Name:

Activity 6: Factoring

# Part I (Paper & pencil, and CAS): Seeing patterns in factors

1. (a) Before using your calculator, try to recall the factorization of each algebraic expression listed in the left column of this table:

|  |  |
| --- | --- |
| Factorization using paper and pencil | Verification using FACTOR (show result displayed by the CAS) |
| *(a+b)(a-b)* |  |
| *(a-b)(a2+ab+b2)* |  |
| *(x-1)(x+1)* |  |
| *(x-1)(x2+x+1)* |  |

**Classroom discussion of Part I, 1a**

(b) Perform the indicated operations (using paper and pencil)

|  |
| --- |
| x2 - 1 |

|  |
| --- |
| x3 - 1 |

2. (a) Without doing any algebraic manipulation, anticipate the result of the following product:

|  |
| --- |
| *x4-1* |

2. (b) Verify the anticipated result above using paper and pencil (in the box below), and then using the calculator.

 *x4+x3+x2+x- x3-x2-x-1*

= *x4-1*

Expand ()

2. (c) What do the following three expressions have in common? And, also, how do they

differ?

, , and .

*(x-1)* is a factor in all three expressions.

The other factor increases by the addition of powers of *x* in each expression, from *x+1* to *x2+x+1* to *x3+x2+x+1*, respectively.

2. (d) How do you explain the fact that the following products result in a binomial: two binomials, a binomial with a trinomial, and a binomial with a quadrinomial?

All products result in a binomial due to the mutual cancellation of positive and negative terms.

**Classroom discussion following Question 2d**

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2. (e) On the basis of the expressions we have found so far, predict a factorization of the expression .

*(x-1)(x4+x3+x2+x+1)*

2. (f) Explain why the product (*x* –1) (*x*15 + *x*14 + *x*13 + … + *x*2 + *x* + *1*) gives the result *x*16–*1* ?

The product of x and each term of the second factor gives *x16 + x15 + …+x*.

The product of –1 and each term of the second factor gives -*x15- x14 - …-x-1*.

Thus, the negative terms cancel with all the corresponding positive terms to yield finally *x16-1*.

2. (g) Is your explanation (in (f), above) also valid for the following equality:

(*x* –*1*) (*x*134 + *x*133 + *x*132 + … + *x*2 + *x* + *1*) = *x*135–*1* ?

Explain:

Yes, it is valid for this equality as well. The same cancellation of inner terms applies.

Classroom discussion of Part I, #1 #2

## Part II: Toward a generalization (activity with paper & pencil and with calculator)

II (A) 1. In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top

row (the cells of the three columns) and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

|  |  |  |
| --- | --- | --- |
| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
| *(x-1)(x+1)* | *(x-1)(x+1)* |  |
| *(x-1)(x2+x+1)* | *(x-1)(x2+x+1)* |  |
| *(x2+1)(x2-1)*  = *(x2+1)(x+1)(x-1)* | *(x2+1)(x+1)(x-1)* | If you factored  as *(x-1)(x3+x2+x+1),*  you may not have realized that the second factor can be factored further. |
| *(x-1)(x4+x3+x2+x+1)* | *(x-1)(x4+x3+x2+x+1)* |  |
| *(x3+1)(x3-1)*  = *(x+1)(x2-x+1)(x-1)(x2+x+1)* | *(x+1)(x2-x+1)(x-1)(x2+x+1)* | *x6-1* can be factored by several approaches: by applying the difference of squares identity; by applying the difference of cubes identity; or by applying the pattern being generated in this activity. However, do not stop your factorization until the expression has been completely factored. |

II.A.2. Conjecture, in general, for what numbers *n* will the factorization of :

1. contain exactly two factors?
2. contain more than two factors?
3. include  as a factor?

Please explain:

From the above examples, we note that when *n* is equal to 2, 3, and 5, there are exactly two factors.

For *n* equal to 4 and 6, there are more than two factors, and one of them is *(x+1)*.

What kind of numbers are 2, 3, and 5?

What kind of numbers are 4 and 6?

Several different conjectures should emerge; but these need to be tested further. See Part II B.

**Classroom discussion of Part II A**

**Part II continued (with paper and pencil, and with calculator)**

II.(B) 1. As with Part A above, each line of the table below must be filled in completely (all three cells), one row at a time before proceeding to the next row. Start from the top row and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

|  |  |  |
| --- | --- | --- |
| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
| *(x-1)(x6+x5+x4+x3+x2+1)* | *(x-1)(x6+x5+x4+x3+x2+1)* |  |
| *(x4+1)(x4-1)*  *= (x4+1)(x2-1)(x2+1)*  *= (x4+1)(x-1)(x+1)(x2+1)* | *(x-1)(x+1)(x2+1)(x4+1)* |  |
| *((x3)3-1)*  *= (x3-1)((x3)2+ x3+1)*  *= (x-1)(x2+x+1)(x6+ x3+1)* | *(x-1)(x2+x+1)(x6+ x3+1)* |  |
| *((x5)2-1)*  *= (x5-1)(x5+1)*  *= (x-1)(x4+x3+x2+x+1)(x+1)(x4-x3+x2-x+1)* | *(x-1)(x+1)(x4+x3+x2+x+1)(x4-x3+x2-x+1)* | It may not have been obvious that *(x5+1)* is factorizable. |
| *(x-1)(x10+x9+ … +x2+x+1)* | *(x-1)(x10+x9+ … +x2+x+1)* |  |
| *((x4)3-1)*  *= (x4-1)((x4)2+x4+1)*  *= (x2-1)(x2+1)(x8+x4+1)*  *= (x-1)(x+1)(x2+1)(x8+x4+1)* | *(x-1)(x+1)(x2+1)(x2+x+1)(x2-x+1)(x4-x2+1)* | EXPAND *((x2+x+1)(x2-x+1))* yields  *(x4+x2+1)*  *and*  EXPAND *((x4+x2+1)(x4-x2+1))* yields  *(x8+x4+1)*  Thus, *(x8+x4+1)* factors to  *(x4+x2+1)(x4-x2+1)*, which in turn factors further to *(x2+x+1)(x2-x+1)(x4-x2+1)*, |
| *(x-1)(x12+x11+ … x+1)* | *(x-1)(x12+x11+ … x+1)* |  |

II.B.2. On the basis of patterns you observe in the table II.(B) above, revise (if necessary) your conjecture from Part A. That is, for what numbers *n* will the factorization of :

i) contain exactly two factors?

ii) contain more than two factors?

ii) include  as a factor?

Please explain:

Prime numbers as values for *n* yield exactly two factors when *xn-1* is factored completely. [These two factors are *(x-1)* and *(xn-1+xn-2+ … +xn-(n-1)+1)* ]

All even values of *n* > 2 yield more than two factors, one of which is always *(x+1)*.

Moreover, since the identity *x2-1 = (x+1)(x-1)* can be applied to the expression *xn-1* when *n* is even, this case also yields *(x-1)* as one of the factors.

For odd values of *n* that are not prime, the factorization of *xn-1* will contain more than two factors, but *(x+1)* will not be one of them.

**II.(C)** Without using your calculator, answer the following questions:

1. Does 

* 1. contain more than two factors?
  2. include  as a factor?

Please explain:

As 2004 is even, there will be more than two factors and *(x+1)* will always be one of those factors.

2. Does 

1. contain more than two factors?

ii) include  as a factor?

Please explain:

As 3003 is an odd number that is a multiple of 3, it is not prime.

So the factorization of *x3003-1* will have more than two factors, but *(x+1)* will not be one of them.

3. Does 

1. contain more than two factors?

ii) include  as a factor

Please explain:

One needs to first test whether 853 is prime.

If we enter FACTOR(853) into the CAS, it displays 853 as a result. This indicates that 853 is prime.

So the complete factorization of *x853-1* will contain exactly two factors.

**Classroom discussion of Part II B and C**

**Part III: Challenge**

Explain why (*x* + 1) is always a factor of  for even values of *n* ≥ 2.

|  |
| --- |
| *xn* – 1 *= x*2*k* – 1(for *n* even)  = (*x*2)*k* – 1  = (*x*2 – 1)(*x*2 + *x*2 … + 1)    = (*x* + 1)(*x* – 1)( … )  Another approach (by grouping): For even values of *n*, take *n*=8, for example:  *x*8 – 1 = (*x* – 1)( *x* 7 + *x* 6 + *x* 5 + *x* 4 + *x* 3 + *x* 2 + *x* + 1)  = (*x* – 1)( *x* 6(*x* + 1) + *x* 4(*x* + 1) + *x* 2(*x* + 1) + 1(*x* + 1))  = (*x* – 1)(( *x* + 1)( *x* 6 + *x* 4 + *x* 2 + 1))  = (*x* – 1)( *x* + 1)( *x* 6 + *x* 4 + *x* 2 + 1)  = … |