An integrability theorem for almost Kähler 4-manifolds

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ABSTRACT. We prove that every compact almost Kähler 4-manifold which satisfies the second curvature condition of Gray [4] is necessarily Kähler.

Un théorème d'intégrabilité pour les variétés presque-kählériennes de dimension 4

RÉSUMÉ. Nous démontrons que chaque variété presque-kählérienne compacte de dimension 4 qui satisfait à la deuxième condition de Gray [4] est kählérienne.

Version française abrégée

Rappelons qu'une structure presque-kählérienne sur une variété M de dimension réelle $2n$ est la donnée d'une structure presque-hermitienne (g, J) dont la 2-forme fondamentale Ω est fermée et fournit alors une structure symplectique; si de plus la structure presque-complexe est intégrable, le triplet (g, J, Ω) définit une structure kählérienne. Un problème naturel consiste à trouver des conditions sur la courbure riemannienne d'une variété presque-kählérienne qui assurent l'intégrabilité de la structure presque-complexe correspondante. Par exemple, une ancienne conjecture de Goldberg [5] — toujours ouverte — affirme que chaque variété presque-kählérienne compacte d'Einstein est kählérienne.

Dans cette Note nous démontrons que sur une variété presque-kählérienne compacte de dimension 4 une condition sur la courbure riemannienne (introduite par A. Gray [4]) implique l'intégrabilité de la structure presque-complexe. En dimension 4 cette "deuxième condition de Gray" signifie que le tenseur de Ricci est J-invariant et que le tenseur de Weyl positif W^+ (considéré comme un endomorphisme symétrique sans trace du fibré de 2-formes autoduales) est de spectre partout dégénéré de manière qu'en chaque point où W^+ ne s'annule pas, Ω est sa forme propre correspondant à la valeur propre de multiplicité 1. Une étude de telles variétés a été récemment developpée dans [1]. Nous complétons ici les résultats obtenus dans [1] par le théorème suivant.

Théorème 1 *Une variété presque-kählérienne compacte de dimension 4 dont la courbure riemannienne satisfait à la deuxième condition de Gray est kählérienne.*

Remarquons qu'un tel résultat n'est plus disponible en dimension $2n$, $n \geq 3$ (*voir* [3]).

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1 Introduction

An almost Kähler structure on a manifold M^{2n} is an almost Hermitian structure (g, J, Ω) for which the fundamental 2-form Ω is closed, and therefore symplectic. If, additionally, the almost complex structure J is integrable, the triple (g, J, Ω) is a Kähler structure on M. There have been many attempts to find sufficient curvature conditions for an almost Kähler structure to ensure integrability. For example, an old, still open conjecture of Goldberg [5] says that a compact almost Kähler, Einstein manifold should be Kähler.

In this note we show that a certain curvature condition first considered by Gray [4] implies integrability for closed manifolds in dimension 4.

It is well known that the curvature of a Kähler metric has strong symmetry properties with respect to the complex structure. On the other hand, an arbitrary almost Kähler metric may have none of these. The condition that the Ricci tensor is J-invariant could be considered to be the minimal degree of resemblance with the Kähler symmetries. This weakly Einstein condition arises naturally from the $U(n)$-decomposition of the curvature [8], and may be even more natural in the context of almost Kähler geometry — interesting variational problems on compact symplectic manifolds lead to almost Kähler metrics with J-invariant Ricci tensor. It was shown in [2] that such almost Kähler metrics are the critical points of the Hilbert functional, the integral of the scalar curvature, restricted to the set of all metrics compatible with a given symplectic form. Compatible Kähler metrics provide absolute maxima for the functional in this setting. It was a natural question [2] to ask if the J-invariance of the Ricci tensor is sufficient for the integrability of an almost Kähler structure on a compact manifold. The answer turns out to be negative in dimension $2n$, $n \geq 3$, by the examples provided in [3], but in dimension 4 the problem is still open, as no examples of compact, non-Kähler, almost Kähler structures with J-invariant Ricci tensor are known yet.

The examples of [3] have, in fact, a higher degree of resemblance to Kähler structures than just the J-invariant Ricci tensor — they are almost Kähler manifolds satisfying the *second curvature condition of Gray [4]*. In dimension 4, this condition just means that the Ricci tensor and the positive Weyl tensor have the same symmetries as they have for a Kähler metric. A study of the local and global geometry of this class of almost Kähler 4-manifolds has been recently started in [1]. In this note we complete the results of [1] in one direction, by proving the following:

Theorem 1 *A compact, 4-dimensional, almost Kähler manifold satisfies the second Gray condition if and only if it is Kähler.*

2 The second Gray condition in dimension 4

In [4], A. Gray considered almost Hermitian manifolds whose curvature tensor R has a certain degree of resemblance to that of a Kähler manifold. The following identities arise naturally:

(G_1) $R_{XYZW} = R_{XYJZJW}$;

(G_2) $R_{XYZW} - R_{JXJYZW} = R_{JXYJZW} + R_{JXYZJW}$;

(G_3) $R_{XYZW} = R_{JXJYJZJW}$.

We shall call the identity G_i the i^{th} Gray condition on the curvature. It is a simple application of the first Bianchi identity to see that $(G_1) \Rightarrow (G_2) \Rightarrow (G_3)$. Also elementary is the fact that a Kähler structure satisfies relation (G_1). Following [4], if $\mathcal{A}K$ is the class of almost Kähler manifolds, let $\mathcal{A}K_i$ be the subclass of manifolds whose curvature satisfies the identity (G_i). Denoting by $\mathcal{K}$ the class of Kähler manifolds, it is easily seen that the equality $\mathcal{A}K_1 = \mathcal{K}$ holds locally [5, 4]. The examples in [3], multiplied by compact Kähler manifolds, show that even in the compact case, the inclusion $\mathcal{A}K_2 \supset \mathcal{K}$ is strict in all dimensions $2n \geq 6$.

Recall that for an almost Kähler 4-manifold (M, g, J, Ω) the conformal scalar curvature κ is defined by $\kappa = 3\langle W^+(\Omega), \Omega\rangle$, where W^+ is the positive Weyl tensor of (M, g) considered as a traceless symmetric endomorphism of the vector bundle $\Lambda^+ M$ of self-dual 2-forms. The local inner product defined by g is extended to $\Lambda^+ M$ and denoted by $\langle ., . \rangle$, so that the induced norm is half of the usual tensor norm on $\Lambda^1 M \otimes \Lambda^1 M$. Let ∇ be the Levi-Civita connection of (M, g). The covariant derivative of the fundamental form $\nabla\Omega$ can be viewed as a section of the real vector bundle underlying $\Lambda^{1,0}M \otimes \Lambda^{2,0}M$, where $\Lambda^{p,q}M$ denotes the vector bundle of complex (p, q)-forms. Thus, locally, for any any real section ϕ of $\Lambda^{2,0}M$ of constant norm $\sqrt{2}$, we can write

$$\nabla\Omega = a \otimes \phi - Ja \otimes J\phi, \tag{1}$$

where the locally defined 1-form a satisfies $|\nabla\Omega|^2 = 4|a|^2$. The action of J extends by Riemannian duality to the bundle of 1-forms by $Ja(X) = -a(JX)$ and to the bundle of (2,0)-forms by $J\phi(X, Y) = -\phi(JX, Y)$. As a simple consequence of the Weitzenböck formula for self-dual 2-forms applied to the harmonic form Ω we get

$$\kappa = s + 6|a|^2, \tag{2}$$

where s is the scalar curvature of (M, g).

Supposing that the almost Kähler 4-manifold satisfies the second Gray condition, a careful investigation of the second Bianchi identity eventually leads to:

Lemme 1 ([1, Propositions 1,2]) *Let (M, g, J, Ω) be a 4-dimensional almost Kähler manifold which satisfies the condition (G_2). Then the smooth function $\kappa - s$ is a non-negative constant. If moreover $\kappa - s$ is a positive constant — that is (M, g, J, Ω) is an almost Kähler,* non-Kähler *4-manifold in the class $\mathcal{A}K_2$ — then the traceless Ricci tensor Ric_0 is given by:*

$$Ric_0 = \frac{\kappa}{4}[-g^{\mathcal{D}} + g^{\mathcal{D}^\perp}],$$

where $g^{\mathcal{D}}$ (resp. $g^{\mathcal{D}^\perp}$) denotes the restriction of g to $\mathcal{D} = \{X \in TM \ : \ \nabla_X\Omega = 0\}$ (resp. to the orthogonal complement $\mathcal{D}^\perp = span\{a, Ja\}$).

It follows from Lemma 1 that if (M, g, J, Ω) is an almost Kähler, *non-Kähler* 4-manifold in the class $\mathcal{A}K_2$, then the *Kähler nullity* $\mathcal{D}$ of (g, J) is a well-defined 2-dimensional distribution on M. If we denote by $\bar{M}$ the manifold M with the reversed orientation, then we may consider the g-orthogonal almost complex structure $\bar{J}$ on $\bar{M}$, defined in the following manner: $\bar{J}$ coincides with J on $\mathcal{D}$ and is equal to $-J$ on $\mathcal{D}^\perp$. Denote by $\bar{\Omega}$ the fundamental form of $(g, \bar{J})$. Our main goal of this note is to improve the following result proved in [1]:

Theorem 2 ([1, Theorem 1]) *Let (M, g, J, Ω) be a closed 4-dimensional almost Kähler,* non-Kähler *manifold satisfying the second Gray condition. Then $(g, \bar{J}, \bar{\Omega})$ is a Kähler structure, such that $(\bar{M}, \bar{J})$ is a minimal properly elliptic surface with vanishing Euler characteristic.*

3 Proof of Theorem 1

Let (M, g, J, Ω) be a closed 4-dimensional almost Kähler, *non-Kähler* manifold satisfying the second Gray condition (G_2). Then the conclusions of Lemma 1 and Theorem 2 above hold. To prove Theorem 1 we have to deduce a contradiction.

Since the fundamental 2-forms Ω and $\bar{\Omega}$ are harmonic, so are the forms

$$\alpha = \frac{1}{2}\big(\Omega - \bar{\Omega}\big) = \frac{6}{(\kappa - s)} a \wedge Ja \quad , \quad \beta = \frac{1}{2}\big(\Omega + \bar{\Omega}\big) = *\alpha \ . \tag{3}$$

A representative for the real first Chern class of (M,J) is given (up to a constant $\frac{1}{2\pi}$) by the Ricci form γ of the *first canonical Hermitian connection* $\nabla^0_X Y = \nabla_X Y - \frac{1}{2}J(\nabla_X J)(Y)$ (see [7]). A short computation of γ gives (see *e.g.* [6]):

$$\gamma(X,Y) = R(\Omega)(X,Y) + \frac{1}{4}g(J(\nabla_X J),(\nabla_Y J)),$$

where the Riemannian curvature R is considered as a symmetric endomorphism operating on the bundle of 2-forms. Using the first Bianchi identity, it is easily seen that under the condition (G_2) the 2-form $R(\Omega)$ is given by

$$R(\Omega)(X,Y) = \frac{(2\kappa+s)}{12}\Omega(X,Y) + Ric_0(JX,Y).$$

Moreover, involving (1), (2) and Lemma 1, the above expression for γ eventually simplifies to (see [1]):

$$\gamma = (s+\kappa)\alpha - \frac{(\kappa - s)}{3}\beta \ . \tag{4}$$

Since $(\bar{M},g,\bar{J},\bar{\Omega})$ is a Kähler surface, a representative for its real first Chern class is given (up to a constant $\frac{1}{2\pi}$) by the Ricci form $\bar{\gamma} = R(\bar{\Omega})$ of $(g,\bar{J},\bar{\Omega})$, *i. e.*

$$\bar{\gamma}(X,Y) = \frac{s}{4}\bar{\Omega}(X,Y) + Ric_0(\bar{J}X,Y) \ .$$

Then, according to Lemma 1 and the definition of $\bar{J}$, we obtain:

$$\bar{\gamma} = -(s+\kappa)\alpha - (\kappa - s)\beta \ . \tag{5}$$

Now, since $(\bar{M},\bar{J})$ is a minimal properly elliptic surface by Theorem 1, we have

$$0 = c_1^2(\bar{M}) = \frac{1}{4\pi^2}\int_M \bar{\gamma}\wedge\bar{\gamma} = \frac{1}{2\pi^2}\int_M (s+\kappa)(\kappa - s)\alpha\wedge\beta \ .$$

As $\kappa - s$ is constant, the above equality together with (5) implies:

$$\int_M \bar{\gamma}\wedge\beta = 0 \ . \tag{6}$$

As $(\bar{M},\bar{J})$ is a minimal properly elliptic surface of zero Euler characteristic, the base of the elliptic fibration is of genus ≥ 2, and there are no fibers with singular reduction. The surface admits an effective canonical divisor K, which by Kodaira's formula is a sum of fibers. Now (6) reads

$$\int_K \beta = 0 \ .$$

But K is a non-zero multiple of the class of the generic fiber, hence the integral of β vanishes on each fiber. As β is a positive semi-definite form of type $(1,1)$ whose kernel is $\mathcal{D}^\perp = span\{a, Ja\}$ (see (3)), this says that the tangent space of any irreducible component of a fiber is just $\mathcal{D}^\perp$. Thus the complex line bundle $\mathcal{D}^\perp$ is tangent to the fibers everywhere. On the other hand, since the only singular fibers of $(\bar{M},\bar{J})$ are multiple fibres with smooth reduction, it follows that $(\bar{M},\bar{J})$ is obtained by logarithmic transformations from a locally trivial elliptic fibration. Moreover, by replacing $(\bar{M},\bar{J})$ with a finite covering if necessary, we may assume that $(\bar{M},\bar{J})$ is obtained by logarithmic transformations from a *principle* elliptic fibre bundle, *i. e.* an elliptic fibre bundle admitting a holomorphic action by translations on its fibres. Clearly this action is preserved by logarithmic transformations, thus $(\bar{M}, \bar{J})$ has a non-vanishing holomorphic vector field tangent to the fibres. This means that $\mathcal{D}^{\perp}$ is a holomorphically trivial sub-bundle of the tangent bundle T of $(\bar{M}, \bar{J})$; in particular, using the splitting $T = \mathcal{D} \oplus \mathcal{D}^{\perp}$ and the definition of $\bar{J}$, we get $c_1(\bar{M}, \bar{J}) = c_1(M, J) \;\; (= c_1(\mathcal{D}))$. But $c_1(\bar{M}, \bar{J})$ and $c_1(M, J)$ are represented by $\frac{1}{2\pi}\bar{\gamma}$ and $\frac{1}{2\pi}\gamma$, respectively. Since α is a closed 2-form, it follows that

$$\int_M \gamma \wedge \alpha = \int_M \bar{\gamma} \wedge \alpha,$$

which, together with (3), (4) and (5), implies

$$\int_M (\kappa - s)\Omega \wedge \Omega = 0 \ .$$

This is a contradiction because $\kappa - s$ is a positive constant according to Lemma 1 and the assumption that (M, g, J, Ω) is not Kähler.

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