a well-defined "polarization" on the (singular) central fiber $p^{-1}((1:0))$. This polarization depends on the parameter x in $(-1, 1)$ and will be therefore denoted by $\tilde{\Omega}^{(x)}$.

3.4. The space $\mathcal{D}(M)$ as a test configuration: $\mathbb{C}^*$-actions. — The $\mathbb{C}^*$-action on $\mathbb{P}^1$ defined by $\zeta \cdot (\lambda : \mu) = (\zeta^{-1}\lambda : \mu)$ determines a $\mathbb{C}^*$-action, denoted by $\boldsymbol{\alpha}$, on $\mathcal{D}(M)$, defined by:

$$\zeta \cdot_{\boldsymbol{\alpha}} \big((z:u), (\lambda:\mu), (\alpha:\beta:v)\big) = \big((z:u), (\zeta^{-1}\lambda:\mu), (\zeta^{-1}\,\alpha:\beta:v)\big). \tag{3.23}$$

This action moves the fibers of p. It fixes the fiber $p^{-1}((0:1))$ (this is smooth, identified with M, and plays no particular role in the story), and the central fiber $p^{-1}((1:0)) = C_1 \cup C_3$: the action $\boldsymbol{\alpha}$ is then trivial on C_1 and coincides with the natural $\mathbb{C}^*$-action on $C_3 = M$.

The natural $\mathbb{C}^*$-action on $M = \mathbb{P}(1 \oplus L)$ — cf. Section 1.1 — induces an $\mathbb{C}^*$-action on $\mathcal{D}(M)$, denoted by $\boldsymbol{\beta}$, defined by

$$\zeta \cdot_{\boldsymbol{\beta}} \big((z:u), (\lambda:\mu), (\alpha:\beta:v)\big) = \big((z:\zeta u), (\lambda:\mu), (\alpha:\beta:\zeta v)\big), \tag{3.24}$$

for ζ in $\mathbb{C}^*$. This action preserves the fibers of p and coincides with the natural $\mathbb{C}^*$-action on each fiber $p^{-1}((\lambda:\mu))$, $(\lambda:\mu) \neq (1:0)$, via the embedding (3.11). On the central fiber $p^{-1}((1:0) = C_1 \cup C_3$, where C_1 and C_3 are both identified with M as explained above, the action $\boldsymbol{\beta}$ coincides with the natural $\mathbb{C}^*$-action on M.

Notice that these actions preserve each fiber of $\tilde{\pi} : \mathcal{D}(M) \to S$ and are therefore entirely determined by their induced actions on $\mathcal{D}(M)_y$ for each y in S. Moreover, on each $\mathcal{D}(M)_y$, both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have natural lifts on the line bundles $\Lambda_{1,y}, \Lambda_{2,y}, \Lambda_{3,y}$. This determines an $\boldsymbol{\alpha}$- and a $\boldsymbol{\beta}$-action on $\mathcal{L}$ as well as on the vector space of its holomorphic sections.

For each fiber $p^{-1}((\lambda : \mu))$, and each positive integer k, the space of holomorphic sections of $\mathcal{L}^k_{|p^{-1}((\lambda:\mu))}$, coincides with the space of holomorphic sections of the holomorphic vector bundle, $\mathbb{E}^{k,(\lambda:\mu)}$, on S whose fiber $\mathbb{E}^{k,(\lambda:\mu)}_y$ at y is the space of holomorphic sections of $\mathcal{L}^k_{|p_y^{-1}((\lambda:\mu))}$.

If $(\lambda;\mu) \neq (1:0)$, we infer from (3.16):

$$\begin{aligned}\mathbb{E}^{k,(\lambda:\mu)}_y &= S_{2k}((\mathbb{C}_2 \oplus L_y)^*) \otimes (\bigotimes_{i=1}^{M} L_i^{1-\epsilon_i\lambda_i})^k_y \\ &= \sum_{j=0}^{2k} L_y^{-j} \otimes (\bigotimes_{i=1}^{M} L_i^{1-\epsilon_i\lambda_i})^k_y,\end{aligned} \tag{3.25}$$

where, in general, $S_\ell(V)$ denotes the ℓ-th symmetric tensor power of V. We thus have

$$H^0(p^{-1}((\lambda:\mu)), \mathcal{L}^k_{|p^{-1}((\lambda:\mu))}) = \sum_{j=0}^{2k} H^0(S, (\bigotimes_{i=1}^{M} L_i^{1-\epsilon_i\lambda_i})^k \otimes L^{-j}). \tag{3.26}$$

On the central fiber $p^{-1}((1:0))$, $\mathbb{E}^{k,(1:0)}_y$ is obtained by considering the direct sum of the spaces of holomorphic sections of $\mathcal{L}^k$ on C_1 and C_3 separately, then removing the common part on $C_1 \cap C_3$. From (3.17), we infer

$$\begin{aligned}H^0(C_{1,y}, \mathcal{L}^k_{|C_{1,y}}) &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes S_{k(2-a)}((\mathbb{C}_2 \oplus L_y)^*) \\ &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \sum_{j=0}^{k(2-a)} L_y^{-j}.\end{aligned} \tag{3.27}$$

Moreover, the infinitesimal weight of $\boldsymbol{\alpha}$, as defined in Definition 3.1, is 0 on this space, whereas the infinitesimal weight of $\boldsymbol{\beta}$ is j on each factor $(\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes L^{-j}$ (for this computation and similar ones in the sequel, compare with Example 3.1 in Section 3.1).

From (3.18), we infer

$$
\begin{aligned}
H^0(C_{3,y},\mathcal{L}^k_{|C_{1,y}}) &= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \mathbb{C}_1^{ka}\otimes L_y^{-k(2-a)}\otimes S_{ka}((\mathbb{C}_1\oplus L_y)^*)\\
&= (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \sum_{j=k(2-a)}^{2k} \mathbb{C}_1^{j-k(2-a)}\otimes L_y^{-j}.
\end{aligned}
\tag{3.28}
$$

Moreover, on each factor $(\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \mathbb{C}_1^{j-k(2-a)}\otimes L_y^{-j}$, the infinitesimal weight of $\boldsymbol{\alpha}$ is $j-k(2-a)$, whereas the infinitesimal weight of $\boldsymbol{\beta}$ is j.

Finally, $H^0(C_{1,y}\cap C_{3,y},\mathcal{L}^k) = (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes L_y^{-k(2-a)}$, which appears in both expressions with weight 0 for $\boldsymbol{\alpha}$.

By removing this term from the rhs of (3.27) or (3.28), and by removing the factors $\mathbb{C}_1^{j-k(2-a)}$ appearing in the rhs of (3.28) — but keeping them in mind for weight issues — we eventually get

$$
\mathbb{E}_y^{k,(1:0)} = (\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k_y \otimes \sum_{j=0}^{2k} L_y^{-j},
\tag{3.29}
$$

hence

$$
H^0(p^{-1}(1:0),\mathcal{L}^k_{|p^{-1}((1:0))}) = \sum_{j=0}^{2k} H^0(S,(\bigotimes_{i=1}^{N} L_i^{1-\epsilon_i\lambda_i})^k\otimes L^{-j}).
\tag{3.30}
$$

It is convenient to rewrite (3.30) as follows

$$
H^0(p^{-1}((1:0)),\mathcal{L}^k_{|p^{-1}((1:0))}) = \sum_{\ell=-k}^{k} H^0(S,(\bigotimes_{i=1}^{N} \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k),
\tag{3.31}
$$

where each $\tilde{L}_i = L_i^{-\epsilon_i}$ is ample and polarizes (S_i,ω_{S_i}) — cf. Section 1.1 — and where we changed the index by setting

$$
\ell = j-k.
\tag{3.32}
$$

Moreover, the infinitesimal weight of $\boldsymbol{\alpha}$ on $H^0(S,(\bigotimes_{i=1}^{N} \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ is

$$
\begin{aligned}
&0 \qquad \text{if } \ell\le k(1-a)=kx\\
&\ell-kx \qquad \text{if } kx\le \ell\le k,
\end{aligned}
\tag{3.33}
$$

whereas the infinitesimal weight of $\boldsymbol{\beta}$ on $H^0(S,(\bigotimes_{i=1}^{N} \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ is

$$
k+\ell,\qquad -k\le \ell\le k.
\tag{3.34}
$$

3.5. The relative Futaki invariant of $\mathcal{D}(M)$. — For any x in $(-1,1)\cap\mathbb{Q}$, the *Futaki invariant* of the $\mathbb{C}^*$-action $\boldsymbol{\alpha}$ on the central fiber $p^{-1}((1:0))$ with respect to the polarization $\tilde{\Omega}^{(x)}_{\boldsymbol{\lambda}}$ is defined by $\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = \mathcal{F}_{\tilde{\Omega}^{(x)}_{\boldsymbol{\lambda}}}(-JX)$, where X denotes the generator of the S^1-action induced by $\boldsymbol{\alpha}$. We similarly define: $\mathcal{F}^{(x)}(\boldsymbol{\beta}) = \mathcal{F}_{\tilde{\Omega}^{(x)}_{\boldsymbol{\lambda}}}(-JY)$, where Y denotes the generator of the S^1-action induced by $\boldsymbol{\beta}$, $B^{(x)}(\boldsymbol{\alpha},\boldsymbol{\beta}) = B_{\tilde{\Omega}^{(x)}_{\boldsymbol{\lambda}}}(-JX,-JY)$ and $B(\boldsymbol{\beta},\boldsymbol{\beta}) = B_{\tilde{\Omega}^{(x)}_{\boldsymbol{\lambda}}}(-JY,-JY)$ (as we shall see below, $\mathcal{F}(\boldsymbol{\beta})$ and $B(\boldsymbol{\beta},\boldsymbol{\beta})$ are independent of x). The *relative Futaki invariant* of $\boldsymbol{\alpha}$ with respect to $\boldsymbol{\beta}$, in the sense of (2.9), is then

$$\mathcal{F}^{(x)}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \mathcal{F}^{(x)}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) - \frac{B^{(x)}(\boldsymbol{\alpha},\boldsymbol{\beta})}{B(\boldsymbol{\beta},\boldsymbol{\beta})}\,\mathcal{F}(\boldsymbol{\beta}). \tag{3.35}$$

The aim of this section is to provide a self-contained computation of $\mathcal{F}^{(x)}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$ by using (3.4)-(3.5) and to prove the following theorem, first established by G. Székelyhidi in [**39**] in the case of *pseudo-Hirzebruch surfaces*, then extended to the general case in [**3**, Section 4.4]:

Theorem 3.1. — *For any x in* (-1, 1)*, we have*

$$\mathcal{F}^{(x)}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = -2\pi V(S)\,\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\int_{-1}^{1} p_{\Omega_{\boldsymbol{\lambda}}}(s)ds}, \tag{3.36}$$

where $V(S) = \prod_{i=1}^{N} V(S_i, g_{S_i})$ denotes the volume of S and, we recall, $p_{\Omega_{\boldsymbol{\lambda}}}$ and $F_{\Omega_{\boldsymbol{\lambda}}}$ denote the characteristic and the extremal polynomial of $\Omega_{\boldsymbol{\lambda}}$ respectively.

Proof. — Denote by $d_k(\ell)$ the (complex) dimension of $H^0(S, (\bigotimes_{i=1}^{N} \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ and by d_k the dimension of $H^0(p^{-1}((1:0)), \mathcal{L}^k_{|p^{-1}((1:0))})$; by (3.31), we then have

$$d_k = \sum_{\ell=-k}^{k} d_k(\ell). \tag{3.37}$$

We denote by $w_k(\boldsymbol{\alpha})$, resp. $w_k(\boldsymbol{\beta})$, the infinitesimal weight of $\boldsymbol{\alpha}$, resp. $\boldsymbol{\beta}$, and by $w_k(\boldsymbol{\alpha},\boldsymbol{\beta})$, resp. $w_k(\boldsymbol{\beta},\boldsymbol{\beta})$, the combined infinitesimal weight — as defined in Section 3.1 — of $\boldsymbol{\alpha},\boldsymbol{\beta}$, resp. of $\boldsymbol{\beta},\boldsymbol{\beta}$, on the space $H^0(p^{-1}((1:0)), \mathcal{L}^k_{|p^{-1}((1:0))})$. From (3.33)-(3.34), we readily infer:

$$w_k(\boldsymbol{\alpha}) = \sum_{\ell=kx}^{k} (\ell - kx)\, d_k(\ell), \quad w_k(\boldsymbol{\beta}) = \sum_{\ell=-k}^{k} (\ell + k)\, d_k(\ell), \tag{3.38}$$

$$w_k(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\ell=kx}^{k} (\ell+k)(\ell-kx)\, d_k(\ell), \quad w_k(\boldsymbol{\beta},\boldsymbol{\beta}) = \sum_{\ell=-k}^{k} (\ell+k)^2\, d_k(\ell). \tag{3.39}$$

Lemma 3.1. — *When k tends to infinity, $d_k(\ell)$ has the asymptotic expansion*

$$\begin{aligned} d_k(\ell) = \frac{V(S)}{(2\pi)^d} \big(k^d p_{\Omega_{\boldsymbol{\lambda}}}(\ell/k) \\ + \frac{k^{d-1}}{4}\big(R(\ell/k) + p_{\Omega_{\boldsymbol{\lambda}}}(\ell/k)(\alpha\,\ell/k+\beta)\big) + O(k^{d-2}\big) \end{aligned} \tag{3.40}$$

where, we recall, $p_{\Omega_{\boldsymbol{\lambda}}}$ denotes the characteristic polynomial of $\Omega_{\boldsymbol{\lambda}}$, defined by (1.7)*; R is the polynomial defined in* (1.42)*; α, β are the normalized leading coefficients of the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$, i.e. the constant appearing in the rhs of* (1.42).

Proof. — Since $\tilde{L}_i$ is ample on S_i, and $0 < \lambda_i - 1 \le \lambda_i + \ell/k\,\epsilon_i \le \lambda_i + 1$ for each $-k \le \ell \le k$, for k large enough $d_k(\ell)$ is equal to $\chi((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$, the *holomorphic Euler characteristic* of $(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k$. By the Riemann-Roch theorem, we have that

$$\chi((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k) = \int_S \mathrm{ch}((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)\,\mathrm{td}(S), \tag{3.41}$$

where $\mathrm{ch}((\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k)$ denotes the *Chern character* of the complex line bundle $(\bigotimes_{i=1}^N \tilde{L}_i^{\lambda_i+\ell/k\,\epsilon_i})^k$ and $\mathrm{td}(S)$ the *Todd class* of the holomorphic tangent bundle of S. Recall that the Chern character of any complex line bundle $\mathcal{L}$ is defined by $\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{r=0}^{\infty} \frac{c_1(\mathcal{L})^r}{r!}$, whereas the Todd class is the multiplicative characteristic class associated to the generating series $x/(1-e^{-x})$; in particular $\mathrm{td}(S) = 1 + c_1(S)/2 + \cdots$, cf. e.g. [**24**]. We thus get:

$$\begin{aligned} d_k(\ell) &= \sum_{r=0}^{d} \frac{k^r}{(2\pi)^r} \int_S \frac{(\sum_{i=1}^N (\lambda_i + \ell/k\,\epsilon_i)\,[\omega_{S_i}])^r}{r!} (1 + c_1(S)/2 + \cdots) \\ &= \frac{k^d}{(2\pi)^d} \int_S \frac{(\sum_{i=1}^N (\lambda_i + \ell/k\,\epsilon_i)\,[\omega_{S_i}])^d}{d!} \\ &+ \frac{k^{d-1}}{(2\pi)^d} \int_S \frac{(\sum_{i=1}^N (\lambda_i + \ell/k\,\epsilon_i)\,[\omega_{S_i}])^{d-1}}{(d-1)!} \wedge \frac{c_1(S)}{2} + O(k^{d-2}) \\ &= \frac{V(S)}{(2\pi)^d} \big(k^d p_{\Omega_{\boldsymbol{\lambda}}}(\ell/k) + k^{d-1} p_{\Omega_{\boldsymbol{\lambda}}}(\ell/k) \sum_{i=1}^N \frac{s_i/4}{\lambda_i + \ell/k\,\epsilon_i} + O(k^{d-2})\big). \end{aligned} \tag{3.42}$$

We conclude by using (1.42). □

In order to evaluate the asymptotic expansions of the sums in (3.37), (3.38) etc. we use the following asymptotic formula, known as the *trapezium rule*:

$$\sum_{\ell=ak}^{bk} f(\ell/k) = k\int_a^b f(t)\,dt + \frac{1}{2}(f(a)+f(b)) + O(k^{-1}) \tag{3.43}$$

for any polynomial f, where $ak \le bk$ are integers, and ℓ runs over all integers between ka and kb.

For convenience, we assume, without loss of generality, that $V(S) = (2\pi)^d$ and we simply write $p(t)$ for $p_{\Omega_{\boldsymbol{\lambda}}}(t)$.

Corollary 3.1. — *When k tends to infinity, d_k has the asymptotic expansion*

$$d_k = k^{d+1}\int_{-1}^1 p(s)\,ds + \frac{k^d}{4}\int_{-1}^1 (\alpha s+\beta)\,p(s)ds + O(k^{d-1}). \tag{3.44}$$

Proof. — Direct consequence of Lemma 3.1 and of the trapezium rule (3.43). □

Corollary 3.2. — *When k tends to infinity, $w_k(\boldsymbol{\alpha})$ has the asymptotic expansion*

$$\begin{aligned} w_k(\boldsymbol{\alpha}) &= -k^{d+2}\int_x^1 (s-x)\,p(s)ds \\ &\quad - \frac{k^{d+1}}{4}\Big(F_{\Omega_{\boldsymbol{\lambda}}}(x) + \int_x^1 (s-x)(\alpha s+\beta)\,p(s)ds\Big) + O(k^d). \end{aligned} \tag{3.45}$$

In particular,

$$\begin{aligned} \frac{w_k(\boldsymbol{\alpha})}{kd_k} &= -\frac{\int_x^1 (s-x)\,p(s)ds}{\int_{-1}^1 p(s)\,ds} - \frac{1}{4}\,\frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\int_{-1}^1 p(s)ds}\,k^{-1} \\ &\quad - \frac{\alpha}{4}\,\frac{\int_x^1 s(s-x)\,p(s)ds\,\int_{-1}^1 p(s)ds - \int_x^1 (s-x)\,p(s)ds\,\int_{-1}^1 s\,p(s)ds}{\big(\int_{-1}^1 p(s)ds\big)^2}\,k^{-1} \\ &\quad + O(k^{-2}) \end{aligned} \tag{3.46}$$

Proof. — (3.45) is a direct consequence of Lemma 3.1 and of (3.43), by using the identity (1.43)-(1.44) and the expression (1.48) of the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$; (3.46) readily follows from (3.45) and (3.44). □

Corollary 3.3. — *When k tends to infinity, $w_k(\boldsymbol{\beta})$ has the asymptotic expansion*

$$w_k(\boldsymbol{\beta}) = -k^{d+2}\int_{-1}^1 (s+1)\,p(s)ds - \frac{k^{d+1}}{4}\int_{-1}^1 (\alpha s+\beta)(s+1)\,p(s)ds + O(k^d). \tag{3.47}$$

In particular,

$$\frac{w_k(\boldsymbol{\beta})}{k\,d_k} = \frac{\int_{-1}^{1}(1+s)\,p(s)ds}{\int_{-1}^{1}p(s)ds} + \frac{\alpha}{4}\,\frac{\left(\int_{-1}^{1}s^2\,p(s)ds\int_{-1}^{1}p(s)ds - \int_{-1}^{1}s\,p(s)ds\int_{-1}^{1}s\,p(s)ds\right)}{\left(\int_{-1}^{1}p(s)ds\right)^2}\,k^{-1} + O(k^{-2}). \tag{3.48}$$

Proof. — Direct consequence of Lemma 3.1 and of (3.43). □

Corollary 3.4. — *When k tends to infinity, $w_k(\boldsymbol{\alpha},\boldsymbol{\beta})$ has the asymptotic expansion*

$$w_k(\boldsymbol{\alpha},\boldsymbol{\beta}) = -k^{d+3}\int_x^1 (s-x)(s+1)\,p(s)ds + O(k^{d+2}). \tag{3.49}$$

In particular,

$$\frac{w_k(\boldsymbol{\alpha},\boldsymbol{\beta})}{k^2 d_k} - \frac{w_k(\boldsymbol{\alpha})}{kd_k}\,\frac{w_k(\boldsymbol{\beta})}{kd_k} = -\frac{\int_x^1 s\,(s-x)\,p(s)ds\ \int_{-1}^{1}p(s)ds - \int_x^1(s-x)\,p(s)ds\ \int_{-1}^{1}s\ p(s)ds}{\left(\int_{-1}^{1}p(s)ds\right)^2} + O(k^{-1}) \tag{3.50}$$

Proof. — Direct consequence of Lemma 3.1 and of (3.43). □

Corollary 3.5. — *When k tends to infinity, $w_k(\boldsymbol{\beta},\boldsymbol{\beta})$ has the following asymptotic expansion:*

$$w_k(\boldsymbol{\beta},\boldsymbol{\beta}) = k^{d+3}\int_{-1}^1 (s+1)^2\,p(s)ds + O(k^{d+2}). \tag{3.51}$$

In particular,

$$\frac{w_k(\boldsymbol{\beta},\boldsymbol{\beta})}{k^2 d_k} - \frac{w_k(\boldsymbol{\beta})}{kd_k}\,\frac{w_k(\boldsymbol{\beta})}{kd_k} = \frac{\int_{-1}^1 s^2\,p(s)ds\ \int_{-1}^{1}p(s)ds - \int_{-1}^1 t\,p(s)ds\ \int_{-1}^{1}s\ p(s)ds}{\left(\int_{-1}^{1}p(s)ds\right)^2} + O(k^{-1}) \tag{3.52}$$

Proof. — Direct consequence of Lemma 3.1 and of (3.43). □

By using (3.4)-(3.5) and $V_{\Omega_{\boldsymbol{\lambda}}} = 2\pi\, V(S) \int_{-1}^{1} p(s)ds$ (deduced from (1.27)), we obtain (by temporarily omitting the overall factor $2\pi V(S)/ \int_{-1}^{1} p(s)ds$)

(3.53)
$$\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = -F_{\Omega_{\boldsymbol{\lambda}}}(x) - \alpha \big(\int_x^1 s(s-x)\, p(s)ds \int_{-1}^1 p(s)ds - \int_x^1 (s-x)\, p(s)ds \int_{-1}^1 s\, p(s)ds \big),$$

$$\mathcal{F}(\boldsymbol{\beta}) = \alpha \big(\int_{-1}^1 s^2\, p(s)ds \int_{-1}^1 p(s)ds - \int_{-1}^1 s\, p(s)ds \int_{-1}^1 s\, p(s)ds \big), \tag{3.54}$$

(3.55)
$$B^{(x)}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = - \int_x^1 s\,(s-x)\, p(s)ds \int_{-1}^1 p(s)ds - \int_x^1 (s-x)\, p(s)ds \int_{-1}^1 s\ p(s)ds,$$

$$B(\boldsymbol{\beta}, \boldsymbol{\beta}) = \int_{-1}^1 s^2\, p(s)ds \int_{-1}^1 p(s)ds - \int_{-1}^1 s\, p(s)ds \int_{-1}^1 s\ p(s)ds. \tag{3.56}$$

Notice that $\mathcal{F}(\boldsymbol{\beta}) = \alpha\, B(\boldsymbol{\beta}, \boldsymbol{\beta})$ — cf. Remark 3.1 below — whereas $\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = -F^{(x)}_{\Omega_{\boldsymbol{\lambda}}}(x) + \alpha\, B^{(x)}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. By restoring the missing factor $2\pi V(S)/ \int_{-1}^{1} p(s)ds$, we get (3.36). □

Remark 3.1. — By comparing (3.54) and (3.56) with (2.31) and (2.32) in Corollary 2.2, we get:

$$\mathcal{F}(\boldsymbol{\beta}) = \mathcal{F}_{\Omega}(-JT), \quad B(\boldsymbol{\beta}, \boldsymbol{\beta}) = B_{\Omega}(-JT, -JT). \tag{3.57}$$

This was in fact quite expected as the $\boldsymbol{\beta}$ action is the same on any fiber $p^{-1}((\lambda : \mu))$ and coincides with the natural S^1-action on M.

Remark 3.2. — The extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ is of degree less than $m+2$ if and only the normalized leading coefficient α is zero. In this case, $\mathcal{F}(\boldsymbol{\beta}) = 0$, by (3.54), and, by (3.53), (3.36) then reduces to

$$\mathcal{F}^{(x)}(\boldsymbol{\alpha}) = -2\pi V(S) \frac{F_{\Omega_{\boldsymbol{\lambda}}}(x)}{\int_{-1}^1 p_{\Omega_{\boldsymbol{\lambda}}}(s)\, ds}. \tag{3.58}$$

Appendix A
The extremal polynomial for $N = 1$

We here compute the extremal polynomial $F_{\Omega_{\boldsymbol{\lambda}}}$ of any (admissible) Kähler class on an admissible ruled manifold $M : \mathbb{P}(1 \oplus L) \to S = \prod_{i=1}^N S_i$ in the case when $N = 1$. The Kähler class $\Omega_{\boldsymbol{\lambda}}$ is then determined by a unique real number $\lambda > 1$, the chosen (constant) scalar curvature s of $S = S_1$ and $\epsilon = \epsilon_1$

which, without loss of generality, will be chosen equal to 1, see Section 1.1. For convenience, we set

$$\kappa = \frac{s}{d(d+1)}, \tag{A.1}$$

where d denotes the complex dimension of S (we then have $\dim_{\mathbb{C}} M = d+1$ and $F_{\Omega_{\boldsymbol{\lambda}}}$ is of degree at most $d+3$) and we replace the variable x in $(-1,1)$ by

$$X := \lambda + x, \tag{A.2}$$

in the interval $(\lambda-1, \lambda+1)$ and we set $P(X) = F_{\Omega_\lambda}(x)$: $P = P(X)$ will be referred to as the *modified* extremal polynomial of $\Omega_{\boldsymbol{\lambda}}$; it will be occasionally denoted by $P_\kappa(X)$ or $P_\kappa(X,\lambda)$ to emphasize the dependence in κ and λ; it will be most often regarded as a polynomial in X with coefficients in the field $R(\lambda)$ of rational fractions in λ; in particular, except for poles, $P_\kappa(X,\lambda)$ is well-defined for any real (or complex) value of λ, not only for admissible $\lambda > 1$. In terms of the modified extremal polynomial $P(X)$, the boundary conditions (1.46)-(1.47) read as follows

$$\begin{aligned} &P(\lambda-1) = P(\lambda+1) = 0,\\ &P'(\lambda-1) = 2(\lambda-1)^d, \qquad P'(\lambda+1) = -2(\lambda+1)^2, \end{aligned} \tag{A.3}$$

whereas the second derivative of P has the form

$$P''(X) = -\alpha X^{d+1} + (\alpha\lambda - \beta)\, X^d + d(d+1)\kappa\, X^{d-1}, \tag{A.4}$$

where α, β are determined by (A.3), cf. Section 1.9. In particular, P is of the form

$$P_\kappa(X,\lambda) = a_0(\lambda) X^{d+3} + a_1(\lambda) X^{d+2} + \kappa\, X^{d+1} + a_3(\lambda) X + a_4(\lambda), \tag{A.5}$$

where a_0, a_1, a_3, a_4 are rational fractions in λ, which depend on κ in an affine way. For convenience, we introduce

$$S_k(\lambda) = (\lambda+1)^k + (\lambda-1)^k, \qquad A_k(\lambda) = (\lambda+1)^k - (\lambda-1)^k. \tag{A.6}$$

Then, a_0, a_1 are solutions of the linear system:

$$\begin{aligned} &(d+3)A_{d+2}(\lambda)\, a_0 + (d+2)A_{d+1}(\lambda)\, a_1 = -(d+1)A_d(\lambda)\,\kappa - 2S_d(\lambda),\\ &((d+3)S_{d+2}(\lambda) - A_{d+3})(\lambda)\, a_0 + ((d+2)S_{d+1}(\lambda) - A_{d+2})(\lambda)\, a_1\\ &\qquad\qquad = (A_{d+1}(\lambda) - (d+1)S_d(\lambda))\,\kappa - 2A_d(\lambda), \end{aligned} \tag{A.7}$$

whereas a_3, a_4 are deduced from a_0, a_1 by

$$\begin{aligned} a_3 &= -\frac{1}{2}(A_{d+3}(\lambda)\, a_0 + A_{d+2}(\lambda)\, a_1 + \kappa\, A_{d+1}(\lambda)),\\ a_4 &= \frac{1}{2}(\lambda^2-1)(A_{d+2}(\lambda)\, a_0 + A_{d+1}(\lambda)\, a_1 + \kappa\, A_d(\lambda)), \end{aligned} \tag{A.8}$$

We thus get (see also [**8**]):

$$
\begin{aligned}
a_0 = & \frac{\kappa}{\Delta(\lambda)} \Big(-S_{2d+2}(\lambda) + 2(\lambda^2-1)^{d+1} + 4(d+1)^2(\lambda^2-1)^d \Big) \\
& + \frac{1}{\Delta(\lambda)} \Big(2\, A_{2d+2}(\lambda) - 8(d+1)\lambda(\lambda^2-1)^d \Big), \qquad \text{(A.9)}
\end{aligned}
$$

(A.10)

$$
\begin{aligned}
a_1 = & \frac{\kappa}{\Delta(\lambda)} \Big(2S_{2d+3}(\lambda) - 4\lambda(\lambda^2-1)^{d+1} - 8(d+1)(d+2)\lambda(\lambda^2-1)^d \Big) \\
& + \frac{1}{\Delta(\lambda)} \Big(-2\, A_{2d+3}(\lambda) + 4(2d+3)(\lambda^2-1)^{d+1} + 16(d+2)(\lambda^2-1)^d \Big),
\end{aligned}
$$

(A.11)

$$
\begin{aligned}
a_3 = & \frac{(\lambda^2-1)^d\,\kappa}{\Delta(\lambda)} \Big(-\frac{1}{2}(\lambda^2-1)^3 A_{d-1}(\lambda) - 2(d+2)^2(\lambda^2-1)A_{d+1}(\lambda) + 2\lambda(\lambda^2-1)A_{d+2}(\lambda) \\
& \qquad + 4(d+1)(d+2)\lambda A_{d+2}(\lambda) - \frac{3}{2}(\lambda^2-1)A_{d+3}(\lambda) - 2(d+1)^2 A_{d+3}(\lambda) \Big) \\
& + \frac{(\lambda^2-1)^d}{\Delta(\lambda)} \Big(-2(\lambda^2-1)^2 A_d - 2(2d+3)(\lambda^2-1)A_{d+2}(\lambda) \\
& \qquad - 8(d+2)A_{d+2}(\lambda) + 4(d+1)\lambda A_{d+3}(\lambda) \Big),
\end{aligned}
$$

(A.12)

$$
\begin{aligned}
a_4 = & \frac{(\lambda^2-1)^{d+1}\,\kappa}{\Delta(\lambda)} \Big(\frac{3}{2}(\lambda^2-1)^2 A_d(\lambda) + 2(d+2)^2(\lambda^2-1)A_d(\lambda) - 2\lambda(\lambda^2-1)A_{d+1}(\lambda) \\
& \qquad - 4(d+1)(d+2)\lambda A_{d+1}(\lambda) + 2(d+1)^2 A_{d+2}(\lambda) + \frac{1}{2}A_{d+4}(\lambda) \Big) \\
& + \frac{(\lambda^2-1)^{d+1}}{\Delta(\lambda)} \Big(4(d+2)(\lambda^2+1)A_{d+1}(\lambda) - 4(d+1)\lambda A_{d+2}(\lambda) \Big),
\end{aligned}
$$

where we have set:

$$
\Delta(\lambda) = -S_{2d+4}(\lambda) + 4(d+2)^2(\lambda^2-1)^{d+1} + 2(\lambda^2-1)^{d+2}. \qquad \text{(A.13)}
$$

Proposition A.1. — *For any real number κ, the discriminant of $P_\kappa(X)$ is non-zero in $R(\lambda)$.*

Proof. — In general, for any polynomial $f(X) = \sum_{i=0}^{n} a_i X^{n-i} = a_0 \prod_{j=1}^{n}(X - t_j)$ with coefficients in some field K, with $a_0 \neq 0$ and $n \geq 1$, the discriminant[10], $D(f)$, of f is defined by

$$D(f) = a_0^{-1} R(f, f') = a_0^{2n-2} \prod_{j \neq k} (t_j - t_k) = a_0^{n-2} \prod_{j=1}^{n} f'(t_j), \tag{A.14}$$

where $R(f, f')$ denotes the resultant[11] of f and its derivative f', and t_j, $j = 1, \ldots, n$, denote the n roots of f in a suitable field extension $\tilde{K}$ of K.

In the present case, we observe that $P_\kappa(X)$, defined by (A.5), can be written as

$$P_\kappa(X) = \Phi(X) + \big(X + \frac{a_4(\lambda)}{a_3(\lambda)}\big)\, P'_\kappa(X), \tag{A.16}$$

by setting $\Phi(X) = -X^d\, Q(X)$ and

$$\begin{aligned} Q(X) = {} & (d+2)\, a_0(\lambda)\, X^3 + \Big((d+3)\, a_0(\lambda)\, \frac{a_4(\lambda)}{a_3(\lambda)} + (d+1)\, a_1(\lambda)\Big)\, X^2 \\ & + \Big((d+2)\, a_1(\lambda)\, \frac{a_4(\lambda)}{a_3(\lambda)} + d\, \kappa\Big)\, X + (d+1)\, \kappa\, \frac{a_4(\lambda)}{a_3(\lambda)}. \end{aligned} \tag{A.17}$$

We then have $R(P, P') = R(\Phi, P')$, hence

$$D(P) = (-1)^d (d+2)^{d+3} a_0(\lambda)^{d+3} a_3(\lambda)^d \prod_{i=1}^{3} P'(\beta_i), \tag{A.18}$$

where $\beta_1, \beta_2, \beta_3$ denote the roots of Q in a suitable field extension, $\widetilde{R(\lambda)}$, of $R(\lambda)$. It follows that $D(P)$ is zero in $R(\lambda)$ if and only if $P'(\beta_i) = 0$ in $\widetilde{R(\lambda)}$ for some $i = 1, 2$ or 3. We show that this cannot happen by considering the

---

(10) We here adopt the definition which appears in [**28**]. The definition in [**6**] differs by a factor $(-1)^{\frac{n(n-1)}{2}}$.

(11) Recall that the *resultant* $R(f, g)$ of two polynomials $f(X) = \sum_{i=0}^{n} a_i X^{n-i} = a_0 \prod_{j=1}^{n}(X - t_j)$ and $g = \sum_{i=0}^{m} b_i X^{m-i} = b_0 \prod_{r=1}^{m}(X - u_r)$, with $a_0 b_0 \neq 0$, has the following expressions:

$$R(f, g) = a_0^m b_0^n \prod_{j=1}^{n} \prod_{r=1}^{m} (t_j - u_r) = a_0^m \prod_{j=1}^{n} g(t_j) = (-1)^{mn} b_0^n \prod_{r=1}^{m} f(u_r). \tag{A.15}$$

behaviour of the product $\prod_{i=1}^{3} P'(\beta_i)$ near $\lambda = \pm 1$. Notice that

$$\begin{aligned} a_0(\lambda) &\cong \frac{1}{4}(\kappa \mp 2), \\ a_1(\lambda) &\cong -\kappa \pm 1, \\ a_3(\lambda) &\cong (-(d+1)\,\kappa) \pm 2)\,(\lambda \mp 1)^d, \\ a_4(\lambda) &\cong (d\,\kappa \mp 2)(\lambda \mp 1)^{d+1}, \end{aligned} \tag{A.19}$$

modulo terms of higher orders in $(\lambda \mp 1)$ near $\lambda = \pm 1$. We temporarily assume that $\kappa \neq \frac{2}{d}$ and $\kappa \neq \frac{2}{d+1}$, so that $\frac{a_4(\lambda)}{a_3(\lambda)}$ is exactly of order 1 in $(\lambda \mp 1)$ near $\lambda = \pm 1$. We also assume $\kappa \neq \pm 2$ and $\kappa \neq 0$. It then follows that one root, β_3 say, of Q is of order 1 as well, with

$$\beta_3 \cong \frac{\kappa \mp \frac{2}{d}}{\kappa \mp \frac{2}{d+1}}\,(\lambda \mp 1), \tag{A.20}$$

whereas the other two, β_1, β_2 tend to the roots, r_1, r_2 say, of the equation

$$\frac{(d+2)}{4}(\kappa \mp 2)\,X^2 + (d+1)(-\kappa \pm 1)\,X + d\kappa = 0, \tag{A.21}$$

which are both finite (as $\kappa \neq \pm 2$) and non zero (as $\kappa \neq 0$). It is easily checked that, for $i = 1, 2$, the limit of $P'(\beta_i)$ at $\lambda = \pm 1$, which is equal to $r_i^d\big(\frac{(d+3)}{4} r_i^2 + (d+2)(-\kappa \pm 1) r_i + (d+1)\kappa\big)$, is non-zero for any value of κ ; indeed, a common root, r, of (A.21) and of the equation

$$\frac{(d+3)}{4}(\kappa \mp 2)\,X^2 + (d+2)(-\kappa \pm 1)\,X + (d+1)\kappa = 0. \tag{A.22}$$

would satisfy $r = -\frac{2(-\kappa \pm 1)}{\kappa \mp 2} = -\frac{2\kappa}{(-\kappa \pm 1)}$, which is clearly impossible. In particular, $P'(\beta_1)$ and $P'(\beta_2)$ are both non zero in K. As for $P'(\beta_3)$, we have

$$\begin{aligned} P'(\beta_3) &\cong a_3(\lambda) + (d+1) a_2 \beta_3^d \\ &= -\frac{(d+1)}{(\kappa \mp \frac{2}{d+1})^d}\,\big((\kappa \mp \frac{2}{d+1})^{d+1} - \kappa(\kappa \mp \frac{2}{d})^d\big)\,(\lambda \mp 1)^d \end{aligned} \tag{)(A.23}$$

modulo terms of higher orders in $\lambda \mp 1$. If $P'(\beta_3)$ was zero in K, the rhs of (A.23) would be zero for $\lambda = -1$ and $\lambda = 1$, meaning that κ and $-\kappa$ would be both a root of the equation

$$(X + \frac{2}{d+1})^{d+1} - X(X + \frac{2}{d})^d = 0. \tag{A.24}$$

On the other hand, if $h(X) = \sum_{j=0}^{d+1} c_j X^{d+1-j}$ denotes the polynomial in the rhs of (A.24), we have that

$$c_j = \frac{2^j \binom{d}{j}}{(d+1-j)(d+1)^{j-1} d^j}\,\varphi_{j-1}(d), \tag{A.25}$$

for $j = 0, \dots, d+1$, by setting $\varphi_k(x) = x^{k+1} - (x-k)(x+1)^k$, for any integer k. It follows that $c_0 = c_1 = 0$, whereas $c_j > 0$ for any $j \geq 2$. To prove the last assertion, it is sufficient to check that $\varphi_k(x)$ is positive on $[1, +\infty)$ for all integers $k \geq 1$. Observe that $\varphi_k'(x) = (k+1)\varphi_{k-1}(x)$. We then conclude by a simple argument by induction: if φ_{k-1} is positive, then φ_k is increasing, hence positive on $[1, +\infty)$, as $\varphi_k(1) = 1$; the argument by induction is then completed by observing that $\varphi_1(x) \equiv 1$. We infer that κ and $-\kappa$ cannot be simultaneously roots of (A.24), proving that $P'(\beta_3)$ is non-zero in K. The case when κ is $\pm 2, \pm\frac{2}{d}, \pm\frac{2}{d+1}$ which were discarded in the argument, is solved by using the same argument at $\lambda = -1$ *or* at $\lambda = 1$ and by observing that none of these values is a root of the equation (A.24). If $\kappa = 0$, we observe that (A.16) holds with $\Phi = -X^{d+1}\,\tilde{Q}(X)$ and

$$\begin{aligned}\tilde{Q}(X) = (d+2)a_0(\lambda)X^2 + \left((d+3)\,a_0(\lambda)\,\frac{a_4(\lambda)}{a_3(\lambda)} + (d+1)\,a_1(\lambda)\right)X^2 \\ + (d+2)\,a_1(\lambda)\,\frac{a_4(\lambda)}{a_3(\lambda)}.\end{aligned} \tag{A.26}$$

This polynomial has two roots, α_1, α_2, in some extension of $R(\lambda)$ and, as before, the discriminant of P is zero if and only if $P'(\alpha_1)$ or $P'(\alpha_2)$ is 0 in this extension. One of these roots, α_2 say, is zero at $\lambda = \pm 1$, with $\alpha_2 \cong \frac{(d+2)}{(d+1)}\,(\lambda \mp 1)$, whereas $\alpha_1 = \frac{2(d+1)}{(d+2)}$. Then, $P'(\alpha_1) = \pm\frac{2^{d+1}(d+1)^{d+1}}{(d+2)^{d+2}} \neq 0$ at $\lambda = \pm 1$, whereas $P'(\alpha_1) \cong a_3(\lambda)$ is non-zero in $R(\lambda)$. This completes the proof of Proposition A.1. □

References

[1] V. Apostolov, D. M. J. Calderbank, Paul Gauduchon, *Hamiltonian 2-forms in Kähler geometry, I General Theory*, J. Differential Geom. **73**, No 3 (2006), 359–412.

[2] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, Christina W. Tønnesen-Friedman *Hamiltonian 2-forms in Kähler geometry, II Global Classification*, J. Differential Geom. **68** No 2 (2004), 277–345.

[3] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, Christina W. Tønnesen-Friedman *Hamiltonian 2-forms in Kähler geometry, III: Extremal metrics and stability*, Invent. math. **173** No 3 (2008), 547–601.

[4] V. Apostolov, D. M. J. Calderbank, Paul Gauduchon, Christina W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry, IV: Weakly Bochner-flat Kähler manifolds*, Comm. Anal. Geom. **16** No 1 (2008), 91–126.

[5] V. Apostolov, C. Tønnesen-Friedman, *A remark on Kähler metrics of constant scalar curvature on ruled complex surfaces*, Bull. London Math. Soc. **38** (2006), 494–500.

[6] N. Bourbaki, *Éléments de Mathématique*, Algèbre, Chapitre 7, Masson, Paris (1981).

[7] J. P. Bourguignon, *Invariants intégraux fonctionnels pour des équations aux dérivées partielles d'origine géométrique*, Partial differential equations, Part 1, 2 (Warsaw, 1990), 65–73,

[8] E. Calabi, *Extremal Kähler metrics*, in *Seminar of Differential Geometry*, ed. S. T. Yau, Annals of Mathematics Studies **102**, Princeton University Press (1982), 259–290.

[9] E. Calabi, *Extremal Kähler metrics, II*, in *Differential Geometry and Complex Analysis*, eds. I. Chavel and H. M. Farkas, Springer Verlag (1985), 95–114.

[10] X. X. Chen, *The space of Kähler metrics*, J. Differential Geom. **56** (2000), 189-234.

[11] X. X. Chen and G.Tian, *Uniqueness of extremal Kähler metrics*, C. R. Math. Acad. Sci. Paris **340** No 4 (2005), 287–290.

[12] X. X. Chen and G. Tian, *Partial regularity for homogeneous complex Monge-Ampère equations* C. R. Math. Acad. Sci. Paris **340** No. 5 (2005), 337–340.

[13] X. X. Chen and G. Tian, *Geometry of Kähler metrics and holomorphic foliation by discs*, Publ. Math. Inst. Hautes tudes Sci. No. 107 (2008), 1–107

[14] S. K. Donaldson, *Symmetric spaces, Kähler geometry and hamiltonian dynamics*, in *Northern California Symplectic Geometry Seminar* (Eliashberg et al. eds.), Amer. Math. Soc. (1999), 13–33.

[15] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349.

[16] W. Fulton, *Intersection theory*, Second Edition, Springer-Verlag (1998)

[17] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), 437–443.

[18] A. Futaki and T. Mabuchi, *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.

[19] P. Gauduchon, *Calabi extremal Kähler metrics: An elementary introduction*, in preparation.

[20] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics, A Wiley-Interscience Series of Texts, Monographs and Tracts, John Wiley & Sons, Inc. (1978).

[21] D. Guan, *Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends*, Trans. Amer. Math. Soc. **347** (1995), 2255–2262.

[22] D. Guan, *On modified Mabuch functional and Mabuchi moduli space of Kähler metrics on toric bundles*, Math. Res. Letters **6** (1999), 547–555.

[23] F. Hirzebruch, *Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann. **124** (1951), 77–86.

[24] F. Hirzebruch, *Topological methods in algebraic geometry*, *Grundlehren der Mathematik*, Springer-Verlag (1978).

[25] A. D. Hwang, *On existence of Kähler metrics with constant scalar curvature*, Osaka J. Math. **31** (1994), 561–595.

[26] A. D. Hwang and M. A. Singer, *A momentum construction for circle-invariant Kähler metrics*, Trans. Amer. Math. Soc. **354** (2002), 2285–2325.

[27] S. Kobayashi, *Transformation groups in differential geometry*, Erg. der Math. Band 70, Springer-Verlag (1972).

[28] S. Lang, *Algebra*, Addison-wesley Publishing Company, Inc. *Second Printing* (1967).

[29] C. R. LeBrun and S. Simanca, *Extremal Kähler metrics and complex deformation theory*, Geom. Funct. Analysis, Vol. **4**, No **3** (1994), 298–336.

[30] C. R. LeBrun and S. Simanca, *On the Kähler classes of Extremal Metrics*, Geometry and Global Analysis, First MSJ Intern. Res. Inst. Sendai, Japan, Eds. Kotake, Nishikawa and Schoen, 1993.

[31] L. Lempert, *Symmetries and other transformations of the complex Monge-Ampère equation*, Duke Math. J. **52** (1985), 869–885.

[32] T. Mabuchi, *Some Symplectic geometry on compact Kähler manifolds*, Osaka J. Math. **24** (1987), 227–252.

[33] R. MacPherson, *Chern classes of singular varieties*, Annals of Math. **100** (1974), 423–432.

[34] J. Ross and R. Thomas, *An obstruction to the existence of constant scalar curvature scalar metrics*, J. Differential Geom. **72** (2006), no 3, 429–466.

[35] J. Ross and R. Thomas, *A study of the Hilbert-Mumford criterion for the stability of projective manifolds*, J. Algebraic Geom. **16** (2007), no 2, 201–255.

[36] S. W. Semmes, *Complex Monge-Ampère and symplectic manifolds*, Amer. J. Math. **114** (1992), 495–550.

[37] S. Simanca, *A note on extremal metrics of nonconstant scalar curvature*, Isreal J. Math. **78** (1992), 85–93.

[38] S. Simanca, *A K-energy characterization of extremal Kähler metrics*, Proc. Amer. Math. Soc. **128**, no 5 (2000), 1531–1535.

[39] G. Székelyhidi, *Extremal metrics and K-stability*, Bull. London Math. Soc., **39** (2007), 76–84.

[40] G. Székelyhidi, *The Calabi functional on a ruled surface*, arXiv math.DG/0703562

[41] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. math. **130** (1997), 1–37.

[42] C. Tønnesen-Friedman, *Extremal metrics on minimal ruled surfaces*, J. reine angew. Math. **502** (1998), 175–197.

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