BIHERMITIAN SURFACES WITH ODD FIRST BETTI NUMBER

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Abstract. Compact bihermitian surfaces are considered, that is, compact, oriented, conformal four-manifolds admitting two distinct compatible complex structures. It is shown that if the first Betti number is odd then, with respect to either complex structure, such a manifold belongs to Class VII in the Enriques-Kodaira classification. Moreover, it must be either a special Hopf or an Inoue surface (in the strongly bihermitian case), or is obtained by blowing-up a minimal, class VII surface with curves (in the non-strongly bihermitian case).

1. Introduction

A compact, connected, oriented, conformal 4-manifold (M, c) is called a *bihermitian surface* if it admits two *distinct* complex structures $J_i, i = 1, 2$, compatible with the conformal structure c and the orientation of M; here and henceforth distinct means that $J_1(x) \neq \pm J_2(x)$ at some point x of M. The triple (c, J_1, J_2) will be then called a (conformal) *bihermitian structure* on M; (c, J_1, J_2) is *strongly bihermitian structure* if $J_1 \neq \pm J_2$ is satisfied everywhere on M.

One of the reasons motivating the study of bihermitian conformal structures is the nice link between conformal and complex geometry that exists in real dimension four: It is known [24] that given two distinct complex structures inducing the same orientation there exists at most one conformal structure compatible with both of them. Conversely, except for the special case of *hyperhermitian* four-manifolds where there is a whole S^2-family of compatible complex structures (see [5] for a classification), a compact oriented conformal four-manifold admits at most two distinct positive orthogonal complex structures.

The classification of compact bihermitian surfaces has been obtained by M. Pontecorvo [24] in the case when the corresponding conformal structure is *anti-self-dual* (ASD for short). It was expected that bihermitian structures could not possibly occur in the non-ASD case. The discovery of P.Kobak [16] of explicit examples of (strongly) bihermitian structures on T^4 thus came as a nice surprise. Further results aiming at a classification in the general case (still to come) have been recently obtained in [2]. It is known now that the spectrum of the self-dual Weyl tensor of a non-ASD bihermitian conformal

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structure (c, J_1, J_2) cannot be everywhere degenerate, and therefore none of the complex structures J_1 or J_2 can be Kähler with respect to any metric in c. This latter remark shows the theory of bihermitian structures to be particularly interesting for complex surfaces of non-Kähler type, i.e., the case when the first Betti number is *odd*.

The present paper is a sequel to [2], where the main results have been obtained for bihermitian surfaces with *even* first Betti number. We are now able to extend the theory to the case when the first Betti number is *odd* and find a number of necessary conditions that a bihermitian surface has to satisfy in this case. These are summarized in the following

Theorem 1. *Let (M, c, J_1, J_2) be a compact bihermitian surface with odd first Betti number. Then the Kodaira dimensions of (M, J_1) and (M, J_2) are $-\infty$. Moreover,*

(i) *if (c, J_1, J_2) is a strongly bihermitian structure, then (M, J_1) (and (M, J_2)) is either a Hopf surface finitely covered by a primary one of the form $(\mathbb{C}^2 \setminus \{0\})/\Gamma$, where $\Gamma = \langle\gamma\rangle$ is the infinite cyclic group generated by*

$$\gamma(z_1, z_2) = (\alpha z_1 + \lambda z_2^m, a\alpha^{-1} z_2),$$

$$\alpha, \lambda \in \mathbb{C}, 0 < |\alpha|^2 \leq a < 1, (a^m - \alpha^{m+1})\lambda = 0,$$

or else (M, J_1) (and (M, J_2)) is one of the Inoue surfaces $S^-_{\mathcal{N},p,q,r}$ and $S^+_{\mathcal{N},p,q,r,t}$;

(ii) *if (c, J_1, J_2) is not a strongly bihermitian structure, then (M, J_1) (and (M, J_2)) is obtained by blowing up a minimal class* VII *surface with curves.*

The Hopf surfaces, studied in detail by Kodaira [17] and Kato [14], are all diffeomorphic to $(S^1 \times S^3)/H$, where H is a certain finite group. Theorem 1(i) shows that if (c, J_1, J_2) is a strongly bihermitian structure on $S^1 \times S^3$, then the complex structures J_1 and J_2 must be quite special — they arise from *taut contact circles* on S^3, cf. [10]. The only known examples of bihermitian structures on $S^1 \times S^3$ are those presented in [2]; the generator of the fundamental group γ is then given as in Theorem 1(i) with $\lambda = 0$ and $|\alpha|^2 = a < 1$.

There are three series of Inoue surfaces constructed in [12] and Theorem 1 shows that at most two of them can admit bihermitian structures. No examples of bihermitian structures are known at present on these Inoue surfaces. If such examples were found they would be examples of strongly bihermitian surfaces which do *not* admit any ASD metric and would thus be of major interest for the development of the theory.

The classification of complex surfaces of Kodaira dimension $-\infty$ (called Class VII surfaces, see *e.g.* [3]) is still an open problem, but some progress was made in the case of minimal surfaces admitting curves, see [23] for an overview. It is conjectured and proved in many situations that any such surface is diffeomorphic to $(S^1 \times S^3)\sharp n\bar{\mathbb{CP}}^2, n \geq 0$. In this direction, Theorem 1(ii) in turn motivates us to conjecture that *the underlying smooth manifold of any non-strongly bihermitian surface is* $(S^1 \times S^3)\sharp n\bar{\mathbb{CP}}^2, n \geq 0$. The only known examples of bihermitian structures on these manifolds are LeBrun's ASD conformal metrics [18], [24].

Combining Theorem 1 with the description of compact bihermitian surfaces with even first Betti number given in [2, Thm.1], we derive that for every compact bihermitian surface (M, c, J_1, J_2) the complex surface (M, J_1) (and (M, J_2)) is either a complex torus or a K3 surface, or else its Kodaira dimension is $-\infty$. As an immediate consequence of this fact we obtain the following generalization (in complex dimension 2) of the well known results of Lichnerowicz [19] for infinitesimal isometries of a compact Kähler manifold.

Corollary 1. *Let (M, g, J) be a compact Hermitian surface. Suppose there exists an orientation preserving conformal isometry of (M, g), which is not a $\pm$-biholomorphism. Then either (M, J) is a complex torus or a K3 surface, or else the Kodaira dimension of (M, J) is $-\infty$.*

Examples of conformal Hermitian structures admitting orientation preserving non-biholomorphic isometries are known to exist on the complex surfaces mentioned in Corollary 1 (see [1], [24] and [2]).

The proof of Theorem 1 relies on results of [2] summarized in Section 2 (see Proposition 1), a vanishing argument (Section 3, Lemma 1) obtained as a consequence of the fundamental ideas of Gauduchon [6, 7, 8, 9], and further analysis involving the Enriques-Kodaira classification of complex surfaces, Bogomolov's theorem [4], and the works of Inoue [12, 13].

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2. Preliminary results

In this section we collect some results of [2] that we shall use later. We thus refer to [2] for further details and proofs.

Let (c, J_1, J_2) be a bihermitian structure on a compact 4-manifold M. Fix a metric g in c and denote by F_1^g and F_2^g the *Kähler forms* of (g, J_1) and (g, J_2), respectively, defined by

$$F_i^g(\cdot, \cdot) = g(J_i \cdot, \cdot), i = 1, 2.$$

Let θ_1^g and θ_2^g be the corresponding *Lee forms*, i.e.

$$dF_i^g = \theta_i^g \wedge F_i^g, i = 1, 2.$$

The *angle function* of (c, J_1, J_2) is defined by (see [24], [2]):

$$J_1 \circ J_2 + J_2 \circ J_1 = -2p \,\mathrm{Id}, \tag{1}$$

or equivalently by

$$p = -\frac{1}{4}\mathrm{trace}(J_1 \circ J_2).$$

Clearly, at any point $x \in M$ the smooth function p satisfies the inequality $|p| \leq 1$ with $p(x) = \pm 1$ if and only if $J_1(x) = \pm J_2(x)$. It follows that for a *strongly* bihermitian structure we have $|p| < 1$ everywhere on M.
We furthermore denote by $[J_1, J_2] = J_1 \circ J_2 - J_2 \circ J_1$ the *commutator* of J_1 and J_2 and consider the *real* J_1-anti-invariant 2-form

$$\Phi^g(\cdot, \cdot) = \frac{1}{2} g([J_1, J_2]\cdot, \cdot),$$

and the corresponding *complex* $(0, 2)$-form

$$\sigma_1^g(\cdot, \cdot) = \Phi^g(\cdot, \cdot) + i\Phi^g(J_1 \cdot, \cdot).$$

Then σ_1^g is a smooth section of the anti-canonical bundle $\mathbf{K}_{J_1}^{-1} \cong \Lambda_{J_1}^{0,2}(M)$ of (M, J_1) such that σ_1^g vanishes at a point $x \in M$ if and only if $\Phi^g(x) = 0$, i.e., iff $J_1(x) = \pm J_2(x)$.

The following properties of θ_1^g, θ_2^g and σ_1^g are taken from [2]:

Proposition 1. *Let (M, c, J_1, J_2) be a compact bihermitian surface. Then, for any metric g in the conformal class c, the 1-forms θ_1^g and θ_2^g and the complex 2-form σ_1^g (viewed as a smooth section of $\mathbf{K}_{J_1}^{-1}$) satisfy*

(i) $d(\theta_1^g + \theta_2^g) = 0$;
(ii) $\int_M |\theta_1^g|^2 dV_g = \int_M |\theta_2^g|^2 dV_g$;
(iii) $\bar{\partial}_{J_1} \sigma_1^g = -\frac{1}{2}(\theta_1^g + \theta_2^g)^{0,1} \otimes \sigma_1^g$,

where $|\cdot|$ stands for the norm with respect to g, $dV_g = \frac{1}{2} F_i^g \wedge F_i^g$ is the volume form, and $(\cdot)^{0,1}$ and $\bar{\partial}_{J_1}$ denote the $(0, 1)$-part and the usual $\bar{\partial}$-operator with respect to J_1.

Moreover, at any point where $J_1(x) \neq \pm J_2(x)$ we have

(iv) $d\sigma_1^g = (\frac{1}{2}(\theta_1^g + \theta_2^g) + d \ln(1 - p^2)) \wedge \sigma_1^g$.

Proof. The equalities (i) and (ii) of Proposition 1 are obtained after integrating the two relations in [2, Lemma 1]; the equility (iii) is proved in [2, Lemma 3]; Proposition 1(iv) easily follows from [2, Lemma 2] (see also the proof of [2, Thm. 2]). **q.e.d**

There are two important corollaries of Proposition 1, which have been already used in [2].

Firstly, it follows from Proposition 1(iv) that if (c, J_1, J_2) is a *strongly* bihermitian structure, and if the conformal structure c contains a metric g such that $\theta_1^g + \theta_2^g = 0$, then the corresponding $(2, 0)$-form $\Omega_1 = \frac{1}{1-p^2}\bar{\sigma}_1^g$ is holomorphic, and therefore trivializes the canonical bundle of (M, J_1) (here and henceforth $\bar{\sigma}_1^g$ denotes the complex conjugated of σ_1^g).

Secondly, as $\theta_1^g + \theta_2^g$ is closed (see Proposition 1(i)), there exists an open covering $\{U_i\}_{i\in I}$ on M and C^∞-functions $\phi_i : U_i \mapsto \mathbb{R}$ such that the (locally defined) metrics $g_i = e^{\phi_i} g$ satisfy $\theta_1^{g_i} + \theta_2^{g_i} = 0$, i.e.

$$(\theta_1^g + \theta_2^g)|_{U_i} = -2d\phi_i.$$

Setting $\sigma_1^i := \sigma_1^{g_i}$, it follows from Proposition 1(iii) that $(U_i, \sigma_1^i), i \in I$ define a holomorphic section, say σ_1, of the holomorphic line bundle $\mathbf{L} \otimes \mathbf{K}_{J_1}^{-1}$, where $\mathbf{L}$ is the (topologically trivial) holomorphic line bundle determined by the closed 1-form $-\frac{1}{2}(\theta_1^g + \theta_2^g)$ via the embedding (see *e.g.* [9])

$$exp : H^1(M, \mathbb{R}) \hookrightarrow H^1(M, \mathcal{O}^*).$$

The holomorphic line bundle $\mathbf{L}$ is (holomorphically) trivial precisely when $\theta_1^g + \theta_2^g$ is exact, i.e., when there exits a metric g in c such that $\theta_1^g + \theta_2^g = 0$. In the latter case σ_1 is, in fact, a non-trivial holomorphic section of the anti-canonical bundle $\mathbf{K}_{J_1}^{-1}$. Notice that there always exists a metric $g \in c$ with $\theta_1^g + \theta_2^g = 0$ when the first Betti number of M is even ([2, Lemma 4]), or when the conformal structure c is ASD and (J_1, J_2) is *not* strongly bihermitian ([24, Prop.3.5 and 3.7]), but this is no longer true for a generic bihermitian surface ([2, Sec.5]).

3. Proof of Theorem 1

3.1. **Kodaira dimension of bihermitian surfaces.** The *Kodaira dimension* of a compact complex surface (M, J) measures the grow of the plurigenra $P_m(M) := \dim_{\mathbb{C}}(H^0(\mathbf{K}^{\otimes m})), m \geq 1$, where $\mathbf{K}$ denotes the canonical bundle of (M, J), cf. *e.g.* [3]; the complex surface (M, J) has Kodaira dimension $-\infty$ if the plurigenera all vanish, i.e. $H^0(\mathbf{K}^{\otimes m}) = 0$ for any $m \geq 1$.

We start with the following vanishing result

Lemma 1. *Let (M, c, J_1, J_2) be a compact bihermitian surface. Then either the Kodaira dimensions of (M, J_1) and (M, J_2) are $-\infty$, or else (M, c, J_1, J_2) is a strongly bihermitian surface and c contains a metric g with $\theta_1^g + \theta_2^g = 0$.*

Proof. We recall that the *degree* of a holomorphic line bundle $\mathbf{E}$ over a compact Hermitian conformal (Riemannian) manifold (M, c, J) is defined by [9]:

$$\deg(\mathbf{E}) = \frac{1}{(m-1)!} \int_M \rho \wedge F_0^{m-1}, \tag{2}$$

where $m = \dim_{\mathbb{C}} M$, ρ is any *pluriharmonic* representative of the real first Chern class of $\mathbf{E}$, and F_0 is the Kähler form of the *standard* metric g_0 of c [6] — i.e. the unique (up to homothety) Hermitian metric in c such that the corresponding Lee form is co-closed . According to [8], the degree of a holomorphic line bundle $\mathbf{E}$ (with respect to c) is equal to the volume (with respect to g_0) of the divisor of any meromorphic section of $\mathbf{E}$. In particular, if $\mathbf{E}$ admits a holomorphic section, then $\deg(\mathbf{E}) \geq 0$ with equality if and only if $\mathbf{E}$ is holomorphically trivial.

We derived from Proposition 1 that for any compact bihermitian surface (M, c, J_1, J_2) the holomorphic line bundle $\mathbf{L} \otimes \mathbf{K}_{J_1}^{-1}$ admits a non-trivial holomorphic section σ_1. It follows

$$\deg(\mathbf{L} \otimes \mathbf{K}_{J_1}^{-1}) = \deg(\mathbf{L}) + \deg(\mathbf{K}_{J_1}^{-1}) \geq 0, \tag{3}$$

with possible equality precisely when the holomorphic section $\sigma_1 = (U_i, \sigma_1^i)_{i \in I}$ (defined in the preceding section) does not vanish, i.e., when (c, J_1, J_2) is a strongly bihermitian structure.

We next compute the degree of the holomorphic line bundle $\mathbf{L}$ with respect to the complex structure $J = J_1$ and the conformal structure c. For that we use a Hermitian metric h on $\mathbf{L}$, defined on $U_i \times \mathbb{C}$ by

$$h = e^{-2\phi_i} h_0,$$

where h_0 is the flat metric on $\mathbb{C}$. A pluriharmonic representative ρ_h of $c_1^{\mathbb{R}}(\mathbf{L})$ is then locally given by [8]

$$\rho_h = -\frac{i}{2\pi} \partial \bar{\partial} \ln(h(s, s)),$$

where s is any (local) holomorphic section of $\mathbf{L}$. By the very definitions of $\mathbf{L}$ and h we obtain therefore

$$\rho_h = -\frac{1}{4\pi} d^C (\theta_1^g + \theta_2^g), \tag{4}$$

where d^C acts on 1-forms by $d^C = i(\overline{\partial} - \partial) = -J \circ d \circ J$; (here the action of J on T^*M is defined by Riemannian duality between TM and T^*M, and is extended to an involution on $\Lambda^2(T^*M)$ by acting on each decomposable element in an obvious manner). Let g be the *standard* metric of (c, J) and F be the corresponding Kähler form. By (2) and (4) the degree of $\mathbf{L}$ is computed to be

$$\begin{aligned} \deg(\mathbf{L}) &= -\frac{1}{4\pi} \int_M d^C (\theta_1^g + \theta_2^g) \wedge F \\ &= -\frac{1}{4\pi} \int_M \langle d^C (\theta_1^g + \theta_2^g), F \rangle dV_g \\ &= -\frac{1}{4\pi} \int_M \langle \theta_1^g + \theta_2^g, \theta_1^g \rangle dV_g, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the inner-product with respect to g. Using Proposition 1(ii) we eventually obtain

$$\deg(\mathbf{L}) = -\frac{1}{8\pi} \int_M |\theta_1^g + \theta_2^g|^2 dV_g,$$

so that $\deg(\mathbf{L}) \leq 0$ with equality if and only if $\theta_1^g + \theta_2^g \equiv 0$. Hence, (3) reads as

$$\deg(\mathbf{K}_{J_1}^{-1}) \geq \frac{1}{8\pi} \int_M |\theta_1^g + \theta_2^g|^2 dV_g. \tag{5}$$

Since $\deg(\mathbf{K}_{J_1}) = -\deg(\mathbf{K}_{J_1}^{-1}) \leq 0$, it follows from [7, Plurigenera Theorem] that either the plurigenera of (M, J_1) all vanish, or else $\deg(\mathbf{K}_{J_1}^{-1}) = 0$; by (3) and (5) the latter situation can occur exactly when (c, J_1, J_2) is strongly bihermitian and $\theta_1^g + \theta_2^g = 0$ is satisfied for the standard metric g of (c, J_1). This completes the proof of the lemma. **q.e.d**

Remark 1. The degree of the anti-canonical bundle with respect to a given conformal class of Hermitian metrics, c, on a compact complex surface (M, J) is called *fundamental constant* of (M, c, J), cf. [7]. We thus have proved in Lemma 1 that the fundamental constants of the two Hermitian structures of a bihermitian surface are either positive or both vanish.

By Lemma 1 and the Enriques-Kodaira classification of compact complex surfaces one easily obtains the first part of Theorem 1:

Corollary 2. *Any compact bihermitian surface with odd first Betti number has Kodaira dimension $-\infty$, and therefore belongs to Class* VII *in the Enriques-Kodaira classification.*

Proof. Suppose for contradiction that the Kodaira dimension of (M, J_1) is non-negative. According to Lemma 1, (c, J_1, J_2) is a strongly bihermitian structure and there is a metric g in c such that $\theta_1^g + \theta_2^g = 0$. It follows from Proposition 1(iv) that the complex $(2,0)$-form $\Omega_1 = \frac{1}{1-p^2}\bar{\sigma}_1^g$ defines a holomorphic-symplectic structure on (M, J_1), i.e., the canonical bundle of (M, J_1) is trivial. Since the first Betti number of M is odd, we conclude that (M, J_1) is a primary Kodaira surface (cf. *e.g.* [3]). Considering J_2 instead of J_1, we derive similarly the existence of a holomorphic-symplectic 2-form Ω_2 on (M, J_2) and we know that Ω_1 and Ω_2 have common real part (equal to $\frac{1}{1-p^2}\Phi^g$, see Section 2). It follows that the real and imaginary parts of Ω_1 and Ω_2 determine three real *harmonic* self-dual forms on (M, c), which are independent at any point where $J_1 \neq \pm J_2$. Therefore, $b^+(M) \geq 3$. But for any primary Kodaira surface $b^+(M) = 2$ (cf. *e.g.* [3]), a contradiction. **q.e.d**

3.2. **Strongly bihermitian surfaces with odd first Betti number.** To prove the statements in Theorem 1(i) we suppose first that (M, c, J_1, J_2) is a compact *strongly* bihermitian surface with odd first Betti number. Since for any metric $g \in c$ the 2-form Φ^g (hence also σ_1^g) nowhere vanishes (see Section 2), the holomorphic line bundle $\mathbf{L} \otimes \mathbf{K}_{J_1}^{-1}$ is trivialized by the holomorphic section $\sigma_1 = (U_i, \sigma_1^i)_{i \in I}$ defined in Section 2. It follows that the real Chern class $c_1(\mathbf{L} \otimes \mathbf{K}_{J_1}^{-1})$ vanishes. The complex line bundle $\mathbf{L}$ is topologically trivial so that the (real) first Chern class $c_1(M)$ of (M, J_1) vanishes as well. By Corollary 2 we know that (M, J_1) is a Class VII surface, and therefore $b_1(M) = 1, b^+(M) = 0$ (see [3]). Thus, by Wu formula we get

$$0 = c_1^2(M) = 2\chi(M) + 3\sigma(M) = -b^-(M) = -b_2(M),$$

i.e., (M, J_1) is a minimal Class VII surface with vanishing second Betti number. Now Bogomolov's theorem [4] states that (M, J_1) is either a Hopf surface or an Inoue surface; recent short proofs of this important result can be found in [21], [22], [25]. The same argument still holds when considering J_2 instead of J_1, so that (M, J_2) is a Hopf or an Inoue surface as well. We next consider separately the two cases.

Case 1: Hopf surfaces. Recall that a Hopf surface is by definition a compact complex surface whose universal covering is $\mathbb{C}^2 \setminus \{0\}$. It was shown by Kodaira [17] that $\pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}_n$, hence any such surface is finitely covered by a primary one – i.e. one whose fundamental group is infinitely cyclic. For simplicity we shall only consider primary Hopf surfaces; they are all diffeomorphic to $S^1 \times S^3$ with cyclic fundamental group $\Gamma = \langle\gamma\rangle$, where γ is an automorphism of $\mathbb{C}^2 \setminus \{0\}$ defined by

$$\begin{aligned} &\gamma(z_1, z_2) = (\alpha z_1 + \lambda z_2^m, \beta z_2), \\ &\alpha, \beta, \lambda \in \mathbb{C}, 0 < |\alpha| \leq |\beta| < 1, (\beta^m - \alpha)\lambda = 0. \end{aligned} \tag{6}$$

Theorem 1(i) says that for strongly bihermitian Hopf surfaces the corresponding complex structures are obtained as above with $\alpha\beta \in \mathbb{R}$, i.e., we have to prove the following

Lemma 2. *Let (c, J_1, J_2) be a strongly bihermitian structure on $M = S^1 \times S^3$. Then (M, J_1) (and (M, J_2)) is a primary Hopf surface of the form $(\mathbb{C}^2 \setminus \{0\})/\Gamma$, where $\Gamma = \langle\gamma\rangle$ is the infinite cyclic group generated by the automorphism*

$$\gamma(z_1, z_2) = (\alpha z_1 + \lambda z_2^m, a\alpha^{-1} z_2),$$

$$\alpha, \lambda \in \mathbb{C},\ 0 < |\alpha|^2 \leq a < 1,\ (a^m - \alpha^{m+1})\lambda = 0.$$

Proof. It is a well known result of Kodaira ([17, III, Thm.42]) that any complex surface diffeomorphic to $S^1 \times S^3$ is a primary Hopf surface. It follows from the considerations in Section 2 that (c, J_1, J_2) lifts to define a (strongly) bihermitian structure (also denoted by (c, J_1, J_2)) on the universal cover $\mathbb{C}^2 \setminus \{0\}$, with J_1 being the standard complex structure on $\mathbb{C}^2$. Since $\mathbb{C}^2 \setminus \{0\}$ is simply connected, the closed 1-form $\theta_1^g + \theta_2^g$ is exact, hence there exists a metric $g \in c$ with $\theta_1^g + \theta_2^g = 0$. As (c, J_1, J_2) is strongly bihermitian, it follows from Proposition 1(iv) that $\Omega_1 = \frac{1}{1-p^2}\bar{\sigma}_1^g$ is a holomorphic 2-form, where, we recall $p = -\frac{1}{4}\mathrm{trace}(J_1 \circ J_2)$ is the *angle function* of (c, J_1, J_2), see (1). It follows that

$$\Omega_1 = h dz_1 \wedge dz_2$$

for a holomorphic function $h(z_1, z_2)$ on $\mathbb{C}^2 \setminus \{0\}$ (and hence on $\mathbb{C}$) which does not vanish on $\mathbb{C}^2 \setminus \{0\}$ (hence on $\mathbb{C}^2$). Moreover, since the complex structures J_1, J_2 and the conformal class $c = [g]$ are all Γ-invariant, we have

$$\gamma^*(\Omega_1) = f\Omega_1$$

for some real-valued, positive function f on $\mathbb{C}^2 \setminus \{0\}$. We then get

$$\alpha\beta(h \circ \gamma) dz_1 \wedge dz_2 = f h dz_1 \wedge dz_2,$$

or equivalently

$$f = \alpha\beta\Big(\frac{h\circ\gamma}{h}\Big).$$

The function in the right-hand side of the latter equality is holomorphic, while f is a real-valued function. It follows that f is a positive real constant, say a, so that

$$h\circ\gamma = \frac{a}{\alpha\beta}h.$$

Then, for all $(z_1, z_2) \in \mathbb{C}^2$,

$$\lim_{n\to\infty}(\frac{a}{\alpha\beta})^n h(z_1, z_2) = \lim_{n\to\infty} h\circ\gamma^n(z_1, z_2) = h(0,0) \neq 0,$$

which implies

$$\alpha\beta = a. \tag{7}$$

The positive constant a must also satisfy $a < 1$ because of (6); the lemma then follows from (7) and (6). **q.e.d**

Remark 2. According to [10], the primary Hopf surfaces described in Lemma 2 are precisely those which arise from a *taut contact circle* on S^3. Following [10], a taut contact circle is by definition a couple of contact structures (ω_0, ω_1) on S^3 satisfying the two orthogonality conditions

(1) $\omega_0 \wedge d\omega_0 = \omega_1 \wedge d\omega_1$,
(2) $\omega_0 \wedge d\omega_1 + \omega_1 \wedge d\omega_0 = 0$.

If we are given such a couple, we may consider the two (real) symplectic forms $\Phi_0 = d(e^t\omega_0), \Phi_1 = d(e^t\omega_1)$ on $\mathbb{R}\times S^3$, which, because of the conditions (1) and (2), satisfy $\Phi_0\wedge\Phi_0 = \Phi_1\wedge\Phi_1, \Phi_0\wedge\Phi_1 = 0$. Then (Φ_0, Φ_1) determine a unique complex structure J_1, such that $\Omega_1 = \Phi_0 + i\Phi_1$ is a holomorphic symplectic structure on $(\mathbb{R} \times S^3, J_1)$ (cf. [20, 11]). Since the vector field $X = \partial_t$ preserves the complex structure J_1, the 2-forms Φ_0 and Φ_1 induce, in fact, a complex structure (denoted also by J_1) on $S^1\times S^3 \cong (\mathbb{R}\times S^3)/\langle\phi_{t_0}\rangle$ where ϕ_t denotes the flow of X, t_0 is a fixed real number, and $\langle\phi_{t_0}\rangle$ is the infinite cyclic group generated by ϕ_{t_0}. The primary Hopf surface $(S^1\times S^3, J_1)$ is then of the form described in Lemma 2 with $a = e^{-t_0}$, cf. [10, Cor. 5.4.].

A possible "contact approach" for looking for strongly bihermitian structures (J_1, J_2) on $S^1 \times S^3$ would be to find a triple of contact forms on S^3, say $(\omega_0, \omega_1, \omega_2)$, such that (ω_0, ω_1) and (ω_0, ω_2) are taut contact circles and

$$\omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 = -p\omega_0 \wedge d\omega_0,$$

where p is a smooth function p on S^3 (the angle function) which must satisfy $|p| < 1$. The corresponding complex structures J_1 and J_2 on $S^1 \times S^3$ would then satisfy the relation (1), and therefore would define a strongly bihermitian structure. The standard *Cartan structure* on S^3 is an example of a triple of contact structures satisfying the above relations with $p \equiv 0$; the corresponding bihermitian structure on $S^1\times S^3$ is then *hyperhermitian.* Moreover, the deformations of strongly bihermitian structures on $S^1 \times S^3$ presented in

[2] correspond to deformations of $(\omega_0, \omega_1, \omega_2)$ in the form $(\omega_0, \omega_1, \psi_s^*\omega_2)$, where ψ_s is a flow which preserves the contact form ω_0.

Case 2: Inoue surfaces. Any such surface is by definition the quotient $(\mathbb{H} \times \mathbb{C})/\Gamma$ where $\mathbb{H}$ is the upper half-plane and Γ is any lattice in one of the three solvable real sub-groups Sol_0^4, Sol_1^4 and Sol'^4_1 of the group $A(2, \mathbb{C})$ of affine transformations of $\mathbb{C}^2$ (see *e.g.* [12, 26, 15]). There are three series of Inoue surfaces which are usually denoted by $S^{\pm}_{\mathcal{M}}, S^{+}_{\mathcal{N},p,q,r,t}, S^{-}_{\mathcal{N},p,q,r}$; the surfaces $S^{\pm}_{\mathcal{M}}$ are obtained as quotients of Sol_0^4 ; the surfaces $S^{+}_{\mathcal{N},p,q,r,t}, t \in \mathbb{R}$ and $S^{-}_{\mathcal{N},p,q,r}$ are quotients of Sol_1^4, while the surfaces $S^{+}_{\mathcal{N},p,q,r,t}, t \in \mathbb{C} \setminus \mathbb{R}$ are quotients of the group Sol'^4_1. Moreover, the parameterization of these surfaces comes from the explicit description of the lattices in the groups Sol_0^4, Sol_1^4 and Sol'^4_1 [12].

Consider first $S^{\pm}_{\mathcal{M}}$. Following Inoue [12], take an unimodular matrix $\mathcal{M} = (m_{kj}) \in SL(3, \mathbb{Z})$ with eigenvalues α, β, $\bar{\beta}$ such that $\alpha > 1$ and $\beta \neq \bar{\beta}$. Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be a (real) eigenvector and an eigenvector of the matrix $\mathcal{M}$ corresponding to α and β, respectively. Let $\Gamma^{+}_{\mathcal{M}}$ be the group of holomorphic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\gamma_0(w, z) = (\alpha w, \beta z), \quad \gamma_k(w, z) = (w + a_k, z + b_k), \ k = 1, 2, 3. \tag{8}$$

If we consider $\bar{\beta}$ instead of β we obtain the group $\Gamma^{-}_{\mathcal{M}}$. It is easy to see that the groups $\Gamma^{\pm}_{\mathcal{M}}$ act on $\mathbb{H} \times \mathbb{C}$ freely and properly discontinuously and the quotient surfaces $S^{\pm}_{\mathcal{M}} = (\mathbb{H} \times \mathbb{C})/\Gamma^{\pm}_{\mathcal{M}}$ are compact complex surfaces which are diffeomorphically, but not biholomorphically equivalent. Realizations of $\Gamma^{\pm}_{\mathcal{M}}$ as discrete subgroups of Sol_0^4 are given in [26].

To construct the surfaces $S^{-}_{\mathcal{N},p,q,r}$ and $S^{+}_{\mathcal{N},p,q,r,t}$ take a matrix $\mathcal{N} \in GL(2, \mathbb{Z})$ with $det(\mathcal{N}) = \epsilon$, $\epsilon = \pm 1$, having real eigenvalues $\alpha > 1$ and ϵ/α. Let (a_1, a_2) and (b_1, b_2) be (real) eigenvectors corresponding to α and ϵ/α, respectively. Fix integers $p, q, r \neq 0$ and a complex number t. Let (c_1, c_2) be the solution of the (algebraic) equation

$$\epsilon(c_1, c_2) = (c_1, c_2)N^t + (e_1, e_2) + \frac{1}{r}(b_1 a_2 - b_2 a_1)(p, q)$$

where $\mathcal{N}^t$ denotes the transpose matrix of $\mathcal{N}$ and

$$e_k = \frac{1}{2} n_{k1}(n_{k1} - 1) a_1 b_1 + \frac{1}{2} n_{k2}(n_{k2} - 1) a_2 b_2 + n_{k1} n_{k2} b_1 a_2, \quad k = 1, 2.$$

Denote by $\Gamma^{(\epsilon)}_{\mathcal{N},p,q,r;t}$ the group of holomorphic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$\gamma_0^{(\epsilon)}(w, z) = (\alpha w, \epsilon z + t), \quad \gamma_k(w, z) = (w + a_k, z + b_k w + c_k), \ k = 1, 2,$$

$$\gamma_3(w, z) = (w, z + \frac{1}{r}(b_1 a_2 - b_2 a_1))$$

Then the groups $\Gamma^+_{\mathcal{N},p,q,r;t}$ and $\Gamma^-_{\mathcal{N},p,q,r;0}$ $(t = 0)$ are isomorphic to discrete subgroups of Sol^4_1 and Sol'^4_1; they act on $\mathbb{H} \times \mathbb{C}$ freely and properly-discontinuously, and the quotient spaces $S^+_{\mathcal{N},p,q,r;t}$ and $S^-_{\mathcal{N},p,q,r}$ are compact complex surfaces. Each surface $S^-_{\mathcal{N},p,q,r}$ is doubly covered by the surface $S^+_{\mathcal{N}^2,p',q',r;0}$ for suitable p', q'.

Theorem 1(i) says that the surfaces $S^{\pm}_{\mathcal{M}}$ do not admit bihermitian structures at all, i.e. we have to prove

Lemma 3. *The smooth 4-manifold underlying the Inoue surfaces $S^{\pm}_{\mathcal{M}}$ does not admit any bihermitian structure.*

Proof. Let (M, c, J_1, J_2) be a bihermitian surface with M being diffeomorphic to the smooth manifold underlying the complex surfaces $S^{\pm}_{\mathcal{M}}$. Since $\pi_1(M) \cong \Gamma^+_{\mathcal{M}}$, we have by [13] that (M, J_1) (hence, also, (M, J_2)) is one of the surfaces $S^{\pm}_{\mathcal{M}}$. As $S^{\pm}_{\mathcal{M}}$ do not admit any curves, the holomorphic section $\sigma_1 = (U_i, \sigma^i_1)_{i \in I}$ of $\mathbf{L} \otimes \mathbf{K}^{-1}_{J_1}$ (see Section 2) does not vanish, i.e. (c, J_1, J_2) is a strongly bihermtian structure. Then (c, J_1, J_2) lifts to a strongly bihermitian structure (still denoted by (c, J_1, J_2)) on the universal cover $\mathbb{H} \times \mathbb{C}$ of $S^{\pm}_{\mathcal{M}}$, with J_1 being the standard complex structure on $\mathbb{H} \times \mathbb{C}$. Arguing as in the case of Hopf surfaces (see the proof of Lemma 2) we show the existence of a metric $g \in c$ with $\theta^g_1 + \theta^g_2 = 0$, a holomorphic 2-form $\Omega_1 = \frac{1}{1-p^2}\bar{\sigma}^g_1$, and a holomorphic function u on $\mathbb{H} \times \mathbb{C}$, such that the following are satisfied

$$\Omega_1 = e^u dw \wedge dz, \tag{9}$$

$$\gamma^*(\Omega_1) = f\Omega_1 = fe^u dw \wedge dz, \tag{10}$$

where f is a real-valued positive function, and γ is any element of $\Gamma^+_{\mathcal{M}}$ (resp. of $\Gamma^-_{\mathcal{M}}$). On the other hand, by (8) we get

$$\gamma^*(dw \wedge dz) = const.dw \wedge dz, \ \forall \gamma \in \Gamma^+_{\mathcal{M}}(resp.\Gamma^-_{\mathcal{M}})$$

so that, as in the proof of Lemma 2, we infer from (9) and (10) that f is a constant, i.e.

$$u \circ \gamma - u = c_\gamma, \ c_\gamma \in \mathbb{C}.$$

But then u is $[\Gamma^+_{\mathcal{M}}, \Gamma^+_{\mathcal{M}}]$-invariant holomorphic function on $\mathbb{H} \times \mathbb{C}$, and therefore constant ([12, Lemma 3], [15, Lemme 4.4]). With respect to the generator γ_0 defined by (8), the relations (9) and (10) imply $\alpha\beta \in \mathbb{R}$, a contradiction. **q.e.d**

3.3. **Non-strongly bihermitian surfaces with odd first Betti number.** To complete the proof of Theorem 1 it remains to consider the case of a *non-strongly* bihermitian surface (M, c, J_1, J_2) and to show that (M, J_1) and (M, J_2) are obtained by blowing up a *minimal* surface admitting a complex curve (Theorem 1(ii)).

Suppose (c, J_1, J_2) is not strongly bihermitian structure. Then the holomorphic section $\sigma_1 = (U_i, \sigma^i_1)_{i \in I}$ of $\mathbf{L} \otimes \mathbf{K}^{-1}_{J_1}$ (see Section 2) does vanish, and therefore defines an *effective* divisor, D, on (M, J_1). The Poincaré-dual class $[D]$ in $H^2(M,\mathbb{Z})$ is $c_1(M,J_1)$ because the holomorphic line bundle $\mathbf{L}$ is topologically trivial; by the adjunction formula we calculate the *virtual genus* $\pi(D)$ of D :

$$\pi(D) = 1 + \frac{1}{2}(\mathbf{K}_{J_1}\cdot[D] + [D]\cdot[D]) = 1,$$

where $\cdot$ denotes the cup-product of $H^2(M,\mathbb{Z})$. Theorem 1(ii) then follows by the more general fact that on any complex surface S an effective divisor D of positive virtual genus defines a non-trivial curve on any minimal model of S. Indeed, if S itself is minimal the claim is trivial. If S is *not* minimal, let $E\subset S$ be a smooth rational curve of self-intersection -1 and $b\ : S\mapsto\tilde{S}$ be the blowing down map which contracts E to a point x and which is biholomorphic elsewhere. Suppose first that $D = mE, m\in\mathbb{N}$. As E is a smooth curve of geometric genus 0 and self-intersection -1 we calculate the virtual genus of $D = mE$ to be $\pi(mE) = 1 - \frac{m(m+1)}{2}$. But this contradicts the assumption that D is effective divisor with $\pi(D) > 0$. We thus have $D = D' + mE$ with D' being an effective divisor which does not contain the exceptional curve E. Then $b(D')$ defines an effective divisor $\tilde{D}$ of $\tilde{S}$, such that the proper pre-image of $\tilde{D}$ is D' and we know [3]

$$[D'] = b^*([\tilde{D}]) + k[E], k\in\mathbb{Z},$$

or, equivalently,

$$[D] = b^*([\tilde{D}]) + \ell[E], \ell\in\mathbb{Z}.$$

Moreover, the canonical bundles $\mathbf{K}_S$ and $\mathbf{K}_{\tilde{S}}$ of S and $\tilde{S}$ are related by [3]

$$\mathbf{K}_S = b^*(\mathbf{K}_{\tilde{S}}) + [E].$$

Taking into account

$$b^*([\tilde{D}])\cdot[E] = b^*(\mathbf{K}_{\tilde{S}})\cdot[E] = 0$$

we get

$$\pi(\tilde{D}) = \pi(D) + \frac{\ell(\ell+1)}{2} \geq \pi(D) > 0.$$

The proof is now complete by induction on the number of blow ups. **q.e.d**

References

[1] D.V. Alekseevsky, M.M. Graev, *Calabi-Yau metric on Fermat surface. Isometries and totally geodesics sub-manifolds*, J. Geom. Physics 7 (1990), 21-43.

[2] V. Apostolov, P. Gauduchon, G. Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc., 79 (1999), 414–429.

[3] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlagh, Berlin Heidelberg New York Tokyo, 1984.

[4] F. A. Bogomolov, *Classification of surfaces of class VII_0 and affine geometry*, Math. USSR-Izv, 10 (1976), 255–269.

[5] C. P. Boyer, *A note on hyperhermitian four-manifolds*, Proc. Amer. Math. Soc., 102, (1988), 157-164.

[6] P. Gauduchon, *Le théorème de l'excentricité nulle*, C.R. Acad.Sci.Paris, Ser.A 285 (1977), 387–390.

[7] P. Gauduchon *Fibrés hermitiens à endomorphisme de Ricci non-négatif*, Bull. Soc. Math. France 105 (1977), 113–140.
[8] P. Gauduchon, *Le théorème de dualité pluriharmonique*, C.R. Acad. Sci. Paris 293 (1981), 59–63.
[9] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. 267, (1984), 495-518.
[10] H. Geiges, J. Gonzalo, *Contact geometry and complex surfaces*, Invent. Math. 121 (1995), 147–209.
[11] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. 55(3) (1987), 59–126.
[12] M. Inoue, *On surfaces of class* VII_0, Invent. Math., 24 (1974), 269–310.
[13] M. Inoue, *An example of an analytic surface*, Sûgaki 27 (1975), 358–364.
[14] M. Kato, *Topology of Hopf surfaces*, J. Math. Soc. Japan 27 (1975), 222–238.
[15] B. Klingler, *Structures affines sur les surfaces complexes*, Ann. Inst. Fourier, 48 (1998), 441–477.
[16] P. Kobak, *Explicit Doubly-Hermitian Metrics*, Diff. Geom. and its Appl., 10 (1999), 179–185.
[17] K. Kodaira, *On the structure of compact complex analytic surfaces*, II, III, Amer. J. Math. 88 (1966), 682-721; 90 (1968), 55–83.
[18] C. LeBrun, *Anti-self-dual hermitian metrics on blown-up Hopf surfaces*, Math. Ann. 289 (1991), 383–392.
[19] A. Lichnerowicz, *Géometrie des groupes de transformations*, Dunod, 1958.
[20] V.V. Lychagin, V.N. Rubtsov, I.V. Chekalov, *A classification of Monge-Ampère equations*, Ann. Sci. École Norm. Sup. (4) 26 (1993), 281–308.
[21] J. Li, S. T. Yau, F. Zheng, *A simple proof of Bogomolov's Theorem on class* VII_0 *surfaces with* $b_2 = 0$, Illinois J. Math. 34 (1990), 217–220.
[22] J. Li, S. T. Yau, F. Zheng, *On projectively-flat hermitian manifolds*, Comm. Anal. Geom. 2 (1994), 103-109.
[23] I. Nakamura, *Towards classification of non-Kähler complex surfaces*, Sagaku Expositions 2 (1989), 209-229.
[24] M. Pontecorvo, *Complex Structures on Riemannian Four-manifolds*, Math. Ann. 309 (1997), 159-177.
[25] A. D. Teleman, *Projectively-flat surfaces and Bogomolov's theorem on class* VII_0 *surfaces*, Int.J.Math. 5 (1994), 253-264.
[26] C. T. C. Wall, *Geometric structures on compact complex analytic surfaces*, Topology 25 (1986), 119-153.

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