BIHERMITIAN STRUCTURES ON COMPLEX SURFACES

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1. *Introduction*

We consider connected, oriented (Riemannian) conformal 4-manifolds (M, c) admitting two independent compatible integrable almost-complex structures J_i, for $i = 1, 2$. Here and henceforth, *compatible* means that J_1 and J_2 are orthogonal with respect to any Riemannian metric g in the conformal class c and induce the chosen orientation of M, and *independent* means that $J_1(x) \neq \pm J_2(x)$ for some point x in M. Then, (c, J_1, J_2) will be called a (conformal) *bihermitian* structure on M and (M, c, J_1, J_2) a *bihermitian surface*.

A bihermitian structure will be called *strongly bihermitian* if $J_1(x) \neq \pm J_2(x)$ everywhere.

In the sequel, an integrable almost-complex structure will occasionally be called a *complex structure*.

A Riemannian conformal structure c is called *anti-self-dual*, ASD for short, if the self-dual part W^+ of the Weyl tensor vanishes identically. It is an easy consequence of Theorem 4.1 in [**5**] that c is ASD if and only if the following holds: *for any point x of M, any compatible complex structure of the tangent space T_xM can be extended to a compatible complex structure on some neighbourhood of x*. Thus, ASD conformal 4-manifolds locally admit infinitely many compatible complex structures, but it may happen that no one of them extends to the whole manifold, as in the case of the standard conformally flat 4-sphere S^4. Kähler complex surfaces with zero scalar curvature provide examples of ASD Hermitian 4-manifolds; conversely, any ASD Hermitian metric on a *compact* 4-manifold is locally conformal to a Kähler metric with zero scalar curvature [**7**].

Compact ASD bihermitian surfaces have recently been classified by M. Pontecorvo [**19**]. Among them, strongly bihermitian surfaces exactly correspond to *hyperhermitian complex surfaces* $(M, c, \mathscr{F})$, where $\mathscr{F}$ is a 2-sphere of compatible complex structures generated by two *anti-commuting* ones; these are (see [**8**]) flat complex tori and Ricci flat K3 surfaces, which are *hyperkähler*, and some class of conformally flat Hopf surfaces.

Certain conformally flat Hopf surfaces actually admit *two* distinct hyperhermitian structures, say $(c, \mathscr{F})$ and $(c, \mathscr{F}')$, compatible with the same (conformally flat) conformal structure and the same orientation; see for example, [**12**] or [**19**]. We thus get two sorts of bihermitian structures (c, J_1, J_2) with respect to the same (flat) conformal metric c, according to whether or not J_1 and J_2 belong to the same sphere $\mathscr{F}$ or $\mathscr{F}'$. In the former case, (c, J_1, J_2) is strongly

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bihermitian, whereas it is *not* in the latter case. More precisely, the subsets where J_1 is equal or conjugate to J_2 are then both non-empty; compare with Proposition 2 below. Other ASD bihermitian surfaces appearing in Pontecorvo's list are obtained from either a Hopf surface or a parabolic Inoue surface by blowing up points along a unique elliptic curve [**19**, Theorem 5.2] (see also Corollary 3 below), that is, they belong to the class VII in the Kodaira classification; in particular, their first Betti number is equal to 1.

In this paper, we are mostly concerned with the *non-ASD* case. Non-ASD connected conformal 4-manifolds admit at most *two* independent compatible complex structures [**19**, Corollary 1.6] so that bihermitian structures are also 'maximally' hermitian. Moreover, non-ASD bihermitian situations cannot occur in the case when W^+ is everywhere *degenerate*, that is, has only two distinct eigenvalues as an operator acting on the bundle Λ^+M of self-dual 2-forms. (For that and for further information concerning the interplay between conformal and complex geometry in the non-ASD case, see for example, [**9**, **20**, **3**].) In particular, if (c, J_1, J_2) is a non-ASD bihermitian structure, neither J_1 nor J_2 can be Kähler with respect to any metric in the conformal class c.

It has been expected for a long time that bihermitian situations could not possibly occur, even locally, in the non-ASD case. It thus came as a (welcome) surprise when P. Kobak [**16**] recently gave a general and explicit method to produce non-ASD (strongly) bihermitian structures (called *doubly-Hermitian* in [**16**]) on $\mathbb{R}^4$ or the torus T^4.

The present paper includes elements of a general theory of (non-ASD) bihermitian 4-manifolds opened up by Kobak's paper and a series of results aiming at a classification in the compact case (still to come), principally in the case when the first Betti number is even. We also provide a general method, partially suggested by D. Joyce, for constructing non-ASD strongly bihermitian conformal structures on (compact) surfaces admitting a hyperkähler metric, that is, on tori and K3 surfaces. A similar procedure also provides non-ASD strongly bihermitian structures on $S^1 \times S^3$.

The paper is organized as follows.

Section 2 is principally devoted to proving the following theorem.

THEOREM 1. *Let (M, c, J_1, J_2) be a connected, compact bihermitian conformal 4-manifold with even first Betti number. Then,*

(i) *either (c, J_1, J_2) is strongly bihermitian, in which case the complex surfaces (M, J_1) and (M, J_2) are either both complex tori or both* K3 *surfaces, or*

(ii) *(M, J_1) (or equivalently (M, J_2)) is obtained from either $\mathbb{C}P^2$ or a minimal ruled surface of genus at most* 1 *by blowing up points along some anti-canonical divisor.*

No examples of bihermitian structures on complex surfaces described in Theorem 1(ii) are known at present. If such examples were found, they would be the first known examples of bihermitian structures on a complex surface *not* admitting any ASD metric and would thus constitute a major step in the development of the theory.

An immediate consequence of Theorem 1 is the following generalization (in complex dimension 2) of the well-known results of Lichnerowicz [**17**] for elements in the connected group of isometries of a compact Kähler manifold.

COROLLARY 1. *Let (M, g, J) be a compact Hermitian surface with even first Betti number. Suppose that there exists an orientation-preserving conformal isometry of (M, g) which is not $\pm$-biholomorphic. Then (M, J) is one of the complex surfaces described in Theorem* 1.

A 'Kählerian' version of the above corollary says that the positive conformal isometries of any compact Kähler, non-hyperkähler surface preserve the Kähler structure [**19**, Theorem 5.3] (see also [**4**, Theorem 2]). On the other hand, it is not difficult to find orientation-preserving isometries on flat tori which are not $\pm$-holomorphic. For some special K3 surfaces such isometries have been shown to exist; see [**1**]. Outside the Kählerian category, we can construct (non-Kähler) Hermitian conformal metrics on some complex tori that admit an orientation-preserving conformal isometry which does not preserve the complex structure (see § 4.1 below). Notice that locally conformally Kähler surfaces with odd first Betti number, admitting orientation-preserving isometries which are not $\pm$-biholomorphisms, are anti-self-dual bihermitian surfaces, so they are either the conformally flat Hopf surfaces or obtained from a Hopf surface or a parabolic Inoue surface by blowing up points along a unique smooth elliptic curve; cf. [**19**, Theorem 5.2].

In § 3, we investigate the situation described in Theorem 1(i) and show that (non-ASD) bihermitian structures on complex tori or K3 surfaces, which are necessarily strongly bihermitian by Theorem 1, actually determine, and are determined by, a special class of *twisted* hyperkähler structures; more precisely, we prove the following.

THEOREM 2. *Let M be either a* 4*-dimensional torus or a* K3 *surface. Then, for any bihermitian structure (c, J_1, J_2) on M, there exist a triple of real symplectic* 2*-forms, Φ_0, Φ_1, Φ_2, such that the complex* 2*-forms $\Omega_1 = \Phi_0 - i\Phi_1$ and $\Omega_2 = \Phi_0 - i\Phi_2$ are holomorphic symplectic* 2*-forms with respect to J_1 and J_2 respectively. Moreover, Φ_0, Φ_1 and Φ_2 are related by*

(a) $\Phi_0 \wedge \Phi_0 = \Phi_1 \wedge \Phi_1 = \Phi_2 \wedge \Phi_2$,

(b) $\Phi_0 \wedge \Phi_1 = \Phi_0 \wedge \Phi_2 = 0$,

(c) $\Phi_1 \wedge \Phi_2 = -p\,\Phi_0 \wedge \Phi_0$,

where the smooth function $p = -\frac{1}{4}\,\mathrm{trace}(J_1 \circ J_2)$ satisfies $|p| < 1$.

Conversely, let M be a 4*-dimensional manifold equipped with a triple of real symplectic forms, Φ_0, Φ_1, Φ_2 that satisfy the conditions* (a), (b), (c) *for some smooth function p with $|p| < 1$. Let M be oriented by any one of these symplectic forms. Then, the triple Φ_0, Φ_1, Φ_2 determines a conformal strongly bihermitian structure (c, J_1, J_2) such that Φ_0, Φ_1 and Φ_2 are self-dual with respect to c and the orientation, and the complex* 2*-forms $\Omega_1 = \Phi_0 - i\Phi_1$ and $\Omega_2 = \Phi_0 - i\Phi_2$ are holomorphic with respect to J_1 and J_2 respectively. Moreover, (c, J_1, J_2) is hyperkähler if and only if p is constant.*

In § 4, we show how Theorem 2 can be used as an effective tool for constructing bihermitian structures. We first recover Kobak's constructions, then, following an idea of D. Joyce, we propose a general procedure for constructing non-anti-self-dual (strongly) bihermitian structures by deforming hyperkähler structures on complex tori or K3 surfaces. A similar construction involving the flat hyperkähler structure of $\mathbb{C}^2$ provides a non-ASD bihermitian conformal class on any Hopf surface admitting a hyperhermitian metric. Bringing together these constructions with the above-mentioned classification of compact hyperhermitian complex surfaces by C. Boyer, we eventually get the following.

PROPOSITION 1. *Any hyperhermitian structure on a compact complex surface can be deformed into a non-ASD strongly bihermitian structure.*

The above proposition shows in particular the existence of non-ASD bihermitian structures on $S^1 \times S^3$. Apart from that, only a little information is available concerning bihermitian situations for a compact 4-dimensional manifold with *odd* first Betti number. Some results pertaining to this case are collected in § 5.

The *angle function* of a bihermitian structure (c, J_1, J_2) is the real function p defined by

$$p = -\tfrac{1}{4}\mathrm{trace}(J_1 \circ J_2), \tag{1}$$

or, equivalently,

$$J_1 \circ J_2 + J_2 \circ J_1 = -2p\,\mathrm{Id}, \tag{2}$$

where Id denotes the identity. Notice that the angle-function p only depends upon the conformal class c. We then have $|p| \leqslant 1$, $p(x) = \pm 1$ if and only if $J_1(x) = \pm J_2(x)$, and $p(x) = 0$ if and only if $J_1(x)$ and $J_2(x)$ anticommute. If p is a constant, of absolute value less than 1, J_1 and

$$J_2' := \frac{1}{\sqrt{1-p^2}}\,(J_2 - pJ_1)$$

anticommute, and hence determine a 2-sphere $\mathscr{F}$ of almost-complex structures, all compatible with c and with the same orientation. Moreover, since p is constant and J_1 and J_2 are both integrable, *all* almost-complex structures in $\mathscr{F}$ are integrable as well (see [**13**, **2**]), so that $(c, \mathscr{F})$ is actually a (conformal) hyperhermitian structure. Then, for any chosen metric g in c, all Hermitian structures (g, J) have the same Lee form θ when J belongs to $\mathscr{F}$, and the hyperhermitian structure $(c, \mathscr{F})$ is (conformal) hyperkähler if θ is zero for some metric g in c.

In § 6, we investigate further properties of the angle-function p. We infer in particular that for any strongly conformal bihermitian structure of a compact complex surface, the self-dual Weyl tensor W^+ must vanish somewhere (see [**2**] for specific properties of p in the ASD case).

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2. *Proof of Theorem* 1

For any almost-hermitian structure (g, J) on a 4-dimensional manifold M, the *Kähler form* and the *Lee form* of (g, J) are the real 2-form, F, and the real 1-form θ, defined by $F = g(J\,\cdot\,,\cdot)$ and $dF = \theta \wedge F$.

Let (c, J_1, J_2) be a bihermitian structure on M.

Fix a metric g in c and denote by F_1 and F_2 the Kähler forms of (g, J_1) and (g, J_2). Let θ_1 and θ_2 be the corresponding Lee forms.

The following lemma is a direct consequence of [**3**, Lemma 2 and Remark 1].

LEMMA 1. *For any metric g in the conformal class c, the Lee forms of J_1 and J_2 satisfy*

$$(d(\theta_1 + \theta_2))_+ = 0 \tag{3}$$

and

$$2\delta\theta_1 + |\theta_1|^2 = 2\delta\theta_2 + |\theta_2|^2, \tag{4}$$

where, for any 2-form Φ, Φ_+ denotes the self-dual part of Φ, δ denotes the codifferential with respect to g, and $|\cdot|$ the norm induced by g.

Let $\mathscr{U}$ be the open subset of points x of M such that $J_1(x) \neq \pm J_2(x)$. Then the *angle function* p (see § 1) satisfies $|p| < 1$ on $\mathscr{U}$. Setting $q = \sqrt{1 - p^2}$, we consider the positive c-compatible almost-complex structures I_0, I_1, I_2 on $\mathscr{U}$ defined by

$$I_1 = \frac{1}{q}(J_2 - pJ_1), \quad I_2 = \frac{1}{q}(J_1 - pJ_2), \quad I_0 = J_1 \circ I_1 = -J_2 \circ I_2. \tag{5}$$

The Kähler forms of (g, I_0), (g, I_1) and (g, I_2) are denoted by Φ_0, Φ_1 and Φ_2 respectively.

Since J_1 and J_2 are both integrable, we clearly have $I_0(\delta\Phi_0) = I_0(J_1\delta\Phi_1) = I_1(\delta\Phi_1)$ and $I_0(\delta\Phi_0) = -I_0(J_2\delta\Phi_2) = I_2(\delta\Phi_2)$. It follows that the almost-hermitian structures (g, I_0), (g, I_1) and (g, I_2) have the same Lee form, say τ.

LEMMA 2. *For any metric g in the conformal class c we have (on $\mathscr{U}$)*

$$\theta_1 + \theta_2 - 2\tau = -2d\ln q. \tag{6}$$

Proof. It follows from the definitions of I_0, I_1 and I_2 that

$$F_1 = pF_2 + q\Phi_2, \tag{7}$$

$$F_2 = pF_1 + q\Phi_1. \tag{8}$$

In the following equalities, ∇ denotes the Levi-Civita connection of g and we freely identify vectors and covectors via g. The integrability of J_1 and J_2 then implies that

$$\nabla F_1 = \tfrac{1}{2} I_0\theta_1 \otimes \Phi_1 - \tfrac{1}{2} I_1\theta_1 \otimes \Phi_0, \tag{9}$$

$$\nabla \Phi_1 = -\tfrac{1}{2} I_0\theta_1 \otimes F_1 + \tfrac{1}{2} I_1(2\tau - \theta_1) \otimes \Phi_0, \tag{10}$$

$$\nabla F_2 = -\tfrac{1}{2} I_0\theta_2 \otimes \Phi_2 + \tfrac{1}{2} I_2\theta_2 \otimes \Phi_0, \tag{11}$$

$$\nabla \Phi_2 = \tfrac{1}{2} I_0\theta_2 \otimes F_2 + \tfrac{1}{2} I_2(2\tau - \theta_2) \otimes \Phi_0. \tag{12}$$

By using (7) and (8), we obtain from (9), (10), (11) and (12),

$$df = f(\theta_1 - \tau) + I_0(\theta_1 - \tau),$$

$$df = f(\theta_2 - \tau) - I_0(\theta_2 - \tau),$$

where $f = p/q$. By adding and subtracting these relations, we get

$$2df = f(\theta_1 + \theta_2 - 2\tau) + I_0(\theta_1 - \theta_2), \tag{13}$$

$$I_0(\theta_1 + \theta_2 - 2\tau) = f(\theta_2 - \theta_1), \tag{14}$$

whence, finally, we obtain (6).

Let $[J_1, J_2] = J_1 \circ J_2 - J_2 \circ J_1$ be the commutator of J_1 and J_2. For any metric g in the conformal class c, we consider the real J_1-anti-invariant 2-form $\Phi(\cdot,\cdot) = \frac{1}{2}g([J_1, J_2]\cdot,\cdot)$ and the corresponding complex $(0,2)$-form $\sigma_1(\cdot,\cdot) = \Phi(\cdot,\cdot) + i\Phi(J_1\cdot,\cdot)$. Thus, σ_1 is a smooth section of $\Lambda^{0,2}_{J_1}(M)$ and $\sigma_1(x) = 0$ if and only if $\Phi(x) = 0$, which is so if and only if $J_1(x) = \pm J_2(x)$. Notice that the complex line bundle $\Lambda^{0,2}_{J_1}(M)$ is identified, via the metric g, with the *anti-canonical* bundle $K^{-1}_{J_1}$. Moreover, the canonical Hermitian connection of the holomorphic line bundle $K^{-1}_{J_1} = \Lambda^{0,2}_{J_1}(M)$ coincides with the connection induced by the Chern connection of (g, J_1); both will be denoted by ∇^1. We then have the following.

LEMMA 3. *For any* (0, 1)*-vector field* X *of* (M, J_1)*,*

$$\nabla^1_X\sigma_1 = -\tfrac{1}{2}(\theta_1 + \theta_2)(X)\,\sigma_1. \tag{15}$$

Proof. Let $\mathcal{U}$ be the open subset of M where $J_1 \neq \pm J_2$. According to [**19**, Proposition 1.3], $\mathcal{U}$ is dense in M, so it is sufficient to verify (15) on $\mathcal{U}$. It is easily checked that $\sigma_1 = q(\Phi_0 + i\Phi_1)$ on $\mathcal{U}$. So by (9) and (10) we obtain

$$(\nabla_X\sigma_1)^{0,2} = (d\ln q - \tau + \tfrac{1}{2}\theta_1)(X)\sigma_1, \tag{16}$$

where X is an arbitrary $(0, 1)$-vector field (with respect to J_1) and $(\cdot)^{0,2}$ denotes the $(0, 2)$ part. Now, by using Lemma 2, we find that the equality (16) reads

$$(\nabla_X\sigma_1)^{0,2} = -\tfrac{1}{2}\theta_2(X)\,\sigma_1.$$

On the other hand, it easily follows from the well-known relation between ∇ and ∇^1 (cf. for example, [**21**]) that, for any $(0, 2)$-form and $(0, 1)$-vector, we have

$$\nabla^1_X\sigma_1 = (\nabla_X\sigma_1)^{0,2} - \tfrac{1}{2}\theta_1(X)\,\sigma_1.$$

Lemma 3 follows directly from the above two equalities.

The above lemma implies that σ_1 is a holomorphic section of $K^{-1}_{J_1}$ provided that $\theta_1 + \theta_2 = 0$ for some metric g in c. The next lemma shows that this always occurs when M is compact with even first Betti number.

LEMMA 4. *Let* (c, J_1, J_2) *be a bihermitian structure on a compact oriented* 4*-manifold* M *with even first Betti number. Then the standard metric* g *of* (c, J_1) *is also standard for* (c, J_2) *and the Lee forms* θ_1 *and* θ_2 *of* J_1 *and* J_2 *with respect to* g *satisfy*

$$\theta_1 + \theta_2 = 0. \tag{17}$$

Proof. Since M is compact, the equality (3) in Lemma 1 is equivalent to the fact that $\theta_1 + \theta_2$ is closed. Let g be the standard metric of (J_1, c), so that $\delta\theta_1 = 0$

(see [**10**]). Let β be the harmonic part in the Hodge decomposition of $\theta_1 + \theta_2$, that is, $\theta_1 + \theta_2 = \beta + d\sigma$, where σ is a smooth function on M. Since $b_1(M)$ is even, we have from [**11**] that $\theta_1 = \delta\alpha$ for some 2-form α on M. Integrating (4) we get

$$\int_M |\theta_1|^2\, dV = \int_M |\theta_2|^2\, dV = \int_M |\beta + d\sigma - \delta\alpha|^2\, dV$$

$$= \int_M (|\theta_1|^2 + |d\sigma|^2 + |\beta|^2)\, dV.$$

Hence $\beta \equiv 0$ and $d\sigma \equiv 0$, that is, $\theta_1 + \theta_2 = 0$.

Putting together Lemmas 1, 3 and 4 we obtain the following.

PROPOSITION 2. *Let (c, J_1, J_2) be a bihermitian structure on a compact oriented 4-manifold M. Set $\mathscr{D} = \{x \in M\colon J_1(x) = \pm J_2(x)\}$. Then $\mathscr{D}$ is a complex curve of both (M, J_1) and (M, J_2). If moreover, the first Betti number of M is even, then each of (M, J_1) and (M, J_2) has an effective anti-canonical divisor.*

Proof. Since M is compact, we infer as above that $\theta_1 + \theta_2$ is closed. Then Lemma 3 shows that σ_1 is a holomorphic section of the holomorphic line bundle $K_{J_1}^{-1} \otimes L$, where L is the topologically trivial holomorphic line bundle, determined by the closed 1-form $\theta_1 + \theta_2$. Then, as the zero set of σ_1, $\mathscr{D}$ is a complex curve of (M, J_1). If moreover, the first Betti number is even, we infer from Lemma 4 that the closed 1-form $\theta_1 + \theta_2$ is exact. Hence L is the trivial holomorphic line bundle, and σ_1 determines an anti-canonical divisor of (M, J_1). The same argument still holds when considering J_2 instead of J_1. This completes the proof of Proposition 2.

We are now ready to prove Theorem 1. Let g be the standard metric of c with respect to J_1 and J_2 (Lemma 4) and let σ_1 be the $(0, 2)$-form with respect to J_1 appearing in Lemma 3, viewed as a section of the anti-canonical bundle $K_{J_1}^{-1}$. By Lemmas 3 and 4, σ_1 is a holomorphic section of $K_{J_1}^{-1}$ and $\mathscr{D}$ is the zero set of σ_1. Then, either $\mathscr{D}$ is empty, and then (c, J_1, J_2) is a strongly bihermitian structure, or, by Proposition 2, the plurigenera of (M, J_1) all vanish, and then the Kodaira dimension of (M, J_1) is equal to $-\infty$. Reversing the roles of J_1 and J_2, we see that the Kodaira dimension of (M, J_2) is then also equal to $-\infty$.

In the former case, σ_1 is a non-vanishing holomorphic section of $K_{J_1}^{-1}$, so that (M, J_1) is either the 4-torus or the K3 surface, and so is (M, J_2) (cf. [**6**]).

In the latter case, the Kodaira classification shows that (M, J_1) (and also (M, J_2)) is either a ruled surface of genus $g \geqslant 1$ or a rational surface (see [**6**]). Moreover, since (M, J_1) has an effective anti-canonical divisor (see Proposition 2), the same is true for the minimal model of M. Note that a minimal ruled surface has a Kähler metric of negative total scalar curvature (and hence admits no holomorphic sections of the anti-canonical bundle) as soon as its genus is greater than 1. So, the minimal model of (M, J_1) is either $\mathbb{C}P^2$ or a minimal ruled surface of genus at most 1. According to the 'blowing down' formula [**6**], every holomorphic section of the anti-canonical bundle of (M, J_1) is the pull-back of an anti-canonical section of the blown down surface, and vanishes at the blown up point; the same holds for (M, J_2). This completes the proof of Theorem 1.

3. *Bihermitian structures and symplectic forms*

Recall that a holomorphic symplectic structure on a complex surface (M, J) is a nowhere-vanishing holomorphic (complex) 2-form Ω. It is well known that the complex structure J is then determined by the real and the imaginary parts of the 2-form Ω; more precisely, we have the following.

LEMMA 5 [**18**, **14**]. *Let M be an oriented 4-manifold and (Φ_1, Φ_2) be a pair of non-degenerate real 2-forms on M satisfying the conditions*

$$\Phi_1 \wedge \Phi_1 = \Phi_2 \wedge \Phi_2, \qquad \Phi_1 \wedge \Phi_2 = 0. \tag{18}$$

Then there is a unique almost-complex structure J on M such that the 2-form $\Omega = \Phi_1 - i\Phi_2$ is of type $(2, 0)$ with respect to J. If moreover Φ_1 and Φ_2 are closed, then J is integrable and Ω defines a holomorphic symplectic structure on (M, J).

Proof. Since Φ_1 and Φ_2 are non-degenerate, we define an automorphism J on M by putting $\Phi_2(\cdot, \cdot) = \Phi_1(J\cdot, \cdot)$. The conditions (18) imply that $J^2 = -\mathrm{Id}$, that is, J is an almost-complex structure and $\Omega := \Phi_1 - i\Phi_2$ is a $(2, 0)$-form with respect to J; moreover, J is clearly the only almost-complex structure such that Ω is of type $(2, 0)$ with respect to J. If Φ_1 and Φ_2 are closed, then J is integrable. Indeed, fix any conformal class c, compatible with J. Since Ω is of type $(2, 0)$ with respect to J, the forms Φ_1 and Φ_2 are both self-dual with respect to c and have the same, non-vanishing, norm, by (18). We can then choose a metric g in c such that Φ_1 and Φ_2 are the Kähler forms of two anti-commuting almost Kähler structures, J_1 and J_2. Then, according to [**20**] we infer that $J = J_1 \circ J_2$ is integrable.

Proof of Theorem 2. (i) Let M be either a 4-dimensional torus or a K3 surface and let (c, J_1, J_2) be a bihermitian structure on M. According to Theorem 1, (c, J_1, J_2) is a strongly bihermitian structure. It follows that the 2-forms Φ_0, Φ_1, Φ_2 defined in the previous section are well defined on M. Recall that Φ_0, Φ_1, Φ_2 are the Kähler forms of the almost-complex structures I_0, I_1, I_2 defined by (5) with respect to some metric g in the conformal class c. If g is chosen to be the common standard metric of J_1 and J_2 (see Lemma 4), then Lemma 2 simply reads

$$\tau = d \ln q.$$

It follows that $q^{-1}\Phi_0$, $q^{-1}\Phi_1$, $q^{-1}\Phi_2$ are harmonic self-dual 2-forms, or equivalently, $\Omega_1 = q^{-1}(\Phi_0 - i\Phi_1)$ and $\Omega_2 = q^{-1}(\Phi_0 - i\Phi_2)$ are holomorphic 2-forms with respect to J_1 and J_2 respectively. Condition (c) in Theorem 2 then follows from (7) and (8).

(ii) Consider the triple of 2-forms

$$\left\{\Phi_0, \Phi_1, \Phi_2' = \frac{1}{\sqrt{1-p^2}}\,(p\Phi_1 + \Phi_2)\right\}.$$

Any two of these three forms satisfy the conditions (18) of Lemma 5. We thus get three almost-complex structures, say I_0, I_1, J_1, corresponding to the pairs $\{\Phi_1, \Phi_2'\}$, $\{\Phi_2', \Phi_0\}$ and $\{\Phi_0, \Phi_1\}$ respectively, where J_1 is integrable. It is easy to check that the almost-complex structures I_0, I_1, J_1 pairwise anticommute, and hence determine a conformal class, c, defined by decreeing that the frame $\{X, I_0 X, I_1 X, J_1 X\}$ is c-orthonormal for any non-vanishing vector field X. By the definition of c, the forms Φ_0, Φ_1, Φ_2' are c-self-dual, and so are the forms Φ_0, Φ_1, Φ_2. It follows that the *integrable* almost-complex structure, J_2, determined by the pair $\{\Phi_0, \Phi_2\}$ via Lemma 5, is also compatible with c. We thus get a conformal bihermitian structure, actually a strongly bihermitian one since $|p| < 1$. The last statement is clear.

4. *Construction of bihermitian structures*

With Theorem 2 to hand, we easily provide a number of examples of strongly bihermitian surfaces. First, we recover Kobak's construction for $\mathbb{R}^4$ and T^4 as follows.

4.1. *Kobak's examples on $\mathbb{R}^4$ and T^4*

Let g_1 be the flat metric of $\mathbb{R}^4$ and let $\{\omega_I, \omega_J, \omega_K\}$ and $\{\omega_{\overline{I}}, \omega_{\overline{J}}, \omega_{\overline{K}}\}$ be the Kähler forms of the canonical flat hyperkähler structure on $(\mathbb{C}^2, g_1)$ and $(\overline{\mathbb{C}}^2, g_1)$ respectively:

$$\begin{aligned} \omega_I &= \mathrm{Re}(dz_1 \wedge dz_2), & \overline{\omega}_I &= \mathrm{Re}(dz_1 \wedge d\bar{z}_2), \\ \omega_K &= \mathrm{Im}(dz_1 \wedge dz_2), & \overline{\omega}_K &= \mathrm{Im}(dz_1 \wedge d\bar{z}_2), \\ \omega_J &= \tfrac{1}{2} i (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), & \overline{\omega}_J &= \tfrac{1}{2} i (dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2), \end{aligned}$$

where (z_1, z_2) are the canonical coordinates of $\mathbb{C}^2$.

Consider the triple

$$\Phi_0 = \omega_I, \quad \Phi_1 = \omega_K, \quad \Phi_2 = \omega_J + p(\omega_K - \overline{\omega}_J), \tag{19}$$

where p is a smooth (real-valued) function on $\mathbb{C}^2$ satisfying $|p| < 1$. According to Theorem 2, the triple $\{\Phi_0, \Phi_1, \Phi_2\}$ determines a (strongly) bihermitian structure on $\mathbb{C}^2$ provided that Φ_2 is closed. This happens if and only if the function p is a solution of the following system:

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)p = \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)p = 0, \tag{20}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

If, moreover, the smooth function p is periodic with respect to all variables x_1, y_1, x_2, y_2, then $\{\Phi_0, \Phi_1, \Phi_2\}$ induce a (strongly) bihermitian structure on the 4-dimensional torus T^4. It is easily seen that the corresponding conformal structure c is determined by the metric $g_f = dz_1 \odot d\bar{z}_1 + f\, dz_2 \odot d\bar{z}_2$, where f is the smooth positive function $(1 - p)/(1 + p)$ (compare with [**16**]).

We may observe that, for some solutions p of (20), the corresponding bihermitian structures (c, J_1, J_2) admit an orientation-preserving isometry ϕ such that $\phi(J_1) = J_2$. Indeed, let $h(t)$ be an arbitrary smooth function on $\mathbb{R}$, satisfying $|h| < 1$. If we put $p = h(y_1 - y_2)$, then p is a solution of (20) and so determines a bihermitian conformal structure (c, J_1, J_2). Let $H(t)$ be a smooth function on $\mathbb{R}$ satisfying $dH(t)/dt = h(t)$ and consider the orientation-preserving diffeomorphism ϕ of $\mathbb{R}^4$ defined by

$$\phi(x_1, y_1, x_2, y_2) = (x_1, \, -y_2 + H(y_1 - y_2), x_2, y_1 - H(y_1 - y_2)).$$

It is easily checked that $\phi(\Phi_0) = \Phi_0$, $\phi(\Phi_1) = \Phi_2$ and $\phi(\Phi_2) = -\Phi_1$. In particular, ϕ preserves the conformal structure c and satisfies $\phi(J_1) = J_2$ and

$\phi(J_2) = -J_1$. We also notice that with respect to the metric g in c which makes Φ_0, Φ_1 and Φ_2 self-dual forms of square-norm 2, ϕ is an orientation-preserving isometry. If moreover H is a 2π-periodic function (take, for example, $H(t) = \frac{1}{2}\sin(t)$), we can perform the above construction on $T^4 = \mathbb{R}^4/2\pi\mathbb{Z}^4$.

4.2 *Strongly bihermitian structures as deformations of hyperkähler structures*

The main idea in this section, consisting in deforming a hyperkähler structure by a Hamiltonian vector field in order to get a family of strongly bihermitian structures via Theorem 2, is due to D. Joyce (private communication).

We start our construction with *any* hyperkähler structure on a 4-dimensional manifold M, determined by a Riemannian metric g_0, a triple of pairwise anticommuting complex structures, K_0, K_1, K_2, and the corresponding triple of Kähler forms, ω_0, ω_1, ω_2. Let ϕ_t be a 1-parameter family of symplectomorphisms of the (real) symplectic manifold (M, ω_0), with $\phi_0 = \mathrm{Id}$, and consider the triple of real symplectic 2-forms Φ_0, Φ_1, Φ_2 defined as follows:

$$\Phi_0 = \omega_0, \quad \Phi_1 = \omega_1, \quad \Phi_2 = \phi_t\,\omega_2. \tag{21}$$

We check easily that this triple satisfies the conditions (a), (b), (c) of Theorem 2 when t is small enough (the latter condition ensures that the function p_t defined by $\Phi_1 \wedge \Phi_2 = -p_t\,\Phi_0 \wedge \Phi_0$ satisfies $|p_t| < 1$). By Theorem 2, we thus obtain a strongly bihermitian structure (c, J_1, J_2), where the complex structures J_1 and J_2 are related to the former ones, K_0, K_1, K_2, by

$$J_1 = -K_2, \quad J_2 = \phi_t^{-1} K_1, \tag{22}$$

whereas the angle function p_t is given by

$$p_t = -\tfrac{1}{4}\,\mathrm{trace}\,(K_2 \circ (\phi_t^{-1} K_1)). \tag{23}$$

We consider the case when $\phi_t = \exp(tV)$, where $V = K_0(\mathrm{grad}\, f)$ is a Hamiltonian vector field with respect to ω_0 (here, f is a real function and $\mathrm{grad}\, f$ is the gradient of f with respect to the hyperkähler metric g_0).

According to the last statement of Theorem 2, we have to show that f can be chosen so that the angle function p_t is non-constant for some t. This follows from the following lemma, where Δ denotes the Laplacian with respect to g_0.

LEMMA 6. *The derivative of p_t with respect to t is given by*

$$\left.\frac{dp_t}{dt}\right|_{t=0} = \tfrac{1}{2}\Delta f. \tag{24}$$

In particular, if Δf is not constant on M, then p_t is non-constant for infinitely many (small) values of t.

Proof. Equation (24) is an easy consequence of the expression (23) of p_t. The last statement follows immediately.

PROPOSITION 3. (i) *Any hyperkähler structure on any* K3 *surface or any complex torus can be deformed as above into a non-ASD strongly bihermitian structure.*

(ii) *Any hyperhermitian structure on a Hopf surface can be deformed into a non-ASD strongly bihermitian structure.*

Proof. (i) It only remains to prove that the strongly bihermitian structure obtained above, when deforming any hyperkähler structure by any *non-constant* function f, is not anti-self-dual. This follows from the fact that any ASD bihermitian structure on a K3 surface or a complex torus is actually (conformally) hyperkähler; see [**19**].

(ii) Let $M = (\mathbb{C}^2 - \{(0,0)\})/\Gamma$ be any Hopf surface admitting a hyperhermitian structure. Then, the induced hyperhermitian structure on the universal covering $(\mathbb{C}^2 - \{(0,0)\})$ is conformal to a flat hyperkähler structure $(g_0, K_0, K_1, K_2; \omega_0, \omega_1, \omega_2)$. Choose any smooth function f on $(\mathbb{C}^2 - \{(0,0)\})$ that scales under Γ so that its symplectic gradient with respect to ω_0 is Γ-invariant (take any smooth function on M, viewed as a Γ-invariant function on $(\mathbb{C}^2 - \{(0,0)\})$, and multiply it by the square of the distance to the origin). By deforming the flat hyperkähler structure by f as above, we get a (conformal) strongly bihermitian structure on $(\mathbb{C}^2 - \{(0,0)\})$, which clearly descends to M. The latter is non-ASD as soon as f is chosen so that Δf is not constant, where Δ is the Laplacian of the flat hyperkähler metric. Indeed, any ASD bihermitian structure on a Hopf surface is actually hyperhermitian and lifts to a (conformal) hyperkähler structure on $(\mathbb{C}^2 - \{(0,0)\})$; see [**19**].

4.3. *Proof of Proposition* 1

By [**8**], each hyperhermitian structure on a compact complex surface belongs to one of the three types considered in Proposition 3. Then, Proposition 1 follows directly from the latter.

5. *Complex surfaces with odd first Betti number*

In this section, we consider the 'odd case', where the (connected) 4-manifold M is compact, with odd first Betti number, endowed with a bihermitian structure (c, J_1, J_2), and we focus our attention on the (closed) 1-form $\theta_1 + \theta_2$, where θ_1 and θ_2 are the Lee forms of J_1 and J_2 with respect to some metric g in the conformal class c. Recall that this 1-form vanishes identically when M is of Kähler type and g is the (common) standard metric of J_1 and J_2; see Lemma 4. This is no longer true in general in the odd case. Indeed, consider the two types of ASD bihermitian structures on a Hopf surface $S^1 \times S^3$ admitting two distinct hyperhermitian structures $(c, \mathscr{F})$ and $(c, \mathscr{F}')$ as in § 1. Then, if g is the standard metric of $S^1 \times S^3$, the common Lee form θ' of $(g, \mathscr{F}')$ is equal to $-\theta$, where θ is the common Lee form of $(g, \mathscr{F})$ (see [**12**]). It follows that, if J_1 and J_2 both belong to $\mathscr{F}$, we have $\theta_1 + \theta_2 = 2\theta \neq 0$; whereas, if J_1 and J_2 belong to distinct families, we have $\theta_1 + \theta_2 = 0$.

As a matter of fact, the only examples known at present where the relation $\theta_1 + \theta_2 = 0$ is satisfied for a bihermitian structure on a compact 4-manifold with odd first Betti number are *anti-self-dual*, namely when either (M, c) is of the latter type or when c is one of the anti-self dual metrics on $M = (S^1 \times S^3) \# n\, \overline{\mathbb{C}P^2}$ constructed by C. LeBrun, and J_1 and J_2 are the corresponding compatible complex structures; see [**19**].

The following proposition shows that the relation $\theta_1 + \theta_2 = 0$ never holds in the odd case for *strongly* bihermitian structures.

Let $\mathscr{D}^+$ and $\mathscr{D}^-$ be the complex curves (for both J_1 and J_2) defined by

$$\mathscr{D}^+ = p^{-1}(1) = \{x \in M \colon J_1(x) = J_2(x)\}$$

and

$$\mathscr{D}^- = p^{-1}(-1) = \{x \in M\colon J_1(x) = -J_2(x)\},$$

where, we recall, $p = -\frac{1}{4}\,\mathrm{trace}(J_1 \circ J_2)$ is the angle function. We then have the following.

PROPOSITION 4. *Let (M, c, J_1, J_2) be a compact bihermitian surface and suppose that there is a metric g in the conformal class c such that $\theta_1 + \theta_2 = 0$. Then the first Betti number of M is odd if and only if the complex curves $\mathscr{D}^+$ and $\mathscr{D}^-$ are both non-empty.*

Proof. Denote by $\mathscr{U}^+$ and $\mathscr{U}^-$ the open sets where $J_1 \neq J_2$ and $J_1 \neq -J_2$ respectively. Keeping the notation of § 2, we consider the self-dual 2-forms

$$\omega_+(\,\cdot\,,\cdot\,) = F_1(\,\cdot\,,\cdot\,) - \frac{1}{1-p}\,\Phi(J_1\,\cdot\,,\cdot\,)$$

and

$$\omega_-(\,\cdot\,,\cdot\,) = F_1(\,\cdot\,,\cdot\,) + \frac{1}{1+p}\,\Phi(J_1\,\cdot\,,\cdot\,),$$

defined on $\mathscr{U}^+$ and $\mathscr{U}^-$ respectively. By using formulae appearing in the proof of Lemma 2, one can easily check that we have $\delta\omega_+ = -\frac{1}{2}\omega_+(\theta_1 + \theta_2)$ and $\delta\omega_- = -\frac{1}{2}\omega_-(\theta_1 + \theta_2)$ on $\mathscr{U}^+ \cap \mathscr{U}^-$. According to [**19**, Proposition 1.3], $\mathscr{U}^+ \cap \mathscr{U}^-$ is a dense subset of $\mathscr{U}^+$ and $\mathscr{U}^-$, so that ω_+ and ω_- are harmonic self-dual 2-forms with respect to the chosen metric g on $\mathscr{U}^+$ and $\mathscr{U}^-$ respectively. Suppose now that $\mathscr{D}^+$ is empty, that is, $\mathscr{U}^+ = M$. Then the harmonic self-dual 2-form ω_+ has non-zero trace, and hence the first Betti number of M is even by [**11**, Proposition II.4].

Conversely, suppose that the first Betti number is even and let $\mathscr{D}^+$ be non-empty. We have to prove that $\mathscr{D}^-$ is empty. Let g be the standard metric of c and consider a harmonic self-dual form ω of trace equal to 1 with respect to (g, J_1) [**11**, Proposition II.4], that is, $\omega = F_1 + \mathrm{Re}(\alpha_1)$, where F_1 is the Kähler form of (g, J_1) and α_1 is a section of $\Lambda^{0,2}_{J_1}M$. Since the metric g is also standard for (c, J_2) (Lemma 4), we have $\omega = \lambda F_2 + \mathrm{Re}(\alpha_2)$, where λ is a constant, F_2 is the Kähler form of (g, J_2) and α_2 is a section of $\Lambda^{0,2}_{J_2}M$. We claim that $\lambda = 1$. Indeed, for any point $x \in \mathscr{D}^+$ we have

$$\omega(x) = F_1(x) + \mathrm{Re}(\alpha_1(x)) = \lambda F_1(x) + \mathrm{Re}(\alpha_2(x));$$

hence $\lambda = 1$, that is, $F_1 + \mathrm{Re}(\alpha_1) = F_2 + \mathrm{Re}(\alpha_2)$. The latter equality shows that there are no points where $J_1 = -J_2$, that is, $\mathscr{D}^-$ is empty.

COROLLARY 2. *Let (M, c, J_1, J_2) be a compact bihermitian surface with odd first Betti number and assume that the conformal class c contains a metric g such that $\theta_1 + \theta_2 = 0$. Then (c, J_1, J_2) is* not *strongly bihermitian and (M, J_1) (equivalently (M, J_2)) is obtained from either a Hopf surface or a parabolic Inoue surface by blowing up points along a unique smooth elliptic curve. In particular, M is diffeomorphic to $(S^1 \times S^3) \# k\,\overline{\mathbb{C}P^2}$ for some $k \geqslant 0$.*

Proof. The proof of Proposition 1 shows that if a bihermitian surface (M, c, J_1, J_2) with odd first Betti number satisfies $\theta_1 + \theta_2 = 0$ for some metric g

in c, then either (M, J_1) (equivalently (M, J_2)) has a trivial (anti-)canonical bundle (that is, (M, J_1) is a primary Kodaira surface, see [**6**]), or else it has an effective anti-canonical divisor, so that (M, J_1) belongs to Class VII in Kodaira's classification. It is easily seen that a primary Kodaira surface (and hence any Kodaira surface) M admits no bihermitian structures at all. Indeed, let (c, J_1, J_2) be such a structure. Since the Kodaira dimension is an oriented smooth invariant (see [**15**]), both (M, J_1) and (M, J_2) are primary Kodaira surfaces, and hence have trivial canonical bundles. There thus exist non-vanishing J_1- and J_2-holomorphic 2-forms, Ω_1 and Ω_2. Since J_1 and J_2 are both compatible with the conformal structure c, the real and the imaginary parts of Ω_1 and Ω_2 define four real harmonic self-dual 2-forms (with respect to c). Since J_1 and J_2 are independent at some point, they span a subspace of $H^2(M, \mathbb{R})$ of dimension at least 3. This contradicts the fact that $b_2^+(M) = 2$. It follows that both (M, J_1) and (M, J_2) belong to Class VII and each has an effective anti-canonical divisor. Moreover, by Proposition 4, $\mathscr{D}^+$ *and* $\mathscr{D}^-$ are non-empty. We then conclude as in [**19**] (see, in particular, the proofs of Proposition 3.11, Corollary 3.12, Corollary 3.14, Theorem 4.1, Proposition 5.1 and Theorem 5.2 in [**19**]).

6. *The angle-function of a bihermitian structure*

LEMMA 7. *Let (M, c, J_1, J_2) be a bihermitian surface. Let p be the angle-function defined by $p = -\frac{1}{4}\,\mathrm{trace}(J_1 \circ J_2)$. Then, the differential dp and the Laplacian Δp with respect to some metric g in c are given by*

$$dp = \tfrac{1}{4}[J_1, J_2](\theta_1 - \theta_2) \tag{25}$$

and

$$\Delta p = -\tfrac{1}{2} g(dp, \theta_1 + \theta_2) + \tfrac{1}{2} p|\theta_1 - \theta_2|^2 + g([J_1, J_2], d\theta_1), \tag{26}$$

where, we recall, $[J_1, J_2] = J_1 \circ J_2 - J_2 \circ J_1$.

Proof. We first observe that $p = \frac{1}{2} g(J_1, J_2)$, for any metric g in c, so that $dp(X) = \frac{1}{2}(g(\nabla_X J_1, J_2) + g(J_1, \nabla_X J_2))$ for any vector field X. Since J_1 and J_2 are both integrable, we have

$$(\nabla_X J_1)(Y) = \tfrac{1}{2}(g(X, Y)J_1\theta_1 + F_1(X, Y)\theta_1 + \theta_1(J_1 Y)X - \theta_1(Y)J_1 X), \tag{27}$$

and

$$(\nabla_X J_2)(Y) = \tfrac{1}{2}(g(X, Y)J_2\theta_2 + F_2(X, Y)\theta_2 + \theta_2(J_2 Y)X - \theta_2(Y)J_2 X), \tag{28}$$

where F_1 and F_2 are the Kähler forms of (g, J_1) and (g, J_2) respectively. Now we easily compute

$$g((\nabla_X J_1), J_2) + g(J_1, (\nabla_X J_2)) = \tfrac{1}{2} g([J_1, J_2](\theta_1 - \theta_2), X),$$

and (25) follows. To obtain (26), we have to calculate $\delta([J_1, J_2](\theta_1 - \theta_2))$. Identifying 2-forms and skew-symmetric endomorphisms by g, we have

$$\delta([J_1, J_2](\theta_1 - \theta_2)) = -(\delta[J_1, J_2])(\theta_1 - \theta_2) + 2g([J_1, J_2], d(\theta_1 - \theta_2)). \tag{29}$$

By using (27), (28) and (2), we get

$$\begin{aligned}\delta(J_1 \circ J_2) &= J_1(\delta J_2) - \sum_{i=1}^{4} (\nabla_{e_i} J_1)(J_2 e_i) \\ &= -(J_1 \circ J_2)\theta_2 - \tfrac{1}{2}(4p\theta_1 + (J_1 \circ J_2 + J_2 \circ J_1)(\theta_1)) \\ &= -(J_1 \circ J_2)\theta_2 - p\theta_1.\end{aligned}$$

Hence

$$\begin{aligned}\delta[J_1, J_2] &= (J_2 \circ J_1)\theta_1 - (J_1 \circ J_2)\theta_2 - p(\theta_1 - \theta_2) \\ &= -\tfrac{1}{2}[J_1, J_2](\theta_1 + \theta_2) - 2p(\theta_1 - \theta_2). \qquad (30)\end{aligned}$$

By using (3) and substituting (30) into (29) we eventually obtain (26).

As a consequence of the above lemma we get the following.

COROLLARY 3. *Let (M, c, J_1, J_2) be a compact strongly bihermitian surface. Then the positive Weyl tensor vanishes at some point of M.*

Proof. It follows from [**3**, Lemma 2 and (13)] that at any point where W^+ does not vanish we have

$$J_2 \circ J_1 = \varepsilon\left[\frac{3\lambda_0}{(\lambda_+ - \lambda_-)}\,\mathrm{Id} + \frac{1}{(\lambda_+ - \lambda_-)}(d\theta_1)_+\right],$$

$$J_1 \circ J_2 = \varepsilon\left[\frac{3\lambda_0}{(\lambda_+ - \lambda_-)}\,\mathrm{Id} + \frac{1}{(\lambda_+ - \lambda_-)}(d\theta_2)_+\right],$$

where $\lambda_+ \geqslant \lambda_0 \geqslant \lambda_-$ are the eigenvalues of W^+ and the sign $\varepsilon \in \{-1, 1\}$ depends on the point. We thus obtain

$$(d\theta_1)_+ = \varepsilon\,\frac{(\lambda_+ - \lambda_-)}{2}[J_1, J_2], \qquad (31)$$

which clearly holds everywhere on M. In general, the sign ε in (31) is not constant, but in the case when (c, J_1, J_2) is strongly bihermitian, W^+ is degenerate if and only if it vanishes. Assume, for a contradiction, that W^+ is everywhere non-degenerate. Then, the right-hand side of (31) does not vanish and ε can be chosen equal to $+1$ or -1 everywhere, in fact equal to 1 by replacing J_1 by $-J_1$ if necessary. It then follows that (26) can be written as

$$\Delta p = -\tfrac{1}{2}g(dp, \theta_1 + \theta_2) + \tfrac{1}{2}p|\theta_1 - \theta_2|^2 + \tfrac{1}{2}(\lambda_+ - \lambda_-)|[J_1, J_2]|^2. \qquad (32)$$

Let $x_0 \in M$ be a point where the angle function p achieves its minimum. We thus have $dp(x_0) = 0$ and $\Delta p(x_0) \leqslant 0$. Since (c, J_1, J_2) is strongly bihermitian, $[J_1, J_2]$ is everywhere non-zero, and hence invertible. We then infer from (25) that θ_1 and θ_2 are equal at x_0, so that

$$(\lambda_+ - \lambda_-)|[J_1, J_2]|^2 \leqslant 0$$

at x_0, by (32). This contradicts the fact that $|[J_1, J_2]|$ and $\lambda_+ - \lambda_-$ are both (strictly) positive.

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