

SELF-DUAL EINSTEIN HERMITIAN FOUR MANIFOLDS

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ABSTRACT. We provide a local classification of self-dual Einstein Riemannian four manifolds admitting a positively oriented Hermitian structure and characterize those which carry a hyperhermitian, non-hyperkähler structure compatible with the negative orientation. We show that self-dual Einstein 4-manifolds obtained as quaternionic quotients of $\mathbb{H}P^2$ and $\mathbb{H}H^2$ are Hermitian.

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INTRODUCTION

This paper is concerned with oriented, four-dimensional Einstein manifolds which are *Hermitian*, i.e. admit a positively oriented (integrable) complex structure, and are *self-dual*, meaning that the anti-self-dual Weyl tensor W^- vanishes identically.

A Riemannian version of the celebrated Goldberg-Sachs theorem of General Relativity implies that a Riemannian Einstein 4-manifold locally admits a positively oriented Hermitian structure if and only if the *self-dual Weyl tensor* W^+ is *degenerate* [44, 12, 40, 2] ; this means that at any point of M at least two of the three eigenvalues of W^+ coincide, when W^+ is viewed as a symmetric traceless operator acting on the three-dimensional space of self-dual 2-forms.

Riemannian Einstein 4-manifolds whose self-dual Weyl tensor W^+ is degenerate have been much studied by A. Derdziński; in particular, we know from [23] that on each connected component W^+ either vanishes identically or else has no zero, i.e. has exactly two distinct eigenvalues at any point (one of them, say λ , is simple; the other one is of multiplicity 2, and therefore equals $-\frac{\lambda}{2}$ as W^+ is trace-free) ; in the latter case, any one of the two normalized generators of the simple eigenspace of W^+ is the Kähler form of an integrable Hermitian structure, say J , and the conformal metric $\bar{g} = |W^+|^{\frac{2}{3}}g$ is *Kähler* with respect to J ; if, moreover, g is *self-dual* the simple eigenvalue λ of W^+ is constant (equivalently, the norm $|W^+|$ is constant) if and only if (M, g) is locally symmetric, i.e., a real or a complex space form.

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From this and from general properties of the Bach tensor of Kähler surfaces, we eventually obtain a natural bijection between the following three classes of Riemannian 4-manifolds :

1. Self-dual Einstein 4-manifolds with degenerate self-dual Weyl tensor W^+ , such that $|W^+|$ is not constant ;
2. Self-dual Einstein Hermitian 4-manifolds which are neither conformally-flat nor Kähler ;
3. Self-dual Kähler manifolds with nowhere vanishing and non-constant scalar curvature.

A precise statement and a proof are provided by Lemma 2 below. In this correspondence, Riemannian metrics are defined on the same manifold and belong to the same conformal class ; we observe that each class is defined by an algebraic closed condition (the vanishing of some tensors) and an open genericity condition.

Since the compact case is completely understood — see e.g. [20] or [23, 8, 34, 12, 1] for a classification — the paper will concentrate on the local situation.

The first known examples of (non-locally-symmetric) self-dual Einstein Hermitian metrics have been metrics of cohomogeneity one under the isometric action of a four-dimensional Lie group. Einstein metrics which are of cohomogeneity one under the action of a four-dimensional Lie group are automatically Hermitian [23]. By using this remark, A. Derdziński constructed [22] a family self-dual Einstein Hermitian metrics of cohomogeneity one under the action of $U(1,1)$ and $U(2)$; this family actually includes (in a rather implicit way) the well-known *LeBrun-Pedersen metrics* [37, 41] which play an important rôle in Section 3 of this paper.

It is *a priori* far from obvious that there are any other examples of self-dual Einstein Hermitian 4-manifolds, since the conditions of being self-dual, Einstein and Hermitian constitute an over-determined second order PDE system for the metric g . We show however that there are actually many of them; more precisely, we classify all local solutions of this system and provide a simple, *explicit* (local) Ansatz for self-dual Einstein Hermitian 4-manifolds (see Theorem 2 and Lemma 3 for a precise statement).

An amazing, *a priori* unexpected, fact comes out from the argument and explains *a posteriori* the integrability of the above mentioned system: *All self-dual Einstein Hermitian metrics admit a local isometric action of \mathbb{R}^2 with two-dimensional orbits* (Theorem 2 and Remark 3). In particular, these metrics locally fall into the more general context of self-dual metrics with torus action considered in [36] and, more recently, in [19, 15] (see Remark 3 (ii)).

It turns out that the property of having more (local) symmetries than expected is actually shared by Kähler metrics with vanishing Bochner tensor in all dimensions, as shown in the recent work of R. Bryant [13] (see [13] for precise statements). Since the Bochner tensor of a Kähler manifold of

real dimension four is the same as the anti-self-dual tensor W^- — so that Bochner-flat Kähler metrics are a natural generalization of self-dual Kähler metrics in higher dimensions — via the correspondence given by Lemma 2 Bryant’s work provides an alternative approach to our classification in Section 2.

The paper is organized as follows:

Section 1 displays the background material; the notation closely follows our previous work [2] — with the exception that the definition of the Lee form is here slightly different from the one in [2] — and we send back the reader to [2] for more details and references.

Section 2.1 provides a complete description of (locally defined) self-dual Einstein Hermitian metrics of cohomogeneity one (Theorem 1). It turns out that they all admit a local isometric action (with three-dimensional orbits) of certain four-dimensional Lie groups, in such a way that the metrics can be put in a *diagonal* form; in other words, these are *biaxial diagonal Bianchi metrics of type A*, see e.g. [47, 18]. Theorem 1 relies on the fact that every (non-locally-symmetric) self-dual Einstein Hermitian metric (g, J) has a distinguished non-trivial Killing field, namely $K = J\text{grad}_g(|W^+|^{-\frac{1}{3}})$, [23]. Then, Jones-Tod reduction with respect to K [49] provides a three-dimensional space of *constant curvature*. The diagonal form of the metrics follows from [49] and [47] (a unified presentation for these cohomogeneity-one metrics also appears in [18]). To the best of our knowledge, apart from these metrics no other examples of self-dual Einstein Hermitian metrics were known in the literature (see however Section 4).

Section 2.2 is devoted to the generic case, when the metric is neither locally-symmetric nor of cohomogeneity one. Our approach is similar to Armstrong’s one in [3]: When considering the Einstein condition alone, the Riemannian Goldberg-Sachs theorem together with Derdziński’s results reported above imply a number of relations for the 4-jet of an Einstein Hermitian metric (Sec. 2.1, Proposition 2); these happen to be the only obstructions for prolonging the 3-jet solutions of the problem to 4-jet and no further obstructions appear when reducing the equations for non-Kähler, non-anti-self-dual Hermitian Einstein 4-manifolds to a (simple) perturbed $SU(\infty)$ -Toda field equation [3, 43]. If, moreover, we insist that g be also *self-dual*, we find further relations for the 5-jet of the metric and we show that they have the form of an integrable closed Frobenius system of PDE’s for the parameter space of the 4-jet of the metric. We thus prove the local existence of non-locally symmetric and non-cohomogeneity-one self-dual Einstein Hermitian metrics (Theorem 2). It turns out that this Frobenius system can be explicitly integrated (Lemma 3). We thus obtain a uniform local description for *all* self-dual Einstein Hermitian metrics in an explicit way.

Section 3 is devoted to the subclass of self-dual Einstein Hermitian metrics which admit a compatible, non-closed, anti-self-dual hypercomplex structure. This is the same, locally, as the class of self-dual Einstein Hermitian metrics which admit a non-closed Einstein-Weyl connection (see Section 1.2). From this viewpoint, it is a particular case of four-dimensional conformal metrics admitting two distinct Einstein-Weyl connections. In our case, one of them is the Levi-Civita connection of the Einstein metric, whereas the other one is *non-closed*, hence attached to a non-closed hyperhermitian structure because of Proposition 3 (recall that a conformal 4-manifold admitting two distinct *closed* Einstein-Weyl structures is necessarily conformally flat (folklore), and that, conversely, every conformally flat 4-manifold only admits closed Einstein-Weyl structures [24], see also Proposition 3 and Corollary 1 below).

It turns out that self-dual Einstein Hermitian metrics which admit a compatible, non-closed, anti-self-dual hypercomplex structure, actually admit a second one and thus fall into the *bi-hypercomplex* situation described by Madsen in [39]; in particular, these metrics admit a local action of $U(2)$, with three-dimensional orbits, and are diagonal Bianchi XI metrics, see Theorem 3 below.

Notice that a general description of (anti-self-dual) metrics admitting two distinct compatible hypercomplex structures appears in [16], see also [5], whereas a family of self-dual Einstein metrics with compatible non-closed hyperhermitian structures, parameterized by holomorphic functions of one variable, has been constructed in [17].

In Section 4, we show that all anti-self-dual, Einstein four dimensional *orbifolds* obtained by quaternionic Kähler reduction from the eight dimensional quaternionic Kähler spaces $\mathbb{H}P^2$ and $\mathbb{H}H^2$ (see [25, 26] and [27]) are actually Hermitian with respect to the opposite orientation, hence locally isomorphic to metrics described in Section 2. These orbifolds include the *weighted projective planes* $\mathbb{C}P^{[p_1, p_2, p_3]}$ for integers $0 < p_1 \leq p_2 \leq p_3$ satisfying $p_3 < p_1 + p_2$, cf. [27, Sec. 4]. On these orbifolds, Bryant has constructed Bochner-flat Kähler metrics with everywhere positive scalar curvature [13, Sec. 4.3], hence also self-dual, Einstein Hermitian metrics according to Lemma 2 below; in view of the results of Section 2, Galicki-Lawson's and Bryant's metrics agree locally.

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1. EINSTEIN METRICS, HERMITIAN STRUCTURES AND EINSTEIN-WEYL GEOMETRY IN DIMENSION 4

1.1. Einstein metrics and compatible Hermitian structures. In the whole paper (M, g) denotes an oriented Riemannian four-dimensional manifold.

A specific feature of the four-dimensional Riemannian geometry is the splitting

$$(1) \quad AM = A^+M \oplus A^-M,$$

of the Lie algebra bundle, AM , of skew-symmetric endomorphisms of the tangent bundle, TM , into the direct sum of two Lie algebra subbundles, $A^\pm M$, derived from the Lie algebra splitting $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ of the orthogonal Lie algebra $\mathfrak{so}(4)$ into the direct sum of two copies of $\mathfrak{so}(3)$.

A similar decomposition occurs for the bundle $\Lambda^2 M$ of 2-forms

$$(2) \quad \Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

given by the spectral decomposition of the Hodge-star operator, $*$, whose restriction to $\Lambda^2 M$ is an involution; here, $\Lambda^\pm M$ is the eigen-subbundle for the eigenvalue \pm of $*$.

Both decompositions are actually determined by the conformal metric $[g]$ only. When g is fixed, $\Lambda^2 M$ is identified to AM by setting: $\psi(X, Y) = g(\Psi(X), Y)$, for any Ψ in AM and any vector fields X, Y ; then, we can arrange signs in (1) so that (1) and (2) are identified to each other. A similar decomposition and a similar identification occur for the bundle $\Lambda^2(TM)$ of bivectors.

Sections of $\Lambda^+ M$, resp. $\Lambda^- M$, are called *self-dual*, resp. *anti-self-dual*, and similarly for sections of AM or $\Lambda^2(TM)$.

In the sequel, the vector bundles AM , $\Lambda^2 M$ and $\Lambda^2(TM)$ will be freely identified to each other; similarly, the cotangent bundle T^*M will be freely identified to TM ; when no confusion can arise, the inner product determined by g will be simply denoted by (\cdot, \cdot) ; we adopt the convention that $(\Psi_1, \Psi_2) = -\frac{1}{2}\text{tr}(\Psi_1 \circ \Psi_2)$, for sections of AM , and the corresponding convention for $\Lambda^2 M$ and $\Lambda^2(TM)$.

The Riemannian curvature, R , is defined by $R_{X,Y} = D_{[X,Y]}^g - [D_X^g, D_Y^g]$, where D^g denotes the Levi-Civita connection of g ; R is thus a AM -values

2-form, but will be rather considered as a section of the bundle $S^2(\Lambda^2 M)$ of symmetric endomorphisms of $\Lambda^2 M$.

The Weyl tensor, W , commutes with $*$ and, accordingly, splits as $W = W^+ + W^-$, where $W^\pm = \frac{1}{2}(W \pm W \circ *)$; W^+ is called the *self-dual Weyl tensor*; it acts trivially on $\Lambda^- M$ and will be considered in the sequel as a field of (symmetric, trace-free) endomorphisms of $\Lambda^+ M$; similarly, the *anti-self-dual Weyl tensor* W^- will be considered as a field of endomorphisms of $\Lambda^- M$.

The Ricci tensor, Ric , is the symmetric bilinear form defined by $\text{Ric}(X, Y) = \text{tr}\{Z \rightarrow R_{X,Z}Y\}$; alternatively, $\text{Ric}(X, Y) = \sum_{i=1}^4 (R_{X,e_i}Y, e_i)$ for any g -orthonormal basis $\{e_i\}$. We then have $\text{Ric} = \frac{s}{4}g + \text{Ric}_0$, where s is the scalar curvature (= the trace of Ric with respect to g) and Ric_0 is the *trace-free Ricci tensor*. The latter can be made into a section of $S^2(\Lambda^2 M)$, then denoted by $\widetilde{\text{Ric}}_0$, by putting $\widetilde{\text{Ric}}_0(X \wedge Y) = \text{Ric}_0(X) \wedge Y + X \wedge \text{Ric}_0(Y)$.

It is readily checked that $\widetilde{\text{Ric}}_0$ satisfies the first Bianchi identity, i.e. $\widetilde{\text{Ric}}_0$ is a tensor of the same kind as R itself, as well as W^+ and W^- ; moreover, $\widetilde{\text{Ric}}_0$ anti-commutes with $*$, so that it can be viewed as a field of homomorphisms from $\Lambda^+ M$ into $\Lambda^- M$, or from $\Lambda^- M$ into $\Lambda^+ M$ (adjoint to each other); we eventually get the well-known Singer-Thorpe decomposition of R , see e.g. [7]:

$$(3) \quad R = \frac{s}{12} \text{Id}_{|\Lambda^2 M} + \frac{1}{2} \widetilde{\text{Ric}}_0 + W^+ + W^-,$$

or, in a more pictorial way

$$R = \begin{pmatrix} W^+ + \frac{s}{12} \text{Id}_{|\Lambda^+ M} & \frac{1}{2} \widetilde{\text{Ric}}_0|_{\Lambda^- M} \\ \frac{1}{2} \widetilde{\text{Ric}}_0|_{\Lambda^+ M} & W^- + \frac{s}{12} \text{Id}_{|\Lambda^- M} \end{pmatrix}$$

The metric g is *Einstein* if $\text{Ric}_0 = 0$ (equivalently, g is Einstein if R commutes with $*$).

The metric g (or rather the conformal class $[g]$) is *self-dual* if $W^- = 0$; *anti-self-dual* if $W^+ = 0$.

An *almost-complex structure* J is a field of automorphisms of TM of square $-\text{Id}_{|TM}$. An *integrable* almost-complex structure is simply called a *complex structure*.

In this paper, the metric g , or its conformal class $[g]$, is fixed and we only consider g -orthogonal almost-complex structures, i.e. almost-complex structure J satisfying the identity $g(JX, JY) = g(X, Y)$, so that the pair (g, J) is an *almost-Hermitian structure*; then, the associated bilinear form, F , defined by $F(X, Y) = g(JX, Y)$ is a 2-form, called the *Kähler form*.

The pair (g, J) is *Hermitian* if J is integrable; it is *Kähler* if J is parallel with respect to the Levi-Civita connection D^g ; if (g, J) is Kähler then J is integrable and F is closed; conversely, these two conditions together imply that (g, J) is Kähler.

A g -compatible almost-complex structure J is either a section of A^+M or a section of A^-M ; it is called *positive*, or *self-dual*, in the former case, *negative*, or *anti-self-dual* in the latter case. Alternatively, the Kähler form is either self-dual or anti-self-dual. Conversely, any section Ψ of A^+M , resp. A^-M , such that $|\Psi|^2 = 2$, is a positive, resp. negative, g -orthogonal almost-complex structure. It follows that any non-vanishing section, Ψ , of A^+M — if any — determines a (positive) almost-complex structure J , defined by $J = \sqrt{2} \frac{\Psi}{|\Psi|}$ (similarly for non-vanishing sections of A^-M).

Whereas the existence of a (positive) g -orthogonal almost-complex structure is a purely topological problem, the similar issue for *complex* structures heavily depends on the geometry of g , and this dependence is essentially measured by the self-dual Weyl tensor W^+ .

This assertion can be made more precise in the following way. Let's denote by $\lambda_+ \geq \lambda_0 \geq \lambda_-$ the eigenvalues of W^+ at some point, x , of M and assume that W^+ does not vanish at x ; equivalently, since W^+ is trace-free, assume that $\lambda_+ - \lambda_-$ is positive; we denote by F_+ an eigenform of W^+ with respect to λ_+ , normalized by $|F_+|^2 = 2$; similarly, F_- denotes an eigenform of W^+ for λ_- , again normalized by $|F_-|^2 = 2$; the *roots*, P , of W^+ at x are then defined by $P = \frac{(\lambda_+ - \lambda_0)^{\frac{1}{2}}}{(\lambda_+ - \lambda_-)^{\frac{1}{2}}} F_- + \frac{(\lambda_0 - \lambda_-)^{\frac{1}{2}}}{(\lambda_+ - \lambda_-)^{\frac{1}{2}}} F_+$; it is easily checked that this expression actually determine *two* distinct pairs of opposite roots in the generic case, when the eigenvalues are all distinct, and *one* pair in the degenerate case, when λ_0 is equal to either λ_+ or λ_- .

It is a basic fact that when J is a positive, g -orthogonal *complex* structure defined on M , the value of J at any point x where W^+ does not vanish must be equal to a root of W^+ at that point. This means that on the open subset of M where W^+ does not vanish, the conjugacy class of a positive, g -orthogonal complex structure — if any — is determined by g (in fact by $[g]$) with at most a 2-fold ambiguity.

On the other hand, it is an easy consequence of the integrability theorem Atiyah-Hitchin-Singer in [4] that A^+M can be locally trivialized by integrable (positive, g -orthogonal) almost-complex structures if and only if $[g]$ is anti-self-dual.

In the sequel, W^+ will be called *degenerate* at some point x if it has at most two distinct eigenvalues at that point. The terms *anti-self-dual* and *non-anti-self-dual* will be abbreviated as ASD and non-ASD respectively.

For a given non-ASD metric g it is a subtle question to decide whether the roots of W^+ actually provide complex structures (this is of course not true in general). The situation is quite different if g is Einstein. It is then settled by the following (weak) Riemannian version of the Goldberg-Sachs theorem, cf. [23, 44, 40, 2]:

Proposition 1. *Let (M, g) be an oriented Einstein 4-manifold; then the following three conditions are equivalent:*

- (i) W^+ is everywhere degenerate;

- (ii) *there exists a positive g -orthogonal complex structure in a neighbourhood of each point of M ;*
- (iii) *(M, g) is either ASD or W^+ has two and only two distinct eigenvalues at each point.*

A consequence of this proposition is that the self-dual Weyl tensor W^+ of a non-ASD Einstein Hermitian 4-manifold nowhere vanishes and has two distinct eigenvalues at any point, one simple, the other one of multiplicity 2; moreover, the Kähler form F is an eigenform of W^+ for the simple eigenvalue. Conversely, for any oriented, Einstein 4-manifold whose W^+ has only two distinct eigenvalues, the generator of the simple eigenspace of W^+ determines a (positive) Hermitian structure.

For any positive g -orthogonal almost-complex structure J , A^+M splits as follows:

$$(4) \quad A^+M = \mathbb{R} \cdot J \oplus A^{+,0}M,$$

where $\mathbb{R} \cdot J$ is the trivial subbundle generated by J and $A^{+,0}M$ is the orthogonal complement (equivalently, $A^{+,0}M$ is the subbundle of elements of A^+M that anticommute with J); $A^{+,0}M$ is a rank 2 vector bundle and will be also considered as a complex line bundle by putting $J\Phi = J \circ \Phi$. We have the corresponding decomposition

$$(5) \quad \Lambda^+M = \mathbb{R} \cdot F \oplus \Lambda^{+,0}M,$$

where $\Lambda^{+,0}M$ is the subbundle of J -anti-invariant 2-forms, i.e. 2-forms satisfying $\phi(JX, JY) = -\phi(X, Y)$; again, $\Lambda^{+,0}M$ is considered as a complex line bundle by putting $(J\phi)(X, Y) = -\phi(JX, Y) = -\phi(X, JY)$. As complex line bundles, both $A^{+,0}M$ and $\Lambda^{+,0}M$ are identified to the *anti-canonical bundle* $K^{-1}M = \Lambda^{0,2}M$ of the (almost-complex) manifold (M, J) .

For an Einstein, Hermitian 4-manifold, the action of W^+ preserves the decompositions (4) and (5).

1.2. The Lee form. The *Lee form* of an almost-Hermitian structure (g, J) is the real 1-form, θ , defined by

$$(6) \quad dF = -2\theta \wedge F;$$

equivalently, $\theta = -\frac{1}{2}J\delta F$, where δ denotes the co-differential with respect to g (here, and henceforth, the action of J on 1-forms is defined via the identification $T^*M \simeq TM$ given by the metric; we thus have $(J\alpha)(X) = -\alpha(JX)$, for any 1-form α). The reason for the choice of the factor -2 in (6) will be clear in the next subsection (notice that a different normalization is used in our previous work [2]).

When (g, J) is Hermitian, it is Kähler if and only if θ vanishes identically; it is conformally Kähler if and only if θ is exact, i.e. $\theta = -d \ln f$ for a positive smooth real function f (then, J is Kähler with respect to the conformal metric $g' = f^{-2}g$); it is locally conformally Kähler if and only if θ is closed, hence locally of the above type.

The Lee form clearly satisfies $(d\theta, F) = 0$; this means that the self-dual part, $d\theta^+$, of $d\theta$ is a section of the rank 2 subbundle, $\Lambda^{+,0}M$.

In the Hermitian case, $d\theta^+$ is an eigenform of W^+ for the mid-eigenvalue λ_0 ; moreover, $\lambda_0 = -\frac{\kappa}{12}$, where κ is the *conformal scalar curvature*, of which a more direct definition is given in the next subsection; κ is related to the (Riemannian) scalar curvature s by

$$(7) \quad \kappa = s + 6(\delta\theta - |\theta|^2),$$

and we also have

$$(8) \quad \kappa = 3(W^+(F), F),$$

see [52, 28]. Notice that, in the Hermitian case, the mid-eigenvalue λ_0 of W^+ is always a *smooth* function (this, however, is not true in general for the remaining two eigenvalues of W^+ , λ_+ and λ_- , which are given by:

$$\lambda_{\pm} = \frac{1}{24}\kappa \pm \frac{1}{8}(\kappa^2 + 32|d\theta^+|^2)^{\frac{1}{2}},$$

cf. [2]).

It follows that for Hermitian 4-manifolds the following three conditions are equivalent (cf. [10, 2]):

- (i) $d\theta^+ = 0$;
- (ii) W^+ is degenerate;
- (iii) F is an eigenform of W^+ .

(In the latter case F is actually an eigenform for the simple eigenvalue of W^+ , which is then equal to $\frac{\kappa}{6}$, also equal to λ_+ or λ_- according as κ is positive or negative). If, moreover, M is compact, any one of the above three conditions is equivalent to (g, J) being locally conformally Kähler; if, in addition, the first Betti number of M is even, (g, J) is then globally conformally Kähler [51].

By Proposition 1 we conclude that for every Einstein Hermitian 4-manifold, we have $d\theta^+ = 0$, i.e. $d\theta$ is self-dual. In fact, a stronger statement is true, see [2, Prop.1] and [23, Prop.4]:

Proposition 2. *Let (M, g, J) be an Einstein, non-ASD Hermitian 4-manifold. Then the conformal scalar curvature κ nowhere vanishes and the Lee form θ is given by $\theta = \frac{1}{3}d \ln |\kappa|$ (in particular, (g, J) is conformally Kähler).*

If, moreover, κ is not constant, i.e. if (g, J) is not Kähler, then $K = J \text{grad}_g(\kappa^{-\frac{1}{3}})$ is a non-trivial Killing vector field with respect to g , holomorphic with respect to J .

1.3. The canonical Weyl structure. Another specific feature of the four-dimensional geometry is that to each conformal Hermitian structure $([g], J)$ is canonically attached a unique *Weyl connection* D such that J is parallel with respect to D ; in other words, any Hermitian structure is “Kähler” in the extended context of Weyl structures (of course, (g, J) is Kähler in the

usual sense — the only one used in this paper — if and only if D is the Levi-Civita connection of some metric in the conformal class $[g]$.

Recall that, given a conformal metric $[g]$, a Weyl connection (with respect to $[g]$) is a torsion-free linear connection, D , on M which preserves $[g]$; the latter condition can be reformulated as follows: for any metric g in $[g]$, there exists a real 1-form θ_g such that $Dg = -2\theta_g \otimes g$; θ_g is called the *Lee form* of D with respect to g ; then, the Weyl connection D and the Levi-Civita connection D^g are related by $D = D^g + \tilde{\theta}_g$, meaning

$$(9) \quad D_X Y = D_X^g Y + \theta_g(X)Y + \theta_g(Y)X - g(X, Y)\theta_g^\sharp,$$

where θ_g^\sharp is the Riemannian dual of θ_g with respect to g . If $g' = f^{-2}g$ is another metric in $[g]$, the Lee form, $\theta_{g'}$, of D with respect to g' is related to θ_g by $\theta_{g'} = \theta_g + d \ln f$.

A Weyl connection D is the Levi-Civita connection of some metric in the conformal class $[g]$ if and only if its Lee form with respect to any metric g in $[g]$ is exact, i.e. $\theta_g = -d \ln f$; then, $D = D^{f^{-2}g}$; such a Weyl connection is called *exact*. More generally, a Weyl connection is said to be *closed* if its Lee form with respect to any metric in $[g]$ is closed; then, D is locally of the above type, i.e. locally the Levi-Civita connection of a (local) metric in $[g]$.

The definitions of the curvature R^D and the Ricci tensor Ric^D of a Weyl connection D are formally identical as the ones we gave for D^g (notice that the derivation of Ric^D from R^D requires no metric); however, R^D is now a $AM \oplus \mathbb{R}\text{Id}|_{TM}$ -valued 2-form, i.e. has a *scalar part* equal to $F^D \otimes \text{Id}|_{TM}$, where the real 2-form F^D , the so-called *Faraday tensor* of the Weyl connection, is equal to $-d\theta_g$ for any metric g in $[g]$; moreover, Ric^D is not symmetric in general: its skew-symmetric part is equal to $\frac{1}{2}F^D$; Ric^D is thus symmetric if and only if D is closed.

A Weyl connection D is called *Einstein-Weyl* if the symmetric, trace-free part of Ric^D vanishes; with respect to any metric g in $[g]$; writing θ instead of θ_g this conditions reads

$$(10) \quad D^g \theta - \theta \otimes \theta + \frac{1}{4}(\delta\theta + |\theta|^2)g - \frac{1}{2}d\theta - \frac{1}{2}\text{Ric}_0 = 0,$$

see e.g. [29]; for a fixed metric g , (10) should be considered as an equation for the unknown 1-form θ .

The *conformal scalar curvature* of D with respect to g , denoted by κ_g , is the trace of Ric^D with respect to g ; it is related to the (Riemannian) scalar curvature s by:

$$(11) \quad \kappa_g = s + 6(\delta\theta - |\theta|^2),$$

see e.g. [29].

A key observation is that the Lee form θ of an almost-Hermitian structure (g, J) is also the Lee form with respect to g of the Weyl connection canonically attached to the conformal almost-Hermitian structure $([g], J)$; in other words, the Weyl connection D defined by $D = D^g + \tilde{\theta}$ is actually independent

of g in its conformal class $[g]$. The Weyl connection D defined in this way is called the *canonical Weyl connection* of the (conformal) almost-Hermitian structure $([g], J)$.

The scalar curvature κ_g of D with respect to g is called the *conformal scalar curvature* of (g, J) ; it coincides with the function κ introduced in the previous paragraph.

The canonical Weyl connection is an especially interesting object when J is integrable, because of the following lemma:

Lemma 1. (i) J is integrable if and only if $DJ = 0$.

(ii) If J_1 and J_2 are two g -orthogonal complex structures, the corresponding canonical connections, D_1 and D_2 , coincide if and only if the scalar product (J_1, J_2) is constant.

Proof. (i) The condition $DJ = 0$ reads

$$(12) \quad D_X^g J = [X \wedge \theta, J];$$

this identity is proved e.g. in [28, 52].

(ii) Let p denote the *angle function* of J_1 and J_2 , defined by $p = -\frac{1}{4}\text{tr}(J_1 \circ J_2) = \frac{1}{2}(J_1, J_2)$; we then have

$$(13) \quad J_1 \circ J_2 + J_2 \circ J_1 = -2p \text{Id}|_{TM}.$$

Let θ_1 and θ_2 be the Lee forms of D_1, D_2 ; from (12) applied to J_1 , we infer $(D^g J_1, J_2) = ([J_1, J_2]X, \theta_1)$; similarly, we have $(D^g J_2, J_1) = ([J_2, J_1]X, \theta_2)$; putting together these two identities, we get

$$(14) \quad dp = -\frac{1}{2}[J_1, J_2](\theta_1 - \theta_2).$$

This obviously implies $dp = 0$ if $D_1 = D_2$; the converse is also true, as the commutator $[J_1, J_2]$ is invertible at each point where $J_2 \neq \pm J_1$. \square

1.4. Hypercomplex and Einstein-Weyl structures. An *almost-hypercomplex structure* on M is the datum of three almost-complex structures, I_1, I_2, I_3 , such that

$$I_1 \circ I_2 = -I_2 \circ I_1 = I_3.$$

Since M is four-dimensional any almost-hypercomplex structure I_1, I_2, I_3 determines a conformal class $[g]$ with respect to which each I_i is orthogonal: $[g]$ is defined by decreeing that, for any non-vanishing (local) vector field X , the frame $X, I_1 X, I_2 X, I_3 X$ is (conformally) orthonormal; for any g in the conformal class defined in this way we thus get an *almost-hyperhermitian structure* (g, I_1, I_2, I_3) ; notice that the I_i 's are pairwise orthogonal with respect to g , so that I_1, I_2, I_3 is a (normalized) direct orthonormal frame of A^+M ; conversely, for a given Riemannian metric g any (normalized) direct orthonormal frame of A^+M can be thought of as an almost-hypercomplex structure and, together with g form an almost-hyperhermitian structure.

An almost-hyperhermitian structure (g, I_1, I_2, I_3) is called *hyperhermitian* if all I_i 's are integrable; it is called *hyperkählerian* if the I_i 's are all parallel with respect to the Levi-Civita connection D^g .

In the hyperhermitian case the canonical Weyl connections, D_1, D_2, D_3 , of the almost-Hermitian structures $(g, I_1), (g, I_2), (g, I_3)$ are the same by Lemma 1; the common Weyl connection, D , is called the *canonical Weyl connection* of the hyperhermitian structure.

Conversely, the condition $D_1 = D_2 = D_3$ implies that (g, I_1, I_2, I_3) is hyperhermitian (this observation is due to S. Salamon and F. Battaglia, see e.g. [32]).

The canonical Weyl connection of a hyperhermitian structure (g, I_1, I_2, I_3) is closed if and only if I_1, I_2, I_3 is locally hyperkähler with respect to some (local) metric belonging to the conformal class $[g]$; for brevity, a hyperhermitian structure will be called *closed* or *non-closed* according as its canonical Weyl connection being closed or non-closed.

Remark 1. In general, for any given hypercomplex structure I_1, I_2, I_3 on a n -dimensional manifold, there exists a *unique* torsion-free linear connection on M that preserves the I_i 's, called the *Obata connection*; the canonical connection thus coincides with the Obata connection; for $n > 4$ however, there is no conformal metric canonically attached to I_1, I_2, I_3 and, in general, the Obata connection is not a Weyl connection.

If (g, I_1, I_2, I_3) is hyperhermitian, we have $DI_1 = DI_2 = DI_3 = 0$, where D is the canonical Weyl connection acting on sections of A^+M ; it follows that the connection of A^+M induced by D is *flat*; conversely, if D is a Weyl connection, whose induced connection on A^+M is flat, then A^+M can be locally trivialized by a D -parallel (normalized) orthonormal frame I_1, I_2, I_3 , which, together with g , constitute a hyperhermitian structure.

The curvature, R^{D, A^+M} , of the induced connection is given by $R_{X,Y}^{D, A^+M} \Psi = [R_{X,Y}^D, \Psi]$, where $R_{X,Y}^D$ is understood as a field of endomorphisms of TM — more precisely a section of $AM \oplus \mathbb{R} \text{Id}|_{TM}$ — and $[R_{X,Y}^D, \Psi]$ is the commutator of $R_{X,Y}^D$ and Ψ ; we easily infer that the vanishing of R^{D, A^+M} is equivalent to the following system of four conditions:

1. $W^+ = 0$;
2. $(F^D)^+ = 0$; if θ denotes the Lee form of D , this also reads $d\theta^+ = 0$;
3. D is Einstein-Weyl, i.e. the Lee form θ is a solution of (10);
4. The scalar curvature of D vanishes identically; in view of (11), this condition reads

$$(15) \quad s = 6(-\delta\theta + |\theta|^2).$$

It follows from the preceding discussion that for an ASD Riemannian 4-manifold the existence of a compatible hypercomplex structure is locally equivalent to the existence of an Einstein-Weyl connection satisfying the

above conditions 2 and 4 (cf. [42] or [32]). In this correspondence, conformally hyperkähler structures correspond to closed Einstein-Weyl structures. The existence of a non locally hyperkähler, hyperhermitian structure is actually (locally) equivalent to the existence of a non-closed Einstein-Weyl connection, in view of the following result of D. Calderbank:

Proposition 3. ([14]) *Let $(M, [g], D)$ be an anti-self-dual Einstein-Weyl 4-manifold. Then either D is closed, or else D satisfies conditions 2 and 4 above, i.e. is the canonical Weyl connection of a hyperhermitian structure.*

Notice that in the case when M is compact, $d\theta^+ = 0$ implies $d\theta = 0$, hence *any* hyperhermitian structure is locally conformally hyperkähler; a complete classification appears in [11].

2. SELF-DUAL EINSTEIN HERMITIAN 4-MANIFOLDS

By Proposition 2, a Hermitian, Einstein 4-manifold, whose self-dual Weyl tensor W^+ has constant eigenvalues is either anti-self-dual or Kähler-Einstein, [23]. If, moreover, the metric g is self-dual, the latter happens precisely when g is locally-symmetric, i.e. when (M, g) is a real or a complex space form, see [50]. More generally, a self-dual Einstein 4-manifold is locally-symmetric if and only if W^+ is degenerate, with constant eigenvalues, [23].

In the opposite case, we have the following lemma:

Lemma 2. *Non-locally-symmetric self-dual Einstein Hermitian metrics are in one-to-one correspondence with self-dual Kähler metrics of nowhere vanishing and non-constant scalar curvature.*

Proof. Every self-dual Einstein Hermitian 4-manifold (M, g, J) of non-constant curvature is conformally related (via Proposition 2) to a self-dual Kähler metric \bar{g} of nowhere vanishing scalar curvature. A self-dual Kähler metric is locally-symmetric if and only if its scalar curvature is constant [23]; thus, one direction in the correspondence of Lemma 2 follows by observing that \bar{g} is locally-symmetric as soon as g is. Since the Bach tensor of a self-dual metric vanishes [31], it follows from [23, Prop.4] that any self-dual Kähler metric of nowhere vanishing scalar curvature gives rise to an Einstein Hermitian metric in the same conformal class. \square

In the remainder of this section, (M, g, J) is an Einstein, self-dual Hermitian 4-manifold, and we assume that g is *not* locally-symmetric; in particular, W^+ is degenerate, but its eigenvalues, $\lambda, -\frac{\lambda}{2}$, or, equivalently, its norm $|W^+| = \sqrt{\frac{3}{2}}|\lambda|$, are not constant.

Since (M, g, J) is not Kähler (Proposition 2), by substituting to M the dense open subset where the Lee form θ does not vanish, we shall assume throughout this section that $D^g J$ nowhere vanishes, see (12).

For convenience, we choose a (local, normalized) orthonormal frame of $\Lambda^{+,0}M$ of the form $\{\phi, J\phi\}$, where $|\phi| = \sqrt{2}$; such a frame will be called a

gauge. Then, the triple $\{F, \phi, J\phi\}$ is a (local, normalized) direct orthonormal frame of Λ^+M .

Recall that by Proposition 1 we have

$$(16) \quad W^+(\psi) = -\frac{\kappa}{12}\psi,$$

for any section ψ of $\Lambda^{+,0}M$, whereas

$$(17) \quad W^+(F) = \frac{\kappa}{6}F.$$

With respect to the gauge $\{\phi, J\phi\}$, the covariant derivative D^gF is written as

$$(18) \quad D^gF = \alpha \otimes \phi + J\alpha \otimes J\phi,$$

where

$$(19) \quad \alpha = \phi(J\theta);$$

equivalently,

$$(20) \quad \phi = -\frac{1}{|\theta|^2}(\alpha \wedge J\theta + J\alpha \wedge \theta); \quad J\phi = \frac{1}{|\theta|^2}(\alpha \wedge \theta - J\alpha \wedge J\theta).$$

We also have

$$(21) \quad D^g\phi = -\alpha \otimes F + \beta \otimes J\phi; \quad D^g(J\phi) = -J\alpha \otimes F - \beta \otimes \phi,$$

for some real 1-form β .

From (18), we infer

$$\begin{aligned} (D^g)^2|_{\Lambda^2M}F &= (d\alpha + J\alpha \wedge \beta) \otimes \phi + (d(J\alpha) - \alpha \wedge \beta) \otimes J\phi \\ &= -R(J\phi) \otimes \phi + R(\phi) \otimes J\phi. \end{aligned}$$

Because of (16) this reduces to

$$(22) \quad \begin{cases} d\alpha - \beta \wedge J\alpha = \frac{(\kappa-s)}{12}J\phi \\ d(J\alpha) + \beta \wedge \alpha = -\frac{(\kappa-s)}{12}\phi. \end{cases}$$

Similarly, because of (17) we infer the following additional relation from (21):

$$(23) \quad d\beta + \alpha \wedge J\alpha = -\frac{(s+2\kappa)}{12}F.$$

Notice that real 1-forms α and β are both *gauge dependent*; if

$$\phi' = (\cos \varphi)\phi + (\sin \varphi)J\phi$$

they transform to

$$\alpha' = (\cos \varphi)\alpha + (\sin \varphi)J\alpha; \quad \beta' = \beta + d\varphi.$$

We next introduce real 1-forms $n_i, m_i, i = 1, 2$ by

$$(24) \quad D^g\theta = m_1 \otimes \theta + n_1 \otimes J\theta + m_2 \otimes \alpha + n_2 \otimes J\alpha.$$

By using (18) and (20) we derive

$$(25) \quad \begin{aligned} D^g(J\theta) &= -n_1 \otimes \theta + m_1 \otimes J\theta - (n_2 + J\alpha) \otimes \alpha + (m_2 + \alpha) \otimes J\alpha; \\ D^g\alpha &= -m_2 \otimes \theta + (n_2 + J\alpha) \otimes J\theta + m_1 \otimes \alpha - (n_1 - \beta) \otimes J\alpha; \\ D^g(J\alpha) &= -n_2 \otimes \theta - (m_2 + \alpha) \otimes J\theta + (n_1 - \beta) \otimes \alpha + m_1 \otimes J\alpha. \end{aligned}$$

A straightforward computation, using identities (22) and the fact that the vector field $K = (\kappa^{-\frac{1}{3}}J\theta)^\sharp_g$, the dual of $\kappa^{-\frac{1}{3}}J\theta$, is Killing (see Proposition 2), gives the following expressions for m_i and n_i :

$$(26) \quad \begin{cases} m_1 = m_0 + (p - \frac{(\kappa-s)}{24|\theta|^2} + \frac{1}{2})\theta \\ n_1 = Jm_0 + (p - \frac{(\kappa-s)}{24|\theta|^2} - \frac{1}{2})J\theta \\ m_2 = J\phi(m_0) - (p + \frac{(\kappa-s)}{24|\theta|^2} + \frac{1}{2})\alpha \\ n_2 = -\phi(m_0) - (p + \frac{(\kappa-s)}{24|\theta|^2} + \frac{1}{2})J\alpha, \end{cases}$$

where p is a smooth function, and m_0 is a 1-form which belongs to the distribution $\mathcal{D}^\perp = \text{span}\{\alpha, J\alpha\}$, the orthogonal complement of $\mathcal{D} = \text{span}\{\theta, J\theta\}$.

Since $m_1 = d \ln |\theta|$, the 1-form m_0 is nothing else than the projection of $d \ln |\theta|$ to the subbundle \mathcal{D}^\perp . Moreover, with respect to any gauge $\{\phi, J\phi\}$, we write

$$(27) \quad m_0 = q\alpha + rJ\alpha,$$

for some smooth functions q and r .

In view of (12), identities (24) and (26) are conditions on the 2-jet of J . Since J is completely determined by W^+ (see Proposition 1), these are the conditions on the 4-jet of the metric referred to in the introduction.

This completes the analysis of the Einstein condition and we are now going to see how the vanishing of W^- interacts on further jets of g .

For that, we introduce the ‘‘mirror frame’’ of Λ^-M :

$$\begin{aligned} \bar{F} &= -F + \frac{2}{|\theta|^2}\theta \wedge J\theta; \quad \bar{\phi} = \phi + \frac{2}{|\theta|^2}J\alpha \wedge \theta; \\ I\bar{\phi} &= J\phi + \frac{2}{|\theta|^2}J\alpha \wedge J\theta; \end{aligned}$$

here, I is the *negative* almost Hermitian structure of which the anti-self-dual 2-form \bar{F} is the Kähler form, equal to J on \mathcal{D} and to $-J$ on \mathcal{D}^\perp . By (25) and the fact that $\theta = \frac{d\kappa}{3\kappa}$, we obtain the following expression for the covariant derivative of the Killing vector field $K = (\kappa^{-\frac{1}{3}}J\theta)^\sharp_g$

$$(28) \quad D^gK = \kappa^{-\frac{1}{3}}|\theta|^2(q\bar{\phi} - rI\bar{\phi} - (p - \frac{1}{2})\bar{F} + \frac{(\kappa-s)}{24|\theta|^2}F).$$

Moreover, since K is Killing, we have

$$(29) \quad D_X^g\Psi = R(K, X),$$

where $\Psi = D^gK$.

By considering the ASD parts of both sides of (29), we infer that the condition $W^- = 0$ is equivalent to

$$(30) \quad D^g(\Psi^-) = \frac{s}{24}(\bar{\phi}(K) \otimes \bar{\phi} + I\bar{\phi}(K) \otimes I\bar{\phi} + IK \otimes \bar{F}),$$

where

$$\Psi^- = \kappa^{-\frac{1}{3}}|\theta|^2(q\bar{\phi} - rI\bar{\phi} - (p - \frac{1}{2})\bar{F})$$

is the ASD part of $\Psi = D^gK$, see (28). Furthermore, by (24) and (25) one gets

$$(31) \quad \begin{aligned} D^g\bar{F} &= -(2m_2 + \alpha) \otimes \bar{\phi} + (2Jm_2 + J\alpha) \otimes I\bar{\phi}; \\ D^g\bar{\phi} &= (2m_2 + \alpha) \otimes \bar{F} + (2n_1 - \beta) \otimes I\bar{\phi}; \\ D^gI\bar{\phi} &= -(2Jm_2 + J\alpha) \otimes \bar{F} - (2n_1 - \beta) \otimes \bar{\phi}. \end{aligned}$$

Keeping in mind that $\theta = \frac{d\kappa}{3\kappa}$ and $m_1 = d \ln |\theta|$, (30) then reduces to

$$(32) \quad \begin{aligned} dp &= -(p - \frac{1}{2})(2m_1 - \theta) + q(m_2 + \alpha) \\ &\quad + r(Jm_2 + J\alpha) - \frac{s}{24|\theta|^2}\theta \end{aligned}$$

$$(33) \quad \begin{aligned} dq &= -(p - \frac{1}{2})(m_2 + \alpha) - q(2m_1 - \theta) \\ &\quad - r(2n_1 - \beta) - \frac{s}{24|\theta|^2}\alpha \end{aligned}$$

$$(34) \quad \begin{aligned} dr &= -(p - \frac{1}{2})(Jm_2 + J\alpha) + q(2n_1 - \beta) \\ &\quad - r(2m_1 - \theta) - \frac{s}{24|\theta|^2}J\alpha. \end{aligned}$$

Now, taking into account (22) and (23), (32)–(34) constitute a closed differential system that a self-dual Einstein Hermitian metric must satisfy; by (22), (23), (25) and (26) one can directly check that the integrability conditions $d(dp) = d(dq) = d(dr) = 0$ are satisfied. This is a first evidence that the existence of self-dual Einstein Hermitian metrics with prescribed 4-jet at a given point can be expected. To carry out this program explicitly, we first consider the case when $q \equiv 0, r \equiv 0$ and show that it precisely corresponds to *self-dual Einstein Hermitian metrics of cohomogeneity one*.

2.1. Self-dual Einstein Hermitian metrics of cohomogeneity one. A Riemannian 4-manifold (M, g) is said to be (locally) *of cohomogeneity one*, if it admits a (local) isometric action of a Lie group G , with three-dimensional orbits. The manifold M is then locally a product

$$M \cong (t_1, t_2) \times G/H.$$

The metric g descends to a left invariant metric $h(t)$ on each orbit $\{t\} \times G/H$, and, by an appropriate choice of the parameter t , can be written as

$$g = dt \otimes dt + h(t).$$

If, moreover, (M, g) is Einstein and self-dual, and G is at least of dimension four, then, according to a result of A. Derdziński [23], the self-dual Weyl tensor W^+ of g is everywhere degenerate, and g is Hermitian with respect to some invariant complex structure.

Here is a way of constructing such metrics, all belonging to the class of *diagonal Bianchi metrics of type A* (see e.g. [47]). Let \tilde{G} be one of the following six three-dimensional Lie groups: \mathbb{R}^3 , Nil^3 , Sol^3 , $\text{Isom}(\mathbb{R}^2)$, $\text{SU}(1,1)$ or $\text{SU}(2)$; let H be a discrete subgroup of \tilde{G} and consider the family of diagonal metrics $h(t)$ of the form

$$(35) \quad h(t) = A(t)\sigma_1^2 + B(t)\sigma_2^2 + C(t)\sigma_3^2,$$

defined on \tilde{G}/H , where A, B, C are positive smooth functions, and σ_i are the standard left invariant generators of the corresponding Lie algebras; we thus have

$$d\sigma^1 = \epsilon_1\sigma_2 \wedge \sigma_3; \quad d\sigma_2 = -\epsilon_2\sigma_1 \wedge \sigma_3; \quad d\sigma_3 = \epsilon_3\sigma_1 \wedge \sigma_2$$

for a triple $(\epsilon_1, \epsilon_2, \epsilon_3)$, $n_i \in \{-1, 0, 1\}$, depending on the chosen group, according to the following table:

class	ϵ_1	ϵ_2	ϵ_3	\tilde{G}
I	0	0	0	\mathbb{R}^3
II	0	0	1	Nil^3
VI_0	1	-1	0	Sol^3
VII_0	1	1	0	$\text{Isom}(\mathbb{R}^2)$
VIII	1	1	-1	$\text{SU}(1, 1)$
IX	1	1	1	$\text{SU}(2)$

Except for Class VI_0 , when $A = B$, all these metrics admit a further (local) symmetry: it comes from the right action of the vector field dual to σ_3 for the metrics in the classes VII_0 , VIII and IX, or from the rotation in the $\{\sigma_1, \sigma_2\}$ -plane for the metrics in the classes I and II. We thus get the so-called *biaxial* Bianchi metrics, i.e. diagonal Bianchi metrics of Type A, admitting a local isometric action of a four-dimensional Lee group G , where G is $\text{U}(2)$, $\text{U}(1, 1)$, $\mathbb{R} \times \text{Isom}(\mathbb{R}^2)$ (corresponding to biaxial Class I and VI_0 metrics), or the non-trivial central extension of $\text{Isom}(\mathbb{R}^2)$ (corresponding to biaxial Class II metrics), see e.g. [18]. Clearly, any such metric admits a positive *and* a negative invariant Hermitian structure, J and I , whose Kähler forms are given by

$$F = \sqrt{C}dt \wedge \sigma_3 + A\sigma_1 \wedge \sigma_2,$$

and

$$\bar{F} = \sqrt{C}dt \wedge \sigma_3 - A\sigma_1 \wedge \sigma_2,$$

respectively. When imposing the Einstein and the self-duality conditions, we obtain an ODE system for the unknown functions A and C , which can be explicitly solved, cf. e.g. [41], [37], [21], [47], [18], [6].

In the sequel, we shall simply refer to these (self-dual, Einstein, Hermitian) metrics as *diagonal Bianchi* metrics.

Notice that 4-dimensional locally symmetric metrics, i.e. real and complex space forms, can also be written (in several ways) as diagonal Bianchi metrics. For example, all self-dual Einstein Hermitian metrics in Class VII₀ are real space forms, while those in Class I are flat, cf. [47].

Our next result shows that, apart from locally symmetric spaces, diagonal Bianchi metrics in the above sense are actually *all* (non-locally symmetric) cohomogeneity-one self-dual Einstein Hermitian metrics, and, in fact, can be characterized by the property $m_0 \equiv 0$ in the notation of the preceding section. More precisely, we have:

Theorem 1. *Let (M, g) be a self-dual Einstein 4-manifold. Suppose that (M, g) is not locally symmetric. Then the following three conditions are equivalent:*

- (i) (M, g) is locally of cohomogeneity one and the spectrum of W^+ is degenerate.
- (ii) (M, g) admits a local isometric action of a Lie group of dimension at least four, with three-dimensional orbits, and is locally isometric to a diagonal Bianchi self-dual Einstein Hermitian metric belonging to one of the classes II, VIII or IX.
- (iii) (M, g) admits a positive, non-Kähler Hermitian structure, J , and a negative Hermitian structure, I , such that I is equal to J on $\mathcal{D} = \text{span}\{\theta, J\theta\}$ and to $-J$ on the orthogonal complement \mathcal{D}^\perp ; equivalently, the 1-form m_0 of (g, J) vanishes identically.

Proof. (i) \Rightarrow (iii). By Propositions 1 and 2, W^+ has two distinct, non-constant eigenvalues at any point and there exists a positive, non-Kähler Hermitian structure J whose Kähler form F generates the eigenspace of W^+ corresponding to the simple eigenvalue. It follows that the Hermitian structure is preserved by the action of G , and therefore both functions $|D^g F|^2 = 2|\theta|^2$ and $|W^+|^2 = \frac{\kappa^2}{24}$ are constant along the orbits of G ; in particular, $d \ln |\theta|$ is colinear to $\theta = \frac{d\kappa}{3\kappa}$, at any point; this means that $m_0 = 0$; by (31) and (26), the vanishing of m_0 is equivalent to the integrability of the negative almost Hermitian structure I .

(iii) \Rightarrow (ii). If $m_0 \equiv 0$ or, equivalently, if the negative almost Hermitian structure I is integrable, then, by (31), the Lie form θ_I of (g, I) reads:

$$(36) \quad \theta_I = \left(2p + \frac{(\kappa - s)}{12|\theta|^2}\right)\theta.$$

According to (26) we also have $m_1 = d \ln |\theta| = \left(p - \frac{(\kappa - s)}{24|\theta|^2} + \frac{1}{2}\right)\theta$ and $\theta = \frac{1}{3}d \ln |\kappa|$; it follows that (locally) $\theta_I = df$ for a positive K -invariant function f , i.e., g is conformal to a Kähler metric $g' = f^2g$. Since $W^- = 0$, the Kähler metric g' is of zero scalar curvature. Clearly, the Killing field K

preserves both J and g , hence, also, the Kähler structure (g', I) . Two cases occur, according as g' is homothetic or not to g .

(a) Suppose g' is *not* homothetic to g ; equivalently, the scalar curvature s of g does not vanish; then, by [23], $K' = I \text{grad}_g(f^{-1})$ is a Killing vector field for g and g' and is holomorphic with respect I . By the very definition of I we have that $J|_{\mathcal{D}} = I|_{\mathcal{D}}$; the Killing vector fields K' and K are thus colinear everywhere (see (36)); it follows that K' is a constant multiple of K . By considering $z = f^2$ as a local coordinate on M and, by introducing a holomorphic coordinate $x + iy$ on the (locally defined) orbit-space for the holomorphic action of $K + \sqrt{-1}IK$ on (M, I) , the metric g can be written in the following form:

$$(37) \quad g = \frac{1}{z^2} [e^u w (dx^{\otimes 2} + dy^{\otimes 2}) + w dz^{\otimes 2} + w^{-1} \omega^2],$$

where $u(x, y, z)$ is a smooth function satisfying the $SU(\infty)$ Toda field equation:

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0,$$

w is a positive function given by

$$w = \frac{6(zu_z - 2)}{s},$$

and ω is a connection 1-form of the \mathbb{R} -bundle $M \mapsto N = \{(x, y, z)\} \subset \mathbb{R}^3$, whose curvature is given by

$$(38) \quad d\omega = -w_x dy \wedge dz - w_y dz \wedge dx - (we^u)_z dx \wedge dy,$$

(see e.g. [49]). Moreover, the Killing field K is dual to $\frac{1}{wz^2}\omega$, and the (anti-self-dual) Kähler form of the negative Hermitian structure I is given by

$$(39) \quad \bar{F} = \frac{1}{z^2} (we^u dx \wedge dy - dz \wedge \omega).$$

By (36) we have that $\mathcal{D} = \text{span}\{\theta, J\theta\} = \text{span}\{\theta_I, I\theta_I\} = \text{span}\{K^{\sharp_g}, IK^{\sharp_g}\}$, so that the Kähler form F of the positive Hermitian structure J is given by

$$(40) \quad F = \frac{1}{z^2} (we^u dx \wedge dy + dz \wedge \omega).$$

It is now easily seen that (39) and (40) simultaneously define integrable almost complex structures if and only if $w_x = w_y = 0$, or equivalently if and only if $u(x, y, z) = u_1(x, y) + u_2(z)$. This means that u is a *separable* solution to the $SU(\infty)$ Toda field equation. Up to a change of the holomorphic coordinate $x + iy$, it is explicitly given by [49]

$$e^u = \frac{4(c + bz + az^2)}{(1 + a(x^2 + y^2))^2},$$

for properly chosen constants a, b, c . Any such solution gives rise to a diagonal Bianchi self-dual Einstein Hermitian metric pertaining to one of classes I, II, VII₀, VIII and IX, depending on the choice of the constants a, b, c

(see e.g. [18, Sec. 8]). The case when $b = 0$ corresponds to Class I or VII₀ metrics; we then have that g' is the product metric of two Riemann surfaces, one of constant curvature a and another one of constant curvature $-a$; correspondingly, the Einstein metric g is conformally flat, i.e. g is a real space form, a contradiction. We thus conclude that g is in one of classes II, VIII or IX.

(b) If g' is homothetic to g , i.e. (g, I) is itself a Kähler structure of zero scalar curvature, then g is locally hyperkähler and K is a Killing vector field preserving the Kähler structure I . Then, one of the two following situations occurs:

(b1) K is triholomorphic, i.e. K preserves each Kähler structure in the hyperkähler family: Then the quotient space, N , for the (real) action of K is flat and is endowed with a field of parallel straight lines. This situation is described by the Gibbons-Hawking Ansatz [33], and the metric g has the form:

$$g = w(dx^{\otimes 2} + dy^{\otimes 2} + dz^{\otimes 2}) + \frac{1}{w}\omega^2,$$

for a positive harmonic function $w(x, y, z)$ on N and a 1-form ω on M satisfying

$$d\omega = -w_x dy \wedge dz - w_y dz \wedge dx - w_z dx \wedge dy.$$

The Killing field K is dual to $\frac{1}{w}\omega$ and one may consider that the positive and negative Hermitian structures, J and I , correspond to the 2-forms

$$F = w dx \wedge dy + dz \wedge \omega; \quad \bar{F} = w dx \wedge dy - dz \wedge \omega,$$

respectively. We again conclude that $w_x = 0, w_y = 0$, and therefore $w = az + b$. The case $a = 0$ corresponds to flat metrics in Class I, whereas, when $a \neq 0$, by putting $at = az + b, \sigma_1 = dx, \sigma_2 = dy, \sigma_3 = \omega$, the metric becomes a diagonal Bianchi metric of Class II.

(b2) K is not triholomorphic: Since, nevertheless, K preserves (g, I) , the metric g takes the form [9]

$$g = e^u w(dx^{\otimes 2} + dy^{\otimes 2}) + w dz^{\otimes 2} + w^{-1}\omega^2,$$

where $u(x, y, z)$ is a solution to the $SU(\infty)$ Toda field equation, $w = au_z$, ω satisfies (38) and a is a constant. Moreover, K is dual to $\frac{1}{w}\omega$, and I is defined by the anti-self-dual form

$$\bar{F} = we^u dx \wedge dy - dz \wedge \omega.$$

Similar arguments as above show that $w_x = w_y = 0$, i.e., u is a separable solution to the $SU(\infty)$ Toda field equation, and therefore our metric is again a diagonal Bianchi metric in one of the classes II, VIII or IX.

The implication (ii) \Rightarrow (i) is clear. □

Remark 2. A weaker version of Theorem 1 was announced in [22] (see [22, Rem. 1.3] and Lemma 2 above).

2.2. The generic case. We now consider the generic case, when m_0 a *non-vanishing* section of \mathcal{D}^\perp , hence determines a gauge ϕ such that $r \equiv 0, q \neq 0$ in (27). According to (26), the 1-form α is then given by

$$(41) \quad m_1 = d \ln |\theta| = q\alpha + \left(p - \frac{(\kappa - s)}{24|\theta|^2} + \frac{1}{2}\right)\theta;$$

moreover, by (32)–(34), we have that

$$(42) \quad \beta = \frac{1}{q} \left(p(2p + \frac{(\kappa - s)}{12|\theta|^2} - 1) - \frac{\kappa}{24|\theta|^2} + 2q^2 \right) J\alpha - \frac{(\kappa - s)}{12|\theta|^2} J\theta,$$

$$(43) \quad dp = \left(2q^2 - p(2p - \frac{(\kappa - s)}{12|\theta|^2} - 1) - \frac{\kappa}{24|\theta|^2} \right) \theta - q \left(4p + \frac{(\kappa - s)}{12|\theta|^2} - 1 \right) \alpha,$$

$$(44) \quad dq = -q \left(4p - \frac{(\kappa - s)}{12|\theta|^2} - 1 \right) \theta - \left(2q^2 - p(2p + \frac{(\kappa - s)}{12|\theta|^2} - 1) + \frac{\kappa}{24|\theta|^2} \right) \alpha.$$

By differentiating (41) and by making use of (43)–(44), we get

$$(45) \quad d\alpha = \frac{(\kappa - s)}{12|\theta|^2} \alpha \wedge \theta = \alpha \wedge J\beta;$$

this is nothing but the first relation in (22), when β is given by (42); by substituting the expression (42) for β into the second relation of (22), we obtain

$$(46) \quad d(J\alpha) = J\alpha \wedge J\beta.$$

In view of (41) and (43)–(44), it is not hard to check that the 1-form $J\beta$ is equivalently given by

$$(47) \quad J\beta = d \ln \left(\frac{|\kappa|}{|q||\theta|^4} \right),$$

so that (46) becomes

$$(48) \quad d \left(\frac{\kappa}{q|\theta|^4} J\alpha \right) = 0;$$

from (25) we get

$$(49) \quad d(J\theta) = J\theta \wedge \left(\frac{1}{3} d \ln |\kappa| - 2d \ln |\theta| \right) + J\alpha \wedge \eta,$$

or, equivalently,

$$(50) \quad d \left(\frac{\kappa^{\frac{1}{3}}}{|\theta|^2} J\theta \right) = \frac{\kappa^{\frac{1}{3}}}{|\theta|^2} J\alpha \wedge \eta,$$

where

$$\eta = -2q\theta + \left(2p + \frac{(\kappa - s)}{12|\theta|^2} - 1\right)\alpha.$$

We are now ready to prove the existence of self-dual Einstein Hermitian metrics with $m_0 \neq 0$. More precisely, we exhibit a 1–1-correspondence between these metrics and the set of solutions of the integrable Frobenius system (43)–(44). We start with the data $(s, \kappa, |\theta|)$ consisting of a constant s (the scalar curvature), a nowhere vanishing smooth function κ (the conformal scalar curvature), and a positive smooth function $|\theta|$ (the norm of the Lie form $\theta = \frac{d\kappa}{3\kappa}$), defined on an open subset U of M , such that $\theta \wedge d|\theta|^2$ has no zero on U (equivalently, m_0 does not vanish on U). We then introduce local coordinates $x = \kappa^{\frac{1}{3}} \neq 0$ and $y = |\theta|^2 > 0$. Observe that x is a *momentum map* for the Killing field K with respect to the self-dual Kähler metric $\bar{g} = \kappa^{\frac{2}{3}}g$ while $y = |K|_{\bar{g}}^2$ is the square-norm of K with respect to \bar{g} (see Proposition 2). The Lee form θ is then given by

$$(51) \quad \theta = \frac{dx}{x},$$

and the 1-form α is given by (41) for some smooth functions $p(x, y)$ and $q(x, y) \neq 0$ of x, y , i.e.

$$(52) \quad \alpha = \frac{1}{q} \left(\frac{dy}{2y} - \frac{1}{x} \left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2} \right) dx \right).$$

Then, (43)–(44) can be made into the following Frobenius system for the (unknown) functions p and q^2 :

$$(53) \quad dp = \frac{1}{x} \left[2q^2 + 2 \left(p + \frac{(x^3 - s)}{24y} \right) \left(p - \frac{(x^3 - s)}{24y} + 1 \right) - \frac{1}{2} - \frac{x^3}{24y} \right] dx \\ - \frac{1}{y} \left[2p + \frac{(x^3 - s)}{24y} - \frac{1}{2} \right] dy$$

$$(54) \quad d(q^2) = -\frac{1}{y} \left[2q^2 - 2p \left(p + \frac{(x^3 - s)}{24y} - \frac{1}{2} \right) + \frac{x^3}{24y} \right] dy \\ - \frac{2}{x} \left[\left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2} \right) \left(2p \left(p + \frac{(x^3 - s)}{24y} - \frac{1}{2} \right) - \frac{x^3}{24y} \right) \right. \\ \left. - 2q^2(1 - p) \right] dx$$

A straightforward computation shows that the integrability condition $d(dp) = d(dq^2) = 0$ is satisfied (as a matter of fact, the explicit solutions are given in Lemma 3 below). The above mentioned correspondence between solutions to (53)–(54) and self-dual Einstein Hermitian metrics with $m_0 \neq 0$ now goes as follows. Since (53)–(54) is integrable, each value of (p, q) at a given point (x_0, y_0) can be extended to a solution of (53)–(54) in some neighborhood V of (x_0, y_0) ; moreover, by choosing $q(x_0, y_0) \neq 0$, we may assume that q has no zero on V ; by (52) and (53)–(54), one immediately obtains (45) for the

corresponding 1-form α . We then introduce a third local coordinate, z , such that

$$(55) \quad J\alpha = \frac{qy^2}{x^3}dz,$$

see (48). Finally, since the 1-form $J\theta$ satisfies (49) or, equivalently, (50), the integrability condition reads as follows:

$$d\left(\frac{qy}{x^2}\eta\right) = 0,$$

see (48) and (49); by using (43)–(46), one easily checks that the integrability condition is actually satisfied, so that

$$(56) \quad J\theta = \frac{y}{x}(dt + hdz),$$

where t is a suitable transversal coordinate to (x, y, z) , and $h(x, y)$ is a smooth function on V , defined by

$$dh = -\frac{qy}{x^2}\eta.$$

It is an easy consequence of (53) that the above equation is solved by

$$(57) \quad h = \frac{yp}{x^2} + \frac{x}{24}.$$

The metric g and the orthogonal almost complex structure J are then given by

$$g = \frac{1}{|\theta|^2}(\theta \otimes \theta + J\theta \otimes J\theta + \alpha \otimes \alpha + J\alpha \otimes J\alpha).$$

According to (51), (52), (55) and (56), and by using the coordinates (x, y, z, t) introduced above, the metric g takes the form

$$(58) \quad g = \frac{1}{y} \left[\frac{dx^{\otimes 2}}{x^2} + \frac{1}{q^2} \left(\frac{dy}{2y} - \frac{1}{x} \left(p - \frac{(x^3 - s)}{24y} + \frac{1}{2} \right) dx \right)^{\otimes 2} + \frac{q^2 y^4}{x^6} dz^{\otimes 2} + \frac{y^2}{x^2} (dt + hdz)^{\otimes 2} \right],$$

where, we recall, s is a constant (equal to the scalar curvature of the metric), (p, q) is a solution of (53)–(54), and h is given by (57). This shows that any self-dual Einstein Hermitian metric with $m_0 \neq 0$ is locally isometric to a metric of the above form for some solution (p, q) to (53)–(54).

Conversely, for any solution to (53)–(54), the corresponding almost-Hermitian metric (g, J) is self-dual Einstein Hermitian metric with $m_0 \neq 0$. Indeed, by (45), (46) and (50), J is integrable and it is easily checked that $\theta = \frac{dx}{x}$ is the Lee form for (g, J) , i.e.,

$$dF = -2\theta \wedge F;$$

moreover, the 1-form α corresponds to the gauge

$$\phi = -\frac{1}{y}(\alpha \wedge J\theta + J\alpha \wedge \theta),$$

meaning that $\alpha = \phi(J\theta)$; one directly computes

$$d\phi = (\theta + J\beta) \wedge \phi,$$

where the 1-form β is given by (42); it follows that β is precisely the 1-form defined by (21) and that (45)–(46) are nothing but the Ricci identities (22); this allows us to recognize the curvature: By (22), the Ricci tensor of (g, J) is J -invariant, and, since $\theta = \frac{dx}{x}$, the dual vector field K of $\kappa^{-\frac{1}{3}}J\theta = \frac{1}{x}J\theta$ is Killing, cf. e.g. [2]; by (50) and (18), the covariant derivative of θ is given by (24) for p and q constructed as above, and $r \equiv 0$; hence, (42) and (43)–(44) (equivalently, (53)–(54)) are the same as (32)–(34); these, in turn, are a way of re-writing (30); it follows that the projection of the curvature to Λ^-M reduces to $\frac{s}{12}\text{Id}|_{\Lambda^-M}$, i.e. the Hermitian metric g is Einstein and self-dual, with scalar curvature equal to s , see (3); turning back to (45), we conclude that the conformal scalar curvature is $\kappa = x^3$, see (22); the metric constructed in this way is not of cohomogeneity one, as $m_0 \neq 0$, see Theorem 1. Finally, different solutions (p, q) of (53)–(54) give rise to non-isometric metrics, as p and q are determined by $|W^+|, d|W^+|$ and $d|D^gW^+|$.

We finally observe that the metric (58) admits two commuting vector fields, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$.

We summarize the results obtained so far as follows:

Theorem 2. *Let (M, g, J) be a self-dual Einstein Hermitian 4-manifold. Suppose that (M, g, J) is neither locally-symmetric nor of cohomogeneity one. Then, on an open dense subset of M , g is locally given by (58). In particular, (M, g) admits a local isometric action of \mathbb{R}^2 almost-everywhere.*

Remark 3. (i) It is easily seen that the metrics (58) have only 2-dimensional continuous symmetries. Moreover, as we already observed, the coordinate $x_1 = x = \kappa^{\frac{1}{3}}$ is a momentum map of the Killing vector field $K_1 = \frac{\partial}{\partial t}$ with respect to the Kähler metric $\bar{g} = x^2g$ while, by (53) and (57), a momentum map x_2 of the second Killing field, $K_2 = \frac{\partial}{\partial z}$, is given by

$$x_2 = \frac{y}{2x} + \frac{x^3 + s}{24x}.$$

The momentum map x_1 is also equal to the scalar curvature of the Kähler metric \bar{g} . A straightforward computation shows that x_2 is related to the Pfaffian of the *normalized Ricci form* $\bar{\sigma}$ of the Kähler metric \bar{g} by $x_2 = 6(\text{Pf}(\bar{\sigma}) + 2b)$, where b is the constant appearing in (61) below. This fits with an observation of R. Bryant in [13]. (Recall that for any 2-form ψ , the Pfaffian of ψ with respect to \bar{g} is defined by: $\psi \wedge \psi = \text{Pf}(\psi) v_{\bar{g}}$, where $v_{\bar{g}}$ is the volume form of \bar{g} ; the normalized Ricci form $\bar{\sigma}$ is the $(1, 1)$ -form associated to the normalized Ricci tensor, \bar{S} , appearing in the usual decomposition $\bar{R} = \bar{S} \wedge \bar{g} + W$ of the curvature operator of \bar{g} ; it is related to the usual Ricci form $\bar{\rho}$ by $\bar{\sigma} = \frac{1}{2}(\bar{\rho}_0 + \frac{x}{12}\bar{\omega})$, where $\bar{\rho}_0$ is the trace-free part of $\bar{\rho}$; since $g = x^{-2}\bar{g}$ is Einstein and $d^c x$ is the dual of a Killing vector field, we have that $\bar{\rho}_0 = -\frac{1}{x}(dd^c x)_0$; the result follows easily).

(ii) It follows from Theorems 1 and 2 that every self-dual Einstein Hermitian 4-manifold admits a (local) isometric \mathbb{R}^2 -action compatible with a *product structure* in the sense of [36]; the general considerations in [36, Sec.2] therefore apply to the present situation; a detailed analysis of self-dual Einstein 4-manifolds admitting \mathbb{R}^2 -continuous symmetry has been carried out by D. Calderbank and H. Pedersen [19] (see also [15]).

We close this section by providing an explicit form for the metric (58).

Lemma 3. *The solutions $p(x, y)$ and $q(x, y)$ of the system (53)–(54) are explicitly given by*

$$(59) \quad p = \frac{f}{y^2} - \frac{(x^3 - s)}{24y} + \frac{1}{4};$$

$$(60) \quad q^2 = \frac{1}{y^2} \left[\frac{x}{2} f' - f + \left(\frac{x^3 - s}{24} \right)^2 \right] - \frac{x^3}{24y} - p^2,$$

where

$$(61) \quad f(x) = ax^2 + bx^4 - \frac{(x^6 - s^2)}{576},$$

s , a and b are constants defined by positivity in (60), and f' stands for the first derivative of f .

Proof. We first observe that (53) can be equivalently written as

$$\begin{aligned} & d \left(y^2 \left(p + \frac{(x^3 - s)}{24y} - \frac{1}{4} \right) \right) = \\ & \frac{y^2}{x} \left[2q^2 + 2 \left(p + \frac{(x^3 - s)}{24y} \right) \left(p - \frac{(x^3 - s)}{24y} \right) + 2 \left(p + \frac{(x^3 - s)}{24y} - \frac{1}{4} \right) + \frac{x^3}{12y} \right] dx; \end{aligned}$$

this shows that $y^2 \left(p + \frac{(x^3 - s)}{24y} - \frac{1}{4} \right)$ is function of x , say f ; from the above equality, we get (59) and (60), where f is a (still unknown) smooth function; in order to determine f , we differentiate (60) by using (59) and substitute into (54); then, cancellations occur and (54) eventually reduces to

$$(62) \quad x^2 f'' - 5x f' + 8f + \frac{(x^6 - s^2)}{72} = 0;$$

the solutions of (62) are given by (61). \square

3. SELF-DUAL EINSTEIN HERMITIAN METRICS WITH HYPERHERMITIAN STRUCTURES

In this section, we consider self-dual, Einstein, Hermitian metrics which in addition admit a *non-closed* hyperhermitian structure compatible with the negative orientation. It is well-known that LeBrun-Pedersen metrics, which are of cohomogeneity one under the action of the unitary group $U(2)$, carry

such hyperhermitian structures; in LeBrun's coordinates [37] these metrics read as follows:

$$(63) \quad g = \frac{1}{(bt^2 + 4c)^2} \left(\left(1 + \frac{8b}{t^2} + \frac{16c}{t^4}\right)^{-1} dt^{\otimes 2} + \frac{t^2}{4} [\sigma_1^2 + \sigma_2^2 + \left(1 + \frac{8b}{t^2} + \frac{16c}{t^4}\right) \sigma_3^2] \right),$$

where b and c are properly chosen constants [39]; more precisely, we have the following

Proposition 4. ([39]) *Let (M, g) be an oriented self-dual Einstein 4-manifold. Assume that (M, g) admits a $U(2)$ isometric action with generically three-dimensional $SU(2)$ -orbits. If g admits a non-closed, $U(2)$ -invariant negative hyperhermitian structure, then g is isometric to (63) with $c > b^2$, and actually admits exactly two distinct invariant hyperhermitian structures.*

We here prove the following more general result:

Theorem 3. *A self-dual Einstein Hermitian 4-manifold (M, g, J) locally admits a non-closed, negative hyperhermitian structure if and only if g is locally isometric to one of the $U(2)$ -invariant metrics (63) with $c > b^2$; then, (M, g) actually carries exactly two distinct hyperhermitian structures, each of them $U(2)$ -invariant.*

We first establish general facts concerning self-dual Einstein 4-manifolds which carry a *non-closed* hyperhermitian structure compatible with the negative orientation. As already observed in Sec.2, a (negative) hyperhermitian structure (g, I_1, I_2, I_3) is determined by a real 1-form θ — the common Lee form of (g, I_i) , also the Lee form of the Obata connection — satisfying conditions (10) and (15), and such that $\Phi := d\theta$ is self-dual; in particular, the 2-form Φ is harmonic. The next Lemma shows that the self-dual Weyl tensor of g is completely determined by θ , Φ and the first covariant derivative $D^g\Phi$ of Φ .

Lemma 4. *Let (M, g) be an oriented self-dual Einstein 4-manifold and assume that (M, g) carries a negative hyperhermitian structure. Then, as a symmetric operator acting on Λ^+M , the self-dual Weyl tensor W^+ is given by*

$$(64) \quad W^+(\psi) = \frac{1}{2}[\psi, \Phi] + \frac{1}{|\theta|^2} D_{\psi(T)}^g \Phi,$$

where ψ is any self-dual 2-form, T is the Riemannian dual vector field of θ , and $[\cdot, \cdot]$ denotes the commutator of 2-forms, viewed as skew-symmetric endomorphisms of the tangent bundle. Moreover, θ and Φ are related by

$$(65) \quad D_T^g \Phi = 2|\theta|^2 \Phi.$$

$$(66) \quad d|\theta|^2 - \left(\frac{s}{12} + |\theta|^2\right)\theta + \Phi(\theta) = 0,$$

Proof. By using (10), the right-hand side of

$$R_{X,Y}\theta = (D^g)_{Y,X}^2\theta - (D^g)_{X,Y}^2\theta$$

is easily computed; we thus obtain:

$$(67) \quad R(\theta \wedge Z) = -\frac{1}{2}d|\theta|^2 \wedge Z - \frac{1}{2}\left(\frac{s}{12} - |\theta|^2\right)\theta \wedge Z \\ - \frac{1}{2}\Phi(Z) \wedge \theta - \frac{1}{2}D_Z^g\Phi + \theta(Z)\Phi.$$

Since g is self-dual and Einstein, $R = \frac{s}{12}\text{Id}|_{\Lambda^2 M} + W^+$, see (3). Then, by projecting (67) to $\Lambda^- M$, we get (66), whereas the projection of (67) to $\Lambda^+ M$ gives (64) and (65). \square

Corollary 1. ([24, 14]) *Every hyperhermitian structure on a conformally flat 4-manifold is closed.*

Proof. If we assume that $\Phi \neq 0$ somewhere on M and that the anti-self-dual Weyl tensor is identically zero, then, after contracting (64) and (65) with Φ , we obtain $\theta = \frac{1}{4}d \ln |\Phi|^2$, which contradicts $\Phi = d\theta \neq 0$. \square

We compute the covariant derivative $D_\theta^g W^+$ of W^+ along T by using (64) together with (65) and (66) (the latter are used for evaluating the term $(D^g)_{T,\psi(T)}^2\Phi$ which appears in the calculation); we thus get

Lemma 5. *Let (M, g) be an oriented self-dual Einstein 4-manifold, admitting a negative hyperhermitian structure; then, the covariant derivative $D_T^g W^+$ of the self-dual Weyl tensor W^+ along the dual vector field T of the Lee form θ is given by*

$$(68) \quad ((D_T^g W^+)(\psi), \phi) = ([W^+(\phi), \psi] + [W^+(\psi), \phi], \Phi) \\ + (4|\theta|^2 - \frac{s}{6})(W^+(\psi), \phi) \\ + |\Phi|^2(\psi, \phi) - 3(\Phi, \psi)(\Phi, \phi),$$

for any sections, ϕ and ψ , of $\Lambda^+ M$.

From Lemma 5 and Propositions 1 and 2, we infer

Proposition 5. *Let (M, g) be an oriented self-dual Einstein 4-manifold, admitting a non-closed hyperhermitian structure compatible with the negative orientation. Then the following three conditions are equivalent:*

- (i) *the spectrum of W^+ is everywhere degenerate;*
- (ii) *W^+ has two distinct eigenvalues at any point;*
- (iii) *the self-dual 2-form Φ is a nowhere vanishing eigenform for W^+ with respect to the simple eigenvalue, and is proportional to a positive Hermitian structure J .*

Proof. (i) \Rightarrow (ii). According to Proposition 1, if the spectrum of W^+ is everywhere degenerate, then either W^+ vanishes identically (and therefore the

hyperhermitian structure is closed by Corollary 1) or W^+ has two distinct eigenvalues λ and $-\frac{\lambda}{2}$ at any point.

(ii) \Rightarrow (iii). By Proposition 1, we know that a normalized generator F of the λ -eigenspace of W^+ is the Kähler form of a positive Hermitian structure J . Let ϕ be any self-dual 2-form orthogonal to F , with $|\phi|^2 = 2$; then, ϕ and $\psi = J\phi$ are orthogonal, $(-\frac{\lambda}{2})$ -eigenforms of W^+ ; by substituting into (68), we get

$$\begin{aligned} 0 &= ((D_\theta^g W^+)(\phi), \psi) = -3(\Phi, \psi)(\Phi, \phi), \\ -d\lambda(\theta) &= ((D_\theta^g W^+)(\phi), \phi) = -(4|\theta|^2 - \frac{s}{6})\lambda + 2|\Phi|^2 - 3(\Phi, \phi)^2, \\ -d\lambda(\theta) &= ((D_\theta^g W^+)(\psi), \psi) = -(4|\theta|^2 - \frac{s}{6})\lambda + 2|\Phi|^2 - 3(\Phi, \psi)^2. \end{aligned}$$

From the last two equalities, we get $(\Phi, \psi) = \pm(\Phi, \phi)$, and by the first one we conclude that $(\Phi, \psi) = (\Phi, \phi) = 0$. This shows that Φ is a multiple of F . It remains to prove that Φ does not vanish on M ; by taking a two-fold cover of M if necessary, we may assume that the Hermitian structure J is globally defined on M ; by Proposition 2, (g, J) is conformally Kähler and $\lambda^{\frac{2}{3}}F$ is the corresponding closed Kähler form; but Φ is also a closed, self-dual 2-form, and a multiple of F , hence a constant (non zero) multiple of $\lambda^{\frac{2}{3}}F$.

(iii) \Rightarrow (i). This is an immediate consequence of Proposition 1. \square

Convention. Up to the end of this section, we assume that (M, g) is an oriented self-dual Einstein 4-manifold whose self-dual Weyl W^+ has degenerate spectrum, and which admits a *non-closed* hyperhermitian structure compatible with the negative orientation of M . According to Proposition 5, W^+ has two distinct eigenvalues which we denote by λ and $-\frac{\lambda}{2}$, and the harmonic self-dual 2-form Φ defines a positive Hermitian structure J on (M, g) whose Kähler form, F , is an λ -eigenform for W^+ . Moreover, as a consequence of Proposition 2, after rescaling the metric if necessary we may assume:

$$(69) \quad \Phi = \frac{\lambda^{\frac{2}{3}}}{2}F.$$

The conformal scalar curvature κ of (g, J) is then equal to 6λ (notation of Sec.2.1); the Lee form θ_J and the Killing vector field K , suitably rescaled by a positive constant, are therefore given by:

$$(70) \quad \theta_J = \frac{d\lambda}{3\lambda}; \quad K = J\text{grad}_g(\lambda^{-\frac{1}{3}}),$$

(see Proposition 2).

At this point, our main technical result reads as follows:

Proposition 6. *A self-dual Einstein Hermitian 4-manifold (M, g, J) admits a non-closed, hyperhermitian structure compatible with the negative*

orientation if and only if the Lee form θ_J satisfies

$$(71) \quad D^g \theta_J = \frac{(1 + \lambda^{\frac{2}{3}})(s + 3\lambda^{\frac{1}{3}})}{12} g + \frac{(1 + 2\lambda^{\frac{2}{3}})}{(1 + \lambda^{\frac{2}{3}})} \theta_J \otimes \theta_J + \frac{\lambda^{\frac{2}{3}}}{(1 + \lambda^{\frac{2}{3}})} J\theta_J \otimes J\theta_J.$$

If this holds, (M, g) actually admits two, and only two, non-closed hyperhermitian structures $\{I'_1, I'_2, I'_3\}$ and $\{I''_1, I''_2, I''_3\}$, whose Lee forms, θ' and θ'' , are given by

$$(72) \quad \theta' = \frac{1}{(1 + \lambda^{\frac{2}{3}})} (\theta_J - \lambda^{\frac{1}{3}} J\theta_J), \quad \theta'' = \frac{1}{(1 + \lambda^{\frac{2}{3}})} (\theta_J + \lambda^{\frac{1}{3}} J\theta_J).$$

Moreover, the Killing vector field K is triholomorphic for both hyperhermitian structures, i.e., K preserves all complex structures I'_i and I''_i , $i = 1, 2, 3$.

Proof. We first show that if (M, g, J) admits a non-closed hyperhermitian structure compatible with the negative orientation, then the corresponding Lee form θ must be one of the forms θ' and θ'' given by (72).

From (65) and the fact that Φ is an λ -eigenform of W^+ , we infer

$$(73) \quad d|\Phi|^2 = 4|\Phi|^2\theta + 4\lambda\Phi(\theta).$$

By differentiating (73) and by using (66) to compute $d(\Phi(\theta))$, we obtain

$$(d\lambda - 3\lambda\theta) \wedge \Phi(\theta) + (|\Phi|^2 + \lambda(\frac{s}{12} + |\theta|^2))\Phi = 0;$$

we infer:

$$(74) \quad |\Phi|^2 = -\lambda(\frac{s}{12} + |\theta|^2).$$

By substituting the above expression of $|\Phi|^2$ in (73), and by using (66) again, we get

$$(75) \quad d\lambda - 3\lambda\theta = \frac{3\lambda^2}{|\Phi|^2} \Phi(\theta).$$

By using (70) and (69) (see the above convention) we finally obtain:

$$(76) \quad \theta = \frac{1}{(1 + \lambda^{\frac{2}{3}})} (\theta_J - \lambda^{\frac{1}{3}} J\theta_J).$$

This shows that every non-closed hyperhermitian structure is completely determined by the self-dual harmonic 2-form Φ . It remains to prove that Φ itself is determined, up to sign, by the metric g ; then, the possible two values of θ appearing in (72) will only differ by conjugating J or, equivalently, by substituting $-\Phi$ to Φ . Notice that, according to our convention, at this stage we have the freedom to rescale Φ by a non-zero constant. In other words, by fixing a non-closed hyperhermitian structure and by following our convention, we know that any other non-closed hyperhermitian structure

corresponds to a harmonic 2-form of the form $a\Phi = \frac{a}{2}\lambda^{\frac{2}{3}}F$, where a is a non-zero constant. Our claim is that $a = \pm 1$; to prove that we calculate

$$|D^g\Phi|^2 = 2|\theta|^2(3|\Phi|^2 + |W^+|^2)$$

by using (64) and (65); in the present situation, when W^+ has degenerate spectrum, the norm of W^+ is given by $|W^+|^2 = \frac{3}{2}\lambda^2$; then, by (74), the above equality reduces itself to

$$(77) \quad |D^g\Phi|^2 = -\left(\frac{|\Phi|^2}{\lambda} + \frac{s}{12}\right)(6|\Phi|^2 + 3\lambda^2);$$

it is readily checked that if Φ and $a\Phi$ simultaneously satisfy (77), then $a = \pm 1$.

We now check that conditions (10)-(15) for either θ' or θ'' are equivalent to (71). Keeping (69) in mind, we see that (75) can be equivalently re-written as

$$(78) \quad \theta_J = \theta + \lambda^{\frac{1}{3}}J\theta;$$

then, the equivalence (71) \Leftrightarrow (10)-(15) follows by a straightforward computation involving the expressions (76) and (78) and using formula (12); 1-forms θ' and θ'' thus correspond to two distinct, non-closed, hyperhermitian structures $\{I'_1, I'_2, I'_3\}$ and $\{I''_1, I''_2, I''_3\}$ provided that (71) holds, see Sec. 1.2.

As a final step, we prove that K is triholomorphic with respect to both hyperhermitian structures. For a general hyperhermitian structure $I_i, i = 1, 2, 3$, with Lee form θ , and for any Killing field K , we have

$$\mathcal{L}_K I_i = D_K I_i - [DK, I_i],$$

where D is the Weyl derivative given by (9); we thus only need to check that in our specific situation DK commutes with I_i ; by using (9), (70), (12) and (71), we get

$$DK = \theta(K)\text{Id}|_{TM} + \frac{(1 + \lambda^{\frac{2}{3}})}{4}J;$$

the claim follows immediately. \square

Corollary 2. ([24]) *A locally-symmetric self-dual Einstein 4-manifold does not admit non-closed hyperhermitian structures.*

Proof. Any such manifold is either a space of constant curvature, hence conformally flat, or a Kähler manifold of constant holomorphic sectional curvature (see Propositions 1 and 2). In the former case, the claim follows by Corollary 1, whereas in the latter case $\theta = 0$; we then conclude by using Proposition 6. \square

Remark 4. In [16], D. Calderbank proved that any conformal selfdual 4-manifold admitting two distinct Einstein-Weyl structures is equipped with a canonical conformal submersion to an Einstein-Weyl 3-manifold. In the

situation described by Proposition 6, this conformal submersion can be described as follows: the hyperhermitian structures $\{I'_1, I'_2, I'_3\}$ and $\{I''_1, I''_2, I''_3\}$ determine an $\text{SO}(3)$ -valued function, p , on M defined by:

$$I''_i = \sum_{j=1}^3 a_{ij} I'_j; \quad A = (a_{ij}) \in \text{SO}(3);$$

we claim that p is a conformal submersion of (M, g) to $\text{SO}(3) = \mathbb{R}P^3$: The differential of p is easily computed by using the fact that I''_i and I'_j are both integrable; we thus obtain:

$$(79) \quad d(a_{ij}) + \frac{\lambda^{\frac{2}{3}}}{2(1 + \lambda^{\frac{2}{3}})} \sum_{k=1}^3 a_{ik} ([I'_k, I'_j] K)^{\sharp_g} = 0;$$

here, $[\cdot, \cdot]$ denotes the commutator of endomorphisms of TM and \sharp_g stands for the Riemannian duality; from (79), we infer:

$$\mathcal{L}_K a_{ij} = 0,$$

$$\sum_{i,j} (da_{ij}(X))^2 = \frac{\lambda^{\frac{4}{3}}}{2(1 + \lambda^{\frac{2}{3}})^2} g(X, X), \quad \forall X \in K^\perp;$$

The first equality shows that p coincides with the projection of M to the space, N , of orbits of K , whereas the second equality means that the K -invariant metric $\bar{g} = \frac{\lambda^{\frac{2}{3}}}{(1 + \lambda^{\frac{2}{3}})} g$ descends to the round metric of $\text{SO}(3) = \mathbb{R}P^3$; in other words, K defines a Riemannian submersion from (M, \bar{g}) to $\text{SO}(3)$.

Proof of Theorem 3. We first notice that the Killing vector field K is trivial if and only if λ is constant (see (70)), or, equivalently, $\theta_J = 0$. Thus, according to Propositions 5 and 6, if (M, g, J) is a self-dual Einstein Hermitian 4-manifold admitting a *non-closed* hyperhermitian structure, the Killing vector field K does not vanish on an open, dense subset of M . It then follows from [32, 17, 18] that self-dual Einstein 4-manifolds admitting two distinct hyperhermitian structures and a non-trivial triholomorphic Killing vector field are locally given by Proposition 4.

For completeness, however, we here give a different and more direct argument adapted to our ‘‘Hermitian’’ situation.

By Proposition 4 it is sufficient to show that our metric can be written in the diagonal form (35). Since the eigenvalues of W^+ are not constant, i.e., $\theta_J \neq 0$ (Proposition 6), we introduce the variable $t = \lambda^{\frac{1}{3}}$; the Lee form θ_J is then equal to $\frac{dt}{t}$, whereas the dual 1-form of the Killing vector field is given by $-\frac{1}{t^2} J dt$. We set: $\sigma_3 = f(t) J dt$, for some smooth function f of t , and we insist that

$$(80) \quad d\sigma_3 = \sigma_1 \wedge \sigma_2,$$

where the 1-forms σ_1 and $\sigma_2 = J\sigma_1$ are both orthogonal to dt and satisfy

$$(81) \quad d\sigma_1 = \sigma_2 \wedge \sigma_3; \quad d\sigma_2 = \sigma_3 \wedge \sigma_1.$$

We then derive f from (80): By differentiating (76) and by making use of (69), we obtain

$$(82) \quad d(Jdt) = -\frac{(1+t^2)t^2}{2}F + \frac{2t}{(1+t^2)}dt \wedge Jdt.$$

By (78), (74) and (69), we also get

$$|dt|^2 = -\left(\frac{t}{2} + \frac{s}{12}\right)(t^4 + t^2);$$

it follows that $(d\sigma_3, dt \wedge Jdt) = 0$ if and only if $(\ln f)' = -\frac{2t}{(1+t^2)} - \frac{1}{(t+\frac{s}{6})}$, where the prime stands for $\frac{d}{dt}$; we then have $f = \frac{a}{(1+t^2)(t+\frac{s}{6})}$, hence

$$(83) \quad \sigma_3 = \frac{a}{(1+t^2)(t+\frac{s}{6})}Jdt$$

for a positive constant a .

In order to determine the 1-forms σ_1 and σ_2 , we choose a gauge ϕ or, equivalently, a 1-form $\alpha = \phi(J\theta) \in \mathcal{D}^\perp$; since σ_1 and $\sigma_2 = J\sigma_1$ are orthogonal to dt , there certainly exist a smooth function h of t and a smooth function φ on M , such that

$$\sigma_1 = h(\cos \varphi \alpha + \sin \varphi J\alpha); \quad \sigma_2 = h(-\sin \varphi \alpha + \cos \varphi J\alpha);$$

by (83) and (80), we obtain the following expression for h :

$$(84) \quad h^2 = \frac{at^2}{(t+\frac{s}{6})^2(1+t^2)};$$

by using (83) and (22), we now see that the conditions (81) are equivalent to

$$(85) \quad d\varphi + \beta + \frac{(\frac{s}{6} - t^3 + at)}{t(1+t^2)(\frac{s}{6} + t)}Jdt = 0;$$

therefore, the existence of a smooth function φ satisfying (85) is equivalent to the following condition:

$$d\left(\beta + \frac{(\frac{s}{6} - t^3 + at)}{t(1+t^2)(\frac{s}{6} + t)}Jdt\right) = 0;$$

a straightforward computation involving (23) and (82) shows that the above equality holds whenever the constant a is chosen equal to $1 + \frac{s^2}{36}$.

4. HERMITIAN STRUCTURES ON QUATERNIONIC QUOTIENTS

Let (N, g) be a quaternionic Kähler manifold of real dimension $4n$, endowed with a non-trivial Killing field K which preserves the quaternionic structure. According to Galicki [25, 26] and Galicki-Lawson [27], under some “non-degeneracy” condition for K one can define a $4(n-1)$ -dimensional quaternionic orbifold (M, g^*) via the so-called *quaternionic reduction construction*. This can be described as follows. We first consider the following orthogonal splitting of the bundle of 2-forms:

$$(86) \quad \Lambda^2 N = \Lambda^+ N \oplus \Lambda^{1,1} N \oplus \Lambda^\perp N,$$

where:

- $\Lambda^+ N$ is the 3-dimensional sub-bundle of “self-dual” 2-forms which determines the quaternionic structure (also identified to a sub-bundle $A^+ N$ of skew-symmetric endomorphism of TN): both $A^+ N$ and $\Lambda^+ N$ are preserved by the Levi-Civita connection D^g and at each point x of N there is an orthonormal basis $\{I_1, I_2, I_3\}$ of $A^+ N \subset \text{End}(T_x N)$ with the property that $I_i \circ I_j = -\delta_{ij} \text{Id}|_{T_x N} + \epsilon_{ijk} I_k$. Then $\Lambda^+ N = \text{span}(\omega_1, \omega_2, \omega_3)$, where ω_i are the orthogonal fundamental 2-forms of the almost Hermitian structures (g, I_l) . In the sequel, we refer to any such choice of I_l 's (resp. ω_l 's) as a *trivialization* of $A^+ N$ (resp. $\Lambda^+ N$);
- $\Lambda^{1,1} N$ is the sub-bundle of 2-forms which are I_i -invariant for any section of $A^+ N$;
- $\Lambda^\perp N$ denotes the orthogonal complement of $\Lambda^+ N \oplus \Lambda^{1,1} N$ in $\Lambda^2 N$.

We denote by Π^+ the projection of $\Lambda^2 N$ onto $\Lambda^+ N$; for any trivialization $\{\omega_1, \omega_2, \omega_3\}$ of $\Lambda^+ N$ we have

$$\Pi^+ = \frac{1}{2n} \sum_l \omega_l \otimes \omega_l,$$

and $\Pi_K^+ := \frac{1}{2n} \sum_l (i_K \omega_l \otimes \omega_l)$ is a section of $T^* N \otimes \Lambda^+ N$.

In [27, Th. 2.4], K. Galicki and H. Blaine Lawson show the existence of a section f_K of $\Lambda^+ N$ such that

$$d^{D^g} f_K = D^g f_K = \Pi_K^+;$$

the section f_K is called *the momentum map* associated to (N, g, K) and it is easily seen that the “level set”

$$L_K := \{x \in N : f_K(x) = 0\}$$

is K -invariant.

If we assume that $K_x \neq 0$ at each point x of L_K , then the level set L_K is regular, i.e., L_K is a smooth submanifold of N (see [27]). When the quotient space $M := L_K/K$ is a $(4n-4)$ -dimensional manifold (or an orbifold) it then becomes a quaternionic Kähler manifold (resp. orbifold) with respect to the “projected” quaternionic structure, g^* , from N . The case of interest here is when N is 8-dimensional; then, the quaternionic reduction gives rise to a four dimensional *anti-self-dual* Einstein manifold (resp. orbifold) with respect to

the canonical orientation induced by N . Note that if K is the generator of an S^1 -quaternionic action on N then, under the non-degeneracy condition as above, M always inherits an orbifold structure, cf. [27, Th. 3.1 & Cor. 3.2].

The above construction applies in particular to $N = \mathbb{H}P^2$ endowed with certain *weighted* S^1 -actions; one thus obtains a wealth of examples of *compact* anti-self-dual Einstein orbifolds; as shown by Galicki-Lawson, the corresponding orbifolds are all weighted projective planes $\mathbb{C}P^{[p_1, p_2, p_3]}$ for some integers $0 < p_1 \leq p_2 \leq p_3$ satisfying $p_3 < p_1 + p_2$, [27, Sec. 4]. Notice that, with respect to the orientation induced by the canonical complex structure, the metric becomes *self-dual*. (In the case when $p_1 = p_2 = p_3$ one obtains the Fubini-Study metric on $\mathbb{C}P^2$).

On the other hand, R. Bryant showed that each weighted projective plane admits a self-dual Kähler metric which, under the above assumption for the weights, has everywhere positive scalar curvature [13, Sec. 4.2]. Therefore, according to Lemma 2, Bryant's metric is conformal to a self-dual *Einstein* Hermitian metric on $\mathbb{C}P^{[p_1, p_2, p_3]}$, $p_3 < p_1 + p_2$. When considering both results together, a natural question arises:

Question. ([38]) Are Galicki-Lawson's metrics on $\mathbb{C}P^{[p_1, p_2, p_3]}$ Hermitian with respect to some anti-self-dual complex structure?

In this section we show that this is indeed the case.

Specifically, we prove the following

Theorem 4. *Let (N, g) be $\mathbb{H}P^2$ or $\mathbb{H}H^2$. Then, any anti-self-dual, non-conformally flat Einstein 4-orbifold (M, g^*) obtained as quaternionic reduction of (N, g) admits a (negatively oriented) Hermitian structure J . In particular, on the smooth part of M , the metric g^* is locally given by the explicit constructions in Section 2.*

The proof is based on the following simple observation.

Lemma 6. *Let (N, g) be a quaternionic Kähler manifold of non-zero scalar curvature and let K be a Killing field on N . Denote by $\Psi(X, Y) = (D_X^g K, Y)$ the 2-form corresponding to $D^g K$ and let $\Psi^+ = \Pi^+(\Psi)$ be the projection of Ψ in $\Lambda^+ N$. Then, up to multiplication by a constant, the momentum map f_K of K is given by Ψ^+ .*

Proof. Since K is Killing, we have the Kostant identity (29). Moreover, for a quaternionic Kähler manifold the curvature operator R acts on $\Lambda^+ N$ by $\lambda \text{Id}|_{\Lambda^+ N}$, where λ is a positive multiple of the scalar curvature, cf. e.g. [46]. Thus, projecting (29) to $\Lambda^+ N$ we get $D_X^g \Psi^+ = \lambda \Pi_K^+$. \square

By Lemma 6 the "level set" L_K of K is the same as the set of points $x \in N$ where $\Psi_x^+ = 0$. Thus, at any point $x \in L_K$ the tangent space $T_x L_K$ is given by $T_x L_K = \{T_x N \ni X : D_X^g \Psi^+ = 0\}$. Since by assumption K does not vanish on L_K , we conclude by (29) and the fact that $R|_{\Lambda^+ N} = \lambda \text{Id}|_{\Lambda^+ N}$

$$T_x L_K = \text{span}(I_1 K, I_2 K, I_3 K)^\perp,$$

where $\{I_1, I_2, I_3\}$ is any trivialization of A^+N .

We also observe that the 2-form Ψ is a section of $\Lambda^+N \oplus \Lambda^{1,1}N$, provided that K preserves the quaternionic structure. Indeed,

$$[D^g K, I_l] = D_K^g I_l - \mathcal{L}_K I_l,$$

where $[\cdot, \cdot]$ stands for the commutator of $\text{End}(TN)$. Since K is quaternionic, the left-hand-side of the above equality is a section of Λ^+N . By summing over l in the above relation we get

$$(87) \quad \Psi - \Pi^{1,1}(\Psi) \in \Lambda^+N,$$

where $\Pi^{1,1}$ denotes the projection to $\Lambda^{1,1}N$:

$$(88) \quad \Pi^{1,1}(\psi)(\cdot, \cdot) = \frac{1}{4} \left[(\psi(\cdot, \cdot) + \sum_l \psi(I_l \cdot, I_l \cdot)) \right], \quad \forall \psi \in \Lambda^2 N.$$

Thus, Ψ is a section of $\Lambda^+N \oplus \Lambda^{1,1}N$, and at $x \in L_K$, Ψ_x actually belongs to $\Lambda_x^{1,1}N$.

Since $\Psi = \frac{1}{2}dK^\sharp$, where K^\sharp is the g -dual 1-form of K , we conclude that

$$\mathcal{L}_K \Psi = d(i_K(\Psi)) = -\frac{1}{2}d(d|K|^2) = 0,$$

i.e. Ψ is a closed K -invariant 2-form. This shows that Ψ projects to $M = L_K/K$ to define an *anti-self-dual* form on (M, g^*) , then denoted by Ψ^* . Considering the Riemannian submersion

$$\pi : L_K \longmapsto M = L_K/K,$$

the *horizontal* space, H , of TL_K is given by

$$H = \text{span}(K, I_1 K, I_2 K, I_3 K)^\perp.$$

Note that H is I_l -invariant for any section I_l of A^+N . Using the above remarks we calculate:

$$(89) \quad (D_{U^*}^{g^*} \Psi^*)(V^*, T^*) = (D_U^g \Psi)(V, T) - \frac{4}{|K|_g^2} \Pi^{1,1}(i_U \Psi \wedge i_K \Psi)(V, T),$$

where D^{g^*} is the Levi-Civita connection of g^* , U^*, V^*, T^* are any vectors on M , and U, V, T are the corresponding horizontal lifts.

By assumption, K has no zero on L_K ; it then follows from (89) and (29) that Ψ^* does not vanish identically on M . Thus, on the open subset of (M, g^*) where $\Psi^* \neq 0$ the normalised ASD form $\frac{\sqrt{2}\Psi^*}{|\Psi^*|_{g^*}}$ determines a *negative* almost Hermitian structure J .

Lemma 7. *The almost-complex structure J is integrable.*

Proof. We denote by Z_i^* any complex (1,0)-vector field of (M, J) and by Z_i the corresponding horizontal lift (considered as complex vector in $T_x^{\mathbb{C}}N$); then, J is integrable if and only if the following identity holds:

$$(90) \quad D_{Z_i^*}^{g^*} \left(\frac{\sqrt{2}\Psi^*}{|\Psi^*|_{g^*}} \right) (Z_j^*, Z_k^*) = \frac{\sqrt{2}}{|\Psi^*|_{g^*}} (D_{Z_i^*}^{g^*} \Psi^*) (Z_j^*, Z_k^*) = 0 \quad \forall i, j, k;$$

by the very definition of J we have $\Psi(Z_i, Z_j) = 0$; moreover, since Ψ belongs to $\Lambda^{1,1}N$ on L_K , the almost complex structure J (defined on H) commutes with I_l 's for any trivialization $\{I_1, I_2, I_3\}$ of A^+N . Then, by (89) and (29) it is easily seen that the integrability condition (90) for J is the same as

$$(91) \quad (D_{Z_i^*}^{g^*} \Psi^*)(Z_j^*, Z_k^*) = (D_{Z_i}^g \Psi)(Z_j, Z_k) = (R(K \wedge Z_i), Z_j \wedge Z_k) = 0.$$

We now derive (91) from the structure of the curvature tensor of the Riemannian symmetric space $\mathbb{H}P^2$ and the corresponding non-compact dual $\mathbb{H}H^2$ (we refer to [46, 30] for a general description of the curvature operator, R , of a Riemannian symmetric space).

When $N = \mathbb{H}P^2 = Sp(3)/(Sp(1)Sp(2))$, the eigenspaces of R are the simple factors $\mathfrak{sp}(1)$ and $\mathfrak{sp}(2)$ of the isotropy Lie sub-algebra $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$, and the orthogonal complement \mathfrak{h}^\perp of \mathfrak{h} in the space $\text{Skew}(\mathfrak{m})$ of the skew-symmetric endomorphisms of $\mathfrak{m} = \mathfrak{sp}(3)/\mathfrak{h}$ (note that R acts trivially on \mathfrak{h}^\perp); the decomposition $\text{Skew}(\mathfrak{m}) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2) \oplus \mathfrak{h}^\perp$ into eigenspaces of R fits in with the splitting (86): Λ^+N is identified to $\mathfrak{sp}(1)$, $\Lambda^{1,1}N$ to $\mathfrak{sp}(2)$, whereas $\Lambda^\perp N$ corresponds to the kernel of R , the space \mathfrak{h}^\perp . This shows that the curvature operator acts on the first two factors in (86) by multiplication with a non-zero constant (a certain multiple of the scalar curvature), and R acts trivially on the third factor (therefore, R has thus three distinct eigenvalues, λ, μ and 0). (This observation also shows that any Killing field on $\mathbb{H}P^2$ is necessarily quaternionic).

As already observed, the almost complex structure J (defined on H) commutes with the I_l 's, so that $I_l(Z_k)$ is again a $(1,0)$ -vector of (H, J) ; we thus get

$$\Pi^+(Z_j \wedge Z_k) = \sum_l (Z_j, I_l(Z_k)) \omega_l = 0,$$

which means that $Z_j \wedge Z_k$ is an element of $\Lambda_x^{1,1}N \oplus \Lambda_x^\perp N$. It then follows that

$$\begin{aligned} (R(K \wedge Z_i), Z_j \wedge Z_k) &= (R(Z_j \wedge Z_k), K \wedge Z_i) \\ &= \mu(\Pi^{1,1}(Z_j \wedge Z_k), K \wedge Z_i). \end{aligned}$$

But $\Pi^{1,1}(Z_j \wedge Z_k)$ is again a $(2,0)$ -vector of (M, J) (see formula (88)), so that $(\Pi^{1,1}(Z_j \wedge Z_k), K \wedge Z_i) = 0$; this implies (91).

The same argument holds for the non-compact dual space $\mathbb{H}H^2$. \square

Proof of Theorem 4. By (89) and Lemma 7, we see that $\frac{1}{|K|^2} \Psi^*$ is a harmonic self-dual 2-form, i.e. it is the Kähler form of a self-dual Kähler metric in the conformal class of g^* , defined on the open subset where $\Psi^* \neq 0$ (see also Proposition 2). Since by hypothesis W^+ is not identically zero, we conclude by Proposition 1 that W^+ has no zero on M and that Ψ^* is an eigenform of W^+ , corresponding to (the unique) simple eigenvalue; according to Proposition 2, the norm of $\frac{1}{|K|^2} \Psi^*$ must be a constant multiple of $|W^+|^{\frac{4}{3}}$, showing that Ψ^* has no zero on M ; thus, the complex structure J is globally

defined on M . The last part of Theorem 4 is a consequence of Theorems 1 and 2.

Remark 5. (i) We cannot expect a similar result to hold for quaternionic quotients of the remaining 8-dimensional quaternionic Kähler Wolf spaces ; we are grateful to D. Calderbank and K. Galicki for pointing this out to us.

(ii) The harmonic 2-form $\frac{2}{|K|^2}\Psi^*$ is the curvature of the bundle $\pi : L_K \mapsto M$. It follows that L_K is a Sasakian manifold fibered over a Kähler self-dual — equivalently, a Bochner-flat — orbifold. It is well known that the corresponding CR-structure of L_K has vanishing fourth-order Chern-Moser curvature; therefore L_K is uniformized over S^5 with respect to $\text{Aut}_{CR}(S^5) = PU(3, 1)$, cf. [53].

(iii) As observed in [27, p. 20], the quaternionic reduction procedure can be applied to the quaternionic hyperbolic space to obtain *smooth, complete* (non locally symmetric) Einstein self-dual metrics of negative scalar curvature, which are necessarily Hermitian by Lemma 7; see also [13] for another construction of complete Einstein self-dual Hermitian metrics.

REFERENCES

- [1] V. Apostolov, J. Davidov and O. Muškarov, *Compact Self-Dual Hermitian Surfaces*, Trans. Amer. Math. Soc. 348 (1996), 3051–3063.
- [2] V. Apostolov and P. Gauduchon, *The Riemannian Goldberg-Sachs Theorem*, Int. J. of Math. 8 (1997), 421–439.
- [3] J. Armstrong, *Almost Kähler Geometry*, Ph.D. Thesis, Oxford, 1998.
- [4] M.F. Atiyah, N.J. Hitchin and I.M. Singer, *Self-duality in four dimensional geometry*, Proc. Roy. Soc. London, A 362 (1979), 425–461.
- [5] F. Belgun, D. Calderbank and A. Moroianu, in preparation.
- [6] L. Bérard Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Publications de l'Institut É. Cartan (Nancy), 4 (1982), 1–60.
- [7] A. L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb.3, Folge 10, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [8] J.-P. Bourguignon, *Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein*, Invent. Math. 63 (1981), 263–286.
- [9] C. Boyer and J. Finley, *Killing vectors in self-dual Euclidean Einstein spaces*, J. Math. Phys. (1982), 1126–1130.
- [10] C. Boyer, *Conformal duality and compact complex surfaces*, Math. Ann. 274 (1986), 517–526.
- [11] C. Boyer, *A note on hyperhermitian four-manifolds*, Proc. Amer. Math. Soc. 102 (1988), 157–164.
- [12] C. Boyer, *Self-dual and anti-self-dual Hermitian metrics on compact complex surfaces*, in “Mathematics and General Relativity” Proceedings, Santa Cruz 1986 (ed. J. Isenberg), Contemp. Math. 71 (1988), 105–114.
- [13] R. Bryant, *Bochner-Kähler metrics*, to appear in J. Amer. Math. Soc., available at arXiv:math.DG/0003099.
- [14] D.M.J. Calderbank, *The Faraday 2-form in Einstein-Weyl geometry*, Math. Scand., to appear.
- [15] D.M.J. Calderbank, *The geometry of the Toda equation*, J. Geom. Phys., to appear.
- [16] D.M.J. Calderbank, *Selfdual Einstein metrics and conformal submersions*, preprint Ecole Polytechnique 99-24 (1999).

- [17] D.M.J. Calderbank and K.P. Tod, *Einstein metrics, hypercomplex structures and the Toda field equation*, Diff. Geom. Appl., to appear.
- [18] D.M.J. Calderbank and H. Pedersen, *Selfdual spaces with complex structures, Einstein-Weyl geometry and geodesics*, Ann. Inst. Fourier, 50 (2000), 921–963.
- [19] D.M.J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, in preparation.
- [20] B.-Y. Chen, *Some topological obstructions to Bochner-Kähler metrics and their applications*, J. Diff. Geom. 13 (1978), 574–588.
- [21] A. Dancer and I. Strachan, *Kähler-Einstein metrics with $SU(2)$ action*, Math. Proc. Cambridge Philos. Soc., 115 (1994), 513–525.
- [22] A. Derdziński, *Exemples de métriques de Kähler et d'Einstein autoduales sur le plan complexe*, in *Géométrie riemannienne en dimension 4*, Séminaire Arthur Besse (eds. L. Bérard-Bergery, M. Berger C. Houzel) CEDIC/Fernand Nathan, 1981.
- [23] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. 49 (1983), 405–433.
- [24] M.G. Eastwood and K.P. Tod, *Local constraints on Einstein-Weyl geometries*, J. reine angew. Math. 491 (1997), 183–198.
- [25] K. Galicki, *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Comm. Math. Phys. 108 (1987), 108, 117–138.
- [26] K. Galicki, *New metrics with $Sp_n Sp_1$ holonomy*, Nucl. Phys. B 289 (1987), 573.
- [27] K. Galicki and H. B. Lawson, *Quaternionic Reduction and Quaternionic Orbifolds*, Math. Ann. 282 (1988), 1–21.
- [28] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. 267 (1984), 495–518.
- [29] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$* , J. reine angew. Math. 469 (1995), 1–50.
- [30] P. Gauduchon, *Complex structures on compact conformal manifolds of negative type*, in “Complex Analysis and Geometry”, (eds. V. Ancona, E. Ballico and A. Silva) Marcel Dekker, New York-Basel-Hong Kong, 1996, 201–212.
- [31] P. Gauduchon, *Connexion canonique et structures de Weyl en géométrie conforme*, Preprint (unpublished).
- [32] P. Gauduchon and K. P. Tod, *Hyper-Hermitian metrics with symmetry*, J. Geom. Phys. 25 (1998), 291–304.
- [33] G. Gibbons and S. Hawking, *Classification of Gravitational Instanton Symmetries*, Comm. Math. Phys. 66 (1979), 291–310.
- [34] M. Itoh, *Self-duality of Kähler surfaces*, Compositio Math. 51 (1984), 265–273.
- [35] P. E. Jones and K. P. Tod, *Minitwistor spaces and Einstein-Weyl spaces*, Class. Quantum Grav. 2(1985), 565–577.
- [36] D. Joyce, *Explicit construction of self-dual 4-manifolds*, Duke Math. J., 77 (1995), 519–552.
- [37] C. LeBrun, *Counter-example to the generalized positive action conjecture*, Commun. Math. Phys. 118 (1988), 591–596.
- [38] C. LeBrun, private communication.
- [39] A. B. Madsen, *Einstein-Weyl structures in the conformal classes of LeBrun metrics*, Class. Quantum Grav. 14 (1997), 2635–2645.
- [40] P. Nurowski, *Einstein equations and Cauchy-Riemann geometry*, Ph.D. Thesis, SISSA/ISAS, Trieste, 1993.
- [41] H. Pedersen, *Einstein metrics, spinning top motions and monopoles*, Math. Ann. 274 (1986), 35–59.
- [42] H. Pedersen and A. Swann, *Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature*, J. reine angew. Math. 441 (1993), 99–113.
- [43] J. Plebański and M. Przanowski, *Hermite-Einstein four-dimensional manifolds with symmetry*, Class. Quantum Grav. 15 (1998), 1721–1735.

- [44] M. Przanowski and B. Broda, *Locally Kähler gravitational instantons*, Acta Phys. Pol. B14 (1983), 637–661.
- [45] S. Salamon, *Special structures on four-manifolds*, Riv. Mat. Univ. Parma 17 (4) (1991), 109–123.
- [46] S. Salamon, Riemannian geometry and holonomy groups. *Pitman Research Notes in Mathematics Series*, 201, New York, 1989.
- [47] K. P. Tod, *Cohomogeneity-one metrics with self-dual Weyl tensor*, in *Twistor Theory* (ed. S. Huggett), Marcel Dekker, New York (1995), 171–184.
- [48] K. P. Tod, *Scalar-flat Kähler and hyper-Kähler metrics from Painlevé-III*, Class. Quantum Grav. 12 (1995), 1535–1547.
- [49] K. P. Tod, *The $SU(\infty)$ -Toda field equation and special four-dimensional metrics*, in *Geometry and Physics* (eds. J.E. Andrsen, J. Dupont, H.Pedersen and A. Swann), Marcel Dekker, New York (1997), 307–312.
- [50] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. 267 (1981), 365–398.
- [51] I. Vaisman, *On locally and globally conformal Kähler manifolds*, Trans. Amer. Math. Soc. 262 (1980), 533–542.
- [52] I. Vaisman, *Some curvature properties of complex surfaces*, Ann. Mat. Pura Appl. 32 (1982), 1–18.
- [53] S. Webster, *On the pseudo-conformal geometry of a Kähler manifolds*, Math. Z., 157 (1977), 265–270.

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