Activity 8: Systems of Equations

Point of insertion for this activity in the course: Volume 1, Chapter 6, p. 114

The aim of this activity is to develop students’ understanding of the algebraic solving methods of comparison and substitution.

**Lesson 1** (Parts I and II, as an introduction to systems of equations)

Part I (with CAS, 25 minutes): Using numerical evaluation to test solutions of given types of equations

(A) First-degree equations in a single unknown

1. The following table contains an equation and some numerical values.

*Without solving* this equation, determine whether the values of the left-hand column are solutions of the equation (using the CAS). But before doing so, describe and justify the strategy you will use to determine whether a given number is a solution (write in the box below). What information will your CAS strategy provide you with?

2. Now the work with the calculator (but no solving, please). Fill in the table below with the appropriate information.



|  |  |  |
| --- | --- | --- |
| Values for *x* | What I entered into the CAS | Result displayed by the CAS |
| *x* = -2 |  |  |
| *x* = 2 |  |  |
| *x* = -5 |  |  |

3. Are there other solutions for this equation? If so, please give one and justify your choice.

(B) First-degree equations in two unknowns

1. The next table contains a new equation and some numerical values.

*Without solving* this equation, but again using the CAS, determine whether each pair of values in the left-hand column is a solution. But before doing so, describe and justify the strategy you will use to determine whether a given number is a solution (write in the box below). What information will your CAS strategy provide you with?

2. Now the work with the calculator (but no solving, please). Fill in the table below with the appropriate information.



|  |  |  |
| --- | --- | --- |
| Values for the pair *x* and *y* | What I entered into the CAS | Result displayed by the CAS |
| *x* = 3 and y = 12 |  |  |
| *x* = -3 and y = 4 |  |  |
| *x* = -18 and y = -6 |  |  |

3. Are there other solution pairs for this equation? If so, please give one and justify your choice.

# (C) Systems of two first-degree equations in two unknowns

1. The next table contains a system of equations and a few ordered pairs of numbers.

*Without solving* this system of equations, determine whether each pair in the left-hand column is a solution of the system (using the CAS). But before doing so, describe and justify the strategy you will use to determine whether a given number is a solution (write in the box below). What information will your CAS strategy provide you with?

2. Now the work with the calculator (but no solving, SVP). Please write in the box below what you entered into the calculator, as well as what the calculator displays as a result.



|  |  |  |
| --- | --- | --- |
| Values for the pair *x* and *y* | What I entered into the CAS | Result displayed by the CAS |
| *x* = 0 and *y* = 2 |  |  |
| *x* = 4 and *y* = 3 |  |  |
| *x* = 2 and *y* = 1 |  |  |

3. Are there other solutions for this system of equations? If so, please give one and justify your choice.

4. Are there any questions or ideas that occurred to you while you were working on these three types of equations? What were they?

###### Classroom discussion of Part I

##### Discussion issues (after the pupils have finished Part I)

* How did you verify whether the given values were solutions?
* From among the given numerical values, which ones were solutions?

How did the calculator give you this information? How, for example, do you

interpret the following display:

“ **|** *x* = 2 false”

• What did you reply when asked: Can you find other solutions for these equations

(or systems of equations)?

* What particular questions came to you while working on the three types of equations?

**Part II (with CAS, 35 minutes):**

**Interpreting CAS solutions for equations in one and two unknowns**

II (A) Solving an equation in one unknown.

Use the command “SOLVE” of the CAS to solve the following equation:

*4(3x-7) = 2(3-x)+5*

|  |  |
| --- | --- |
| What you enter into the CAS | What the CAS displays for the result |
|  |  |

II (B) Solving an equation in two unknowns.

The six following questions concern the equation: *2x+7 = 8y+11*.

1. What do you predict as a result if you use the CAS to solve this equation for *x*?

2. Use the CAS to solve this equation for *x:*

|  |  |
| --- | --- |
| What you enter into the CAS | What the CAS displays for the result |
|  |  |

3. How do you interpret the result displayed by the CAS?

4. What do you predict for the result if you use the CAS to solve this equation for *y*?

5. Use the CAS to solve the equation *2x+7 = 8y+11* for *y:*

|  |  |
| --- | --- |
| What you enter into the CAS | What the CAS displays for the result |
|  |  |

6. How do you interpret the result displayed by the CAS?

II (C) Distinctions between solutions of equations in one and two unknowns

1. You probably noticed that, in Part II (A), the CAS displayed a numerical value as the solution for x. In contrast, for Part II (B), the calculator displayed the result for *x* in the form of an algebraic expression. How do you explain this difference?

2. How can we use these expressions displayed by the CAS to find numerical solutions of the equation *2x + 7 = 8y + 11* ?

##### **Classroom discussion of Part II A, B, C**

Discussion issues (after pupils have finished Part II A, B, C)

• Why did the CAS display an expression in *y* or in *x* when we used the SOLVE

command to solve the equation *2x+7 = 8y+11* ?

• How are we to interpret the expressions produced by the SOLVE command for

equations in two unknowns?

• How can we use these expressions to find numerical solutions to this equation?

**[**Points to raise: a) The numerical solution pairs are not random pairs; they are determined by the constraints of the equation.

b) The numerical solution pairs generated by the rule *x* = f(y)… are the same as those generated by the rule *y* = f-1(x)… **]**

**Example:**

Use the CAS to generate numerical solution pairs, using the expressions obtained with the command “SOLVE”: *x= 2(2y+1)* and *y* =

(i) To generate some numerical solution pairs with the rule *x= 2(2y+1*):

*x= 2(2y+1)* **|** *y* = 1 →x = 6

*x= 2(2y+1)* **|** *y* = 2 →x = 10

*x= 2(2y+1)* **|** *y* = 3 →x = 14

(ii) To generate the same solution pairs with the rule *y* =:

*y* = **|** *x* = 6 →y = 1

*y* = **|** *x* = 10 →y = 2

*y* = **|** *x* = 14 →y = 3

(iii) To verify one of these solutions in the original equation:

*2x + 7* = *8y + 11* **|** x = 6 and y = 1 →true

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II **(**D**)** Using CAS to generate and verify solutions to equations in two unknowns (to be completed for homework, if necessary).

1. Use the CAS to generate three solutions for each of these equations (keep track of what you enter into the CAS as you go along, and of what the CAS displays as a result at each step). Verify at least one solution for each equation (with the CAS).

(a) 

|  |  |
| --- | --- |
| What I entered into the CAS | Result displayed by the CAS |
|  |  |

(b) 

|  |  |
| --- | --- |
| What I entered into the CAS | Result displayed by the CAS |
|  |  |

2. Give at least one question or idea that occurred to you while doing this work (for example, a question concerning difficulties you experienced).

**Lesson 2** (Parts IIIA, IIIB, IIIC)

**Point of insertion**: after students have worked with the Comparison and Substitution methods as presented in the textbook, pp. 126-128

Preamble concerning yesterday’s homework (questions, comments, etc.) (5 minutes)

**Part IIIA (paper & pencil, 15 minutes):**

**Review of Comparison and Substitution methods**

Introductory discussion:

“You remember that we already discussed what is meant by a system of equations”.

(a) “How would you explain it to someone who has never heard of them?”

(b) “Give an example of a system of equations”

(c) “What can we say concerning the number of solutions to a system of linear

equations?”

Individual student work:

1. Here is the COMPARISON method for solving a system of linear equations (adapted from p. 126 of your textbook):

|  |  |
| --- | --- |
| COMPARISON METHOD  The algebraic comparison method consists of: | x + 3y = 5  7x + 6y = 20 |
| 1. Isolating the same unknown in each of the equations, thereby creating two expressions each in a common single unknown; | y = (5-x)/3  y = (20-7x)/6 |
| 2. Setting the two expressions obtained in step 1 equal to each other to construct an equation in one unknown; | (5-x)/3 = (20-7x)/6 |
| 3. Solving the resulting equation; | (5-x)/3 = (20-7x)/6  2(5-x) = (20-7x)  10-2x = 20-7x  7x-2x = 20-10  5x = 10  x = 2 |
| 4. Replacing the resulting value in one of the system’s equations to calculate the value of the other unknown of the solution pair. | y = (5 – x)/3 = (5 – 2)/3 = 1  Solution pair is therefore (x, y) = (2, 1)  Verify this! |

Question : Why do you think that this method is called the Comparison method (in other words, in what sense is a comparison being made in this method)?

1. Here is the SUBSTITUTION method for solving a system of linear equations (adapted from p. 128 of your textbook):

|  |  |
| --- | --- |
| SUBSTITUTION METHOD  The algebraic substitution method consists of: | 2x + 3y = 25  5x + y = 30 |
| 1. Isolating, if necessary, one of the unknowns in one of the equations; | y = 30 – 5x |
| 2. Substituting the expression obtained in step 1 for the appropriate unknown in the other equation, thereby creating an equation in a single unknown; | 2x + 3(30 – 5x) = 25 |
| 3. Solving the equation obtained in step 2; | 2x + 90 – 15x = 25  -13x = 25-90  -13x=-65  x = 65/13 |
| 4. Substituting the value obtained in one of the system’s equations to calculate the value of the other unknown in the ordered pair solution. | y = 30 – 5(65/13)  = 65/13  Solution pair is (x, y) = (65/13, 65/13)  Verify this! |

Question: Why do you think that this method is called the Substitution method?

3. In what way do these two methods (the Comparison and the Substitution methods) allow us to reduce the given situation to one that we already know how to handle?

# **Classroom discussion of Part IIIA**

Brief discussion of students’ responses to Questions 1, 2, and 3

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**Part IIIB (with CAS, 25 minutes): Using the Comparison method with the CAS**

**Remark for teacher:** It is possible that pupils will have difficulty with this part. If some have conceptual difficulties with the Comparison method, they will find it quite challenging to translate from the paper-and-pencil method to a corresponding calculator method. So, we have two aims for this section: (a) to use the activity as an indicator of their difficulties, and (b) to furnish them with an occasion for confronting these difficulties/obstacles. Concerning (b), a potential obstacle might be the absence of a principle justifying the creation of an equality involving two algebraic expressions that express one unknown in terms of the other. In this case, a discussion of the transitive property of equality might be helpful.

Here is a system of linear equations: 

1. With the CAS, use the Comparison method to solve this system (keep track of what you enter into the calculator as you go along, and of what the calculator displays in response to your commands).

|  |  |  |
| --- | --- | --- |
| The Comparison method involves: | What you enter into the CAS | What the CAS displays for the result |
| 1. Isolating the same unknown in each of the equations, thereby creating two expressions each in a common single unknown |  |  |
| 2 & 3. Setting the two expressions obtained in step 1 equal to each other to construct an equation in one unknown; solving the resulting equation |  |  |
| 4. Replacing the resulting value in one of the system’s equations to calculate the value of the other unknown of the solution pair |  |  |

2. How do you verify with the CAS that your solution is correct?

1. In Step 4 of Question 1 above, you replaced the value obtained in Step 3 (for the first unknown) in one of the equations. Now substitute this same value obtained in Step 3 into the other equation. What do you notice? Why is this so?

# **Classroom discussion of Part IIIB**

Discussion issues (after the pupils have completed Part IIIB)

* How did you reply to the question 2 above: “How do you verify with the CAS that your solution is correct?”

* What did you notice when you substituted the same value obtained in Step 3 into the other equation? How do we explain this phenomenon (to obtain the same value in the two cases – perhaps some pupils will speak about the point of intersection of the two straight lines, but we are interested in seeing if any of them suggest an algebraic explanation: that is, that in substituting a particular value of x in the two equations, they obtain the same value of y)?
* The issue of which equation one should use for Step 4 will likely arise. With the CAS, we do not substitute back into the original equation because the substitution will not immediately yield the value of the other unknown. It simply displays another form of the given equation, having carried out only the substitution, but no simplification.
* What is the underlying logic of the first two steps of the method of comparison and how is it linked to steps 3 and 4 of this method? (points to raise: the principles supporting the first two steps are based on equivalence and the axiom of transitivity of equality; the first two steps permit us thus to obtain an equation with only one unknown, something we are already familiar with – and finding the value of this unknown allows us to obtain the value of the other by substitution)

**Part IIIC (with CAS, 15 minutes): Using the Substitution method with the CAS**

**Remark** **for teacher:** A possible obstacle in this section concerns the notion of replacing a unknown by an algebraic expression. Suggestion: If, based on the rule

*x* = *5 - 3y*, that satisfies the first equation, pupils are able to conceptualize “*5 - 3y*” as an object and not just as a process, they will be in a better position to make sense of substituting *5-3y* for *x* in the second equation.

1. With the CAS, use the Substitution method to solve this system (keep track of what you enter into the calculator as you go along, and of what the calculator displays in response to your commands).



|  |  |  |
| --- | --- | --- |
| The Substitution method involves: | What you enter into the CAS | What the CAS displays for the result |
| 1. Isolating, if necessary, one of the unknowns in one of the equations; |  |  |
| 2. Substituting the expression obtained in step 1 for the appropriate unknown in the other equation, thereby creating an equation in a single unknown |  |  |
| 3. Solving the equation obtained in step 2 |  |  |
| 4. Substituting the value obtained in one of the system’s equations to calculate the value of the other unknown in the ordered pair solution. |  |  |

2. How do you verify with the CAS that your solution is correct?

3. Which of the two methods (COMPARISON and SUBSTITUTION) do you prefer and

why?

4. What do these two methods have in common (don’t just rewrite the steps of the two methods)?

# **Classroom discussion of Part IIIC**

Discussion issues (after the pupils have finished Part IIIC)

Concerning Question 2, we suggest inquiring further into students’ ideas on verification and on the notion of solution by asking: “What does it mean when we say that a certain pair is a solution to a system of equations?”

A possible question related to question 4 above would be: “If we speak about the underlying logic of the two methods, in which ways do these two methods resemble each other?”

### Homework

With the CAS, use the more appropriate method (Comparison or Substitution) to solve the following systems of linear equations:

(1)

*y + 1 = x + 6*

*y – 4 = -x + 3*

(2)

*3x + y = 23*

#### 2x + 3y = 48