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Constructing Knowledge Via a Peer Interaction in a CAS Environment with Tasks Designed from a Task–Technique–Theory Perspective

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Abstract Our research project aimed at understanding the complexity of the construction of knowledge in a CAS environment. Basing our work on the French instrumental approach, in particular the Task–Technique–Theory (T–T–T) theoretical frame as adapted from Chevallard's Anthropological Theory of Didactics, we were mindful that a careful task design process was needed in order to promote in students rich and meaningful learning. In this paper, we explore further Lagrange's (2000) conjecture that the learning of techniques can foster conceptual understanding by investigating at close range the task-based activity of a pair of 10th grade students—activity that illustrates the ways in which the use of symbolic calculators along with appropriate tasks can stimulate the emergence of epistemic actions within technique-oriented algebraic activity.

Keywords Constructing knowledge · Task design · Task-technique-theory framework · CAS technology · Theoretical thinking in algebra · Peer interaction · Epistemic actions

1 Introduction

Researchers, for a decade or so, have been asking themselves why teachers of mathematics make little use of technology in the classroom. For example, Guin and Trouche (1999, pp. 195–196) have pointed out: "No more than 15% of teachers include graphic calculators in their teaching, in spite of the fact that all students have a graphic calculator in scientific classrooms (in the Fifth and Sixth Forms). Teachers appear to resist the integration of new technologies even at elementary level". Why do they resist the use of technological tools (calculators, computers, etc.) in the teaching of mathematics?

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One of the reasons concerns tasks—the need for new tasks, the need to design tasks that can make use of the technology so as to improve mathematical learning. In responding to the above question in informal conversations, some teachers expressed that with technology, when presented with a "problem" or an "exercise", students do not need to know much about the mathematical content. They added that, unfortunately, it is enough to know how to use the calculator or computer to obtain an immediate answer. Consequently, they believe that technology does not help in the construction of mathematical knowledge. We note that a great majority of the old problems, which teachers used in the past, become inadequate in technological environments. Other teachers, with boundless enthusiasm, think that many problems are solved with the use of technology. They might focus on the teaching of techniques and developing in their students the false belief that mathematical problems are solved using a technological tool ("button-press techniques") without worrying deeply about the problem itself. Few teachers are centred on analyzing the pros and cons of the use of technology and the importance of designing new activities that allow for deeper understanding when constructing knowledge. We do not have to point out either that there is not a great deal produced, on the part of researchers, with respect to didactic situations that could support teachers in these new classroom environments.

The reflections of some investigators on the resistance of teachers to using technology in the classroom have shown that there are many variables to take into account, and that the problem is much more complex than we initially believed. Artigue (2000, pp. 8–9), who has analyzed why it is that in the last 20 years of instruction in computational environments there has not been a real impact in the mathematics classroom, points to four reasons:

1. The poor educational legitimacy of computer technologies as opposed to their social and scientific legitimacy;
2. The underestimation of issues linked to the computerization of mathematical knowledge;
3. The dominant opposition between the technical and conceptual dimensions of mathematical activity;
4. The underestimation of the complexity of instrumentation processes.

Both the underestimation of issues linked to the computerization of mathematical knowledge and the dominant opposition between the technical and conceptual dimensions of mathematical activity—Artigue's second and third points above—are suggested by, for example, the following two studies.

> Let us refer to a significant example, concerning the following question to 100 students 18 years old: $\lim_{x\to\infty} \ln x + 10\sin x$. All students' responses were correct in the modality without the calculator (50 students). On the other hand, confronted with the rather disturbing graph produced by the calculator, students could not come to terms with the inconsistency of the results displayed by the machine: in this case only 10% of the answers were correct (another group of 50 students). (Guin and Trouche 1999, p. 197)

From here, it seems that students' theoretical understandings in a paper-and-pencil environment were adequate to answer this kind of question; but for the other group, their understanding was rather limited. This example suggests that it might have proved useful for the researchers to have designed tasks that were both more elaborated and which might have promoted interactions between paper-and-pencil and machine activity.

Another instance is drawn from Tall (2000), who describes a group of students using Derive, within a purely technical approach to the calculus of limits, students who could not provide a theoretical explanation of the concept at stake. Tall has stated:

> The same phenomenon occurred when the students were asked to explain the meaning of $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Here, both the Derive group and one of the paper-and-pencil groups had participated in a discussion of the meaning of the notation. All of the non-Derive group gave a satisfactory theoretical explanation of the concept, but none of the Derive group could give any theoretical explanation. (p. 213)

Here, we think that the non-Derive group had developed a conceptual approach to understanding the concept of derivative, while the Derive group had paid more attention to mastering the software and less to the conceptual aspects. Also, as in the French study, it seems that the Derive group might have benefited from task activity involving both paper and pencil and machine.

While both of these examples touch upon aspects related to tasks and their design, at a deeper level they lay bare some fundamental questions regarding the relation between technical and conceptual aspects of mathematical understanding. For example, what is the role of technique in mathematical conceptualization? Is there an interaction between the two? Can we speak about a conceptual understanding of a technique? If so, what might it look like? Skemp's (1976) classic article on instrumental and relational understanding, which was controversial at the time, advanced the notion of two different ways of learning. The point is interesting since from a perspective of the constructivism of the 1970s and 1980s, it seemed that there was only a single way to approach the construction of mathematical knowledge and it was one of relational understanding (a conceptual approach). Skemp expressed recognition of these two kinds of learning. For him, relational understanding involves thinking about what to do and why it should be done, and instrumental understanding involves having rules without reasons. In the past, the two kinds of knowledge were viewed as separate entities.

This recognition of different ways of constructing mathematical knowledge marked a new path of investigation centred on the search for relationships between these two types of knowledge (see, e.g., conceptual and procedural knowledge in Hiebert 1986, among others). Hiebert and Lefevre (1986, p. 16) have stated that knowledge of procedures can promote conceptual growth: "It appears that on occasion procedural knowledge takes the lead and spurs the development of new concepts. For example, Gelman and Meck (1986, in the same volume) present a scenario in which children use already acquired counting skills to promote the development of an ordinal concept of number." Although the aim of the Hiebert monograph was to describe the relationships between conceptual and procedural knowledge, it was primarily for the domain of school arithmetic that these potential relationships were in fact discussed. For algebra and calculus, the traditional dichotomy between the procedural and the conceptual was to remain for several years.

Then, a major step forward was achieved in the 1990s by the French school of didactical research. The so-called instrumental approach to tool use emerged. It is noted that the use of the term *instrumental* in this context is not at all the same as that given by Skemp. According to Artigue (2002), one can clearly distinguish two central influences within the instrumental approach, both of which had begun to develop at about the same time. One influence came from cognitive ergonomics (Vérillon and Rabardel 1995); the other from the anthropological theory of didactics (Chevallard 1999). The main emphasis of the cognitive-ergonomic approach is instrumental genesis and the formation of mental schemes related to the processes whereby an artifact becomes an instrument of thought for the user. On the other hand, the main emphasis coming from the didactical anthropological approach is a focus on the techniques and theories that students develop, within institutional settings.

Artigue (2000, p. 10) has discussed Chevallard's anthropological theory of didactics, one in which mathematical objects emerge from systems of practice (or praxeologies) that involve:

- *tasks* in which the objects are embedded,
- *techniques* used to solve these tasks,
- *technology*, a discourse that explains and justifies the techniques, and finally
- *theory*, discourse justifying the technological discourse.

Artigue (2002) and her collaborators have reduced Chevallard's four components of practice (Task, Technique, Technology, Theory) to three: Task, Technique, and Theory. In so doing, *technology* and *theory* have been combined into the one term, *theory*, thus giving the theoretical component a wider meaning than is usual in the anthropological approach, but also reserving the term *technology* for the use of computational artefacts within the learning environment. In their adaptation of Chevallard's theory, Artigue and her fellow researchers have also pointed out that, within the T-T-T theoretical framework, *technique* too must be given a wider meaning than is usually the case in educational discourse. Techniques are, according to Artigue (2002, p. 248), "a complex assembly of reasoning and routine work"—a stance that is consistent with that of Chevallard. In other words, theoretical elements are threaded through techniques. Thus, there is not an exact parallel between the terms *conceptual/procedural* and *theoretical/technical*. The wider meaning given in the T–T–T theoretical frame to *techniques* as including some theoretical aspects is not generally found in the current usage of the terms *procedures* and *skills*.

Lagrange (2003, p. 271) has emphasized that it is particularly during the process of learning a new technique that theoretical/conceptual elements come into play: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for conceptual reflections when compared with other techniques and when discussed with regard to consistency". The notion that technique plays an epistemic role, especially during its elaboration, is an important one. Hershkowitz et al. (2001, p. 203), in their research on mathematical abstraction, have defined epistemic actions as "mental actions by means of which knowledge is used or constructed". They have identified the following three epistemic actions as central components in the genesis of abstraction: constructing, recognizing, and building-with. According to Pontecorvo and Girardet (1993, p. 368), the epistemic actions practiced by novices when learning a particular domain resemble the particular epistemic actions that are carried out by experts when interpreting objects and phenomena of that domain. So if, as Lagrange (2003) argues, technique plays an epistemic role—especially during its elaboration—then several questions arise. For example, in which ways does this epistemic role manifest itself? How do students try to make sense of and understand a new technique within a CAS environment? How do theoretical/conceptual and technical aspects interact within this technical genesis? To what extent do the epistemic actions of students within a particular mathematical domain resemble those of mathematical experts?

From his classroom research with Mounier and Aldon (1996) on the task of factoring $x^n - 1$, Lagrange (2000) found evidence to support his conjecture regarding the epistemic role played by technique; the learning of specific techniques did foster students' conceptual understanding in that it led them to develop certain proofs. Similarly, results reported by

Kieran et al. (2006) showed how techniques gave rise to theoretical thinking and vice versa within the context of the designed tasks. However, neither the Lagrange nor the Kieran and Drijvers studies analyzed their data with respect to specific *epistemic actions*; rather the results were presented in terms of the conceptual understandings that emerged from the dialectical relation between the technical and the theoretical within the classroom settings in which the tasks were engaged. The study that is being reported in the present paper builds upon the previous work in this area by focusing on a pair of students—their mathematical discourse and actions when working together on one of the same CAS tasks that was designed for the study described by Kieran and Drijvers (an elaboration of the Mounier and Aldon task of factoring $x^n - 1$). The present study distinguishes itself from the classroom study of Kieran and Drijvers, first, by its attention to a peer interaction in order to understand more deeply the processes of articulating paper-and-pencil and CAS techniques in generating conceptual knowledge, second, by its focus on the epistemic actions involved in this knowledge construction and the ways in which both the task sequence and the CAS tool contribute to the provocation of these epistemic actions. The moments during the peer interactions when these epistemic actions seemed to be most crucial with respect to the construction of knowledge will be referred to within the paper as *epistemic moments*.

In sum, the present paper has the precise intention of understanding the complexity of the process of construction of knowledge as per Artigue's second and third points above (i.e., the underestimation of issues linked to the computerization of mathematical knowledge and the dominant opposition between the technical and conceptual dimensions of mathematical activity), within the T–T–T theoretical framework. Tasks and their design are a necessary adjunct to coming to understand this complexity. In addition, the fact that current curriculum reforms support the use of technology in students' learning of mathematics, it becomes even more important to understand the nature of the tasks that can promote mathematical learning.

2 Objective

In the past, a great deal of research was carried out related to paper-and-pencil tasks. Also, for a couple of decades, research concentrated on how the use of technology fosters the construction of knowledge, but without taking into account the importance of the interaction between a paper-and-pencil task and a related CAS task. Then, with our theoretical approach, we were interested in the interactions among the processes related to the triad Task–Technique–Theory. But, what kinds of relationships are important to analyse? And, how? To be more specific, we intended to use our theoretical approach to better understand the relationships of the triad TASK–TECHNIQUE–THEORY, and how students construct articulations between techniques and theories when working on a task. Under this approach, the technique used in a task involving paper and pencil needs to be articulated with the corresponding CAS technique. From this point of view, our analysis takes into account the students' production of techniques.

3 Methodology

In a large project involving CAS use in the mathematics classroom, we designed eight activities (i.e., multitask sequences, the sixth of which is the focus of this paper) related to the algebra curriculum in Québec, for the Secondary 4 level (10th grade). We agreed with Artigue's (2002) notion that, in an instrumental learning environment, it is important to conceive long sequences of didactical activities so as to promote a better understanding. Our activities thus constitute sequences of interrelated tasks in paper-and-pencil and CAS environments. The activities were experimented in:

- three classes (two in an anglophone school, and one in a francophone school) in Montreal, Canada, during the period from September to February (2004–2005);
- one class in Portland, USA;
- two classes in Toluca, Mexico;
- Repetition of the experiment in the Montreal anglophone school with a different classroom group of students (September–December 2005).

The first few activities were designed to promote not only a dialectic between the technical and theoretical, and the interaction between the paper-and-pencil and CAS media, but also the process of instrumental genesis (see Trouche 2002), that is, to promote within students the processes whereby a technological artefact becomes an instrument of thought; but a description of those processes is not part of this particular report.

We initially asked the three teachers in Montreal who were to be involved in the study to work on the activities and to give us feedback regarding the content and the amount of time that students would need to work each one. They also gave us their opinion about the difficulties that students might have with them. We then reconstructed the activities with their feedback in mind.

For the study, each student had a CAS calculator (TI-92 Plus) during the experimentation (at school and at home). All the activities in the classroom were audio- and video-recorded.

In our experimental approach we interviewed students in different modalities:

(a) Interviews with pairs of students, while at the same time all the rest of the students worked on the activity in the classroom setting;
(b) Interviews with individual students, while at the same time the activity was worked on by the rest of the students in the classroom;
(c) Individual interviews with some students, after they had worked on an activity in the classroom.

Our reflections in this paper are restricted to modality (a) above. The interviewees are a pair of students who usually work together in class. From this point of view, the results are closely related to those that might occur in the more typical sociocultural learning setting of the classroom. This marks a difference from studies that analyze exclusively individual actions within an interview. More specifically, we were interested in: (1) the construction of knowledge during a peer interaction in a CAS environment; (2) the kinds of techniques that emerge when solving a task, sometimes resulting in different representations; (3) the kinds of theories that are generated; and (4) the ways in which the students constructed articulations among representations, techniques, and theories.

When we were in the process of generating our sets of activities, we came across an article involving the task: "*Factor* $x^n - 1$" (Mounier and Aldon 1996). The authors showed how rich this task was with students in 11th grade in France. However, Mounier and Aldon's aims were related to students' proving general factorizations of $x^n - 1$. They thought that the computer could liberate the technical aspects of computing by hand, with the expectation that students would:

– use various heuristic strategies, e.g., organized trials, searching for regularities or counterexamples,
– control their investigations by organizing checks and using feedback,
– keep sight of the main goal.

But the students, who were working in pairs in the Derive laboratory, did not go as far in their proving attempts as their instructors had anticipated. Lagrange (2000, p. 19), in his analysis of the various phases of the Mounier and Aldon study, remarked that:

> The 'Derive' factorizations just hide the general factorizations that constitute the mathematical objective. … It is therefore not surprising that [the student] is not able to take some distance from the Derive factorizations so as to conceive more general factorizations. [our translation]

We agree with Lagrange's analysis; however, contrary to Mounier and Aldon's vision of "liberating the technical aspects of computing by hand," (Mounier and Aldon 1996, p. 52), we believe that it is important to focus on the reconciling of techniques used in a paper-and-pencil task with those used in a corresponding CAS task.

As we mentioned above, our objective was related to the Québec curriculum and we were working with students at the 10th-grade level. At this level, pupils are not sure what a proof is (as opposed to the 11th grade students in the Mounier and Aldon study); in general, students in Québec at this level are asked to give arguments to support and justify their findings. The multitask sequence that we elaborated around the Mounier and Aldon factorization task was designed to take at least 2 sessions of 65 minutes each. For the single interview situation, which took place while the rest of the class was working on the same activity, it was also planned that the two participants **C** and **P** would, on the day following the interview, return to their class for the completion of the activity. This completion was to include whole-class discussion on the arguments to support the conjectures that had been generated by **C** and **P**, and by the other students of the class, the day before.

4 Rationale for the Design for the Activity

Given that the theme of factorization was touched upon in some of our prior activities, we thought that, for the given activity, we could give the students an opportunity to further develop their previous knowledge of factoring and construct more complex mathematical knowledge with the assistance of CAS technology.

Our research group in their preliminary approach to the issue of task design took into consideration the following aspects of algebraic activity as indicated by Kieran (2004, pp. 23–24):

- Generational activity;
- Transformational activity;
- Global/meta-level activity.

In thinking about the design of our activities, Kieran's frame regarding the nature of transformational activity (dealing with factoring, expanding, substituting, etc.), and global/meta-level activity (dealing with problem-solving, modelling, noticing structure, generalizing, analyzing relationships, justifying, proving, and predicting) was taken into account, in combination with Artigue's reflections regarding the two points mentioned earlier, as well as Lagrange's (2000) perspective with respect to the role of the techniques. Regarding the latter, Kieran (2004, p. 31) has emphasized that, “The fact that conceptual understanding can come with technique will surely put the study of algebraic transformations among the fruitful and interesting areas of research to be carried out in algebra learning during the years to come.” But she adds that, “from the point of view of the curriculum, the global/meta-level activities cannot be separated from the other activities of algebra” (p. 24).

Thus, the main goal of the activity, which centred on the factoring of the $x^n - 1$ family of polynomials, n being a positive integer, was related to the construction of knowledge. As such, it also involved conjecturing, generalizing, and promoting a need for argumentation. We were also mindful of the need to foster an articulation between the knowledge generated using one kind of technique (a CAS technique) with the knowledge generated using another kind of technique (a paper-and-pencil technique), and using cognitive conflict as a motor for solving a mathematical problem.

4.1 What is a Task? What is a Technique? and, What is a Theory?

In general, the word *task* refers to an undertaking, a piece of work to be done. Our tasks had mathematical objects embedded within them. Chevallard (1999) notes that a mathematical object emerges in a system of practices, its evolution being explained in terms of human action (praxeology). Chevallard takes a task to be a problem.

A *technique* is a method for carrying out, or the ability to perform, a task. As mentioned earlier, we agree with Artigue and Chevallard that using a technique involves not only routine work. That is, the ability to perform a task could imply a complex reasoning. In the past, mathematics educators had a tendency to view the technical aspects involved in solving a task as routine work, forgetting the possible epistemic value with respect to generating knowledge (Lagrange 2000). It is also the case that, sometimes, a student can generate his/her own technique to accomplish a task, because the technique needed to accomplish the task is not at hand. Then, complex reasoning is required. In our study, we try to detect and explain epistemic moments such as these.

From Artigue (2000, 2002) and Lagrange’s (2000, 2003) view of Chevallard’s anthropology theory, a *theory* is seen as a discourse justifying or explaining a technique. Specifically, in our case, a theory is the cognitive structure constructed by the students, which emerges through the process of resolution of an activity (multitask sequence), and which can become visible to the observer by means of the student’s actions and discourse. Indeed, this process is related to an articulation among representations and techniques.

Then, in this document, when referring to *activity*, we mean the set of sequenced tasks that have been designed to promote not only the use of certain techniques and the articulation among them, but also the evocation of conjecturing and the use of argumentation to give support to the conjecturing (i.e., theory).

It is well known that in executing a task with paper and pencil, like multiplying 300×198, we can use a technique that is completely different from that used with a mental approach. For mental computation, we might multiply 300×200 and then subtract 600 [i.e., $300 \times 198 = 300(200 - 2)$]. In other words, given a task, we can often choose from among different techniques. Similarly, in our work, we distinguish between a task in a paper-and-pencil environment and a task in a CAS environment. For example, if a task asks students to factor the expression $x^4 - 1$, then one technique a student could use is the telescoping technique, that is $x^4 - 1 = (x - 1)(x^3 + x^2 + x + 1)$, whereby when these two factors are multiplied together the intermediate terms of the product *telescope together* to leave just the first and the last terms of the product. But the use of a CAS calculator, like

TI-92 *Plus*, TI-Voyage 200 or TI-*nspire* CAS, leads to the result: $x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$. The question then arises as to how to reconcile the two results obtained with different techniques. One possibility in this case is to do the following:

$$\begin{aligned} x^4 - 1 &= (x-1)\left(x^3 + x^2 + x + 1\right) \\ &= (x-1)\left(x^2(x+1) + 1(x+1)\right) \\ &= (x-1)\left(\left(x^2+1\right)(x+1)\right) \\ &= (x-1)(x+1)\left(x^2+1\right). \end{aligned}$$

That is more or less what some students in the classroom study reported by Kieran et al. (2006) did, as is shown in Fig. 1. Taking this as an example, we say that a student can be confronted with different representations of the same object [in the example: $x^4 - 1$, $(x - 1)(x^3 + x^2 + x + 1)$, and $(x - 1)(x + 1)(x^2 + 1)$] when trying to construct a cognitive structure (related to the equivalence of expressions). This requires a "coordination of representations" at the same time that the student is constructing a "coordination between techniques". That is, through a treatment of and/or conversion between representations, students could construct a cognitive structure involving a "coordination of representations" that could in turn promote a "coordination between techniques". In our case, the treatment of representations is related to the construction of a coordination between a paper-and-pencil technique and a CAS technique (as suggested by the action depicted in Fig. 1).

Regarding the issue of tasks and their design, it is important to add that, when we are presenting a task to students, and when they in turn generate a conjecture from their results, they will, at times, create their own tasks and techniques to validate their conjecture. This point is vital to our methodology, and we will come back to it later.

4.2 Notation

The notations we use in this paper are the following:

- Task with paper and pencil $\mathbf{TASK_{P\text{-}P}}$;
- Task with CAS $\mathbf{TASK_{CAS}}$;
- Technique involving paper and pencil $\mathbf{TECH_{P\text{-}P}}$;
- Technique involving CAS $\mathbf{TECH_{CAS}}$;
- Semiotic production with paper and pencil $\mathbf{PROD_{P\text{-}P}}$;
- Semiotic production with CAS $\mathbf{PROD_{CAS}}$;
- A P–P task provoking a semiotic production $TASK_{P\text{-}P} \xrightarrow[TECH_{P-P}]{} PROD_{P\text{-}P}$ that depends on a technique $\mathbf{TECH_{P\text{-}P}}$.

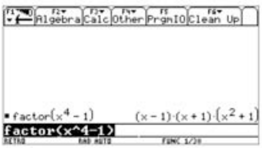
Paper-and-pencil approach	CAS approach	Constructing a coordination between two techniques
$x^4-1=(x-1)(x^3+x^2+x+1)$		$(x-1)(x^3+x^2+x+1)$ $x^2(x+1)+1(x+1)$ $(x-1)(x^2+1)(x+1)$

Fig. 1 Different techniques depending on the environment (from Kieran et al. 2006)

- A CAS task provoking a semiotic production $TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS}$ that depends on a technique $\mathbf{TECH_{CAS}}$.
- Conversion between productions: $\mathbf{PROD_{P\text{-}P}} \rightarrow \mathbf{PROD_{CAS}}$ and vice versa $\mathbf{PROD_{P\text{-}P}} \leftarrow \mathbf{PROD_{CAS}}$.
- Articulation between techniques:

$(TASK_{P\text{-}P} \xrightarrow[TECH_{P\text{-}P}]{} PROD_{P\text{-}P}) \leftarrow\cdots\rightarrow (TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS})$ (the dotted arrows mean an internal construction).

- Construction of a theory related to a technique:

$\left(TASK_{P\text{-}P} \xrightarrow[TECH_{P\text{-}P}]{} PROD_{P\text{-}P}\right)$ ----> $\mathbf{THEO_{P\text{-}P}}$ (the dotted arrow means an internal construction)

$\left(TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS}\right)$ ----> $\mathbf{THEO_{CAS}}$ (the dotted arrow means an internal construction)

- Articulation between techniques and construction of a theory:

$$\begin{array}{c} THEO \\ \uparrow \\ (TASK_{P\text{-}P} \xrightarrow[TECH_{P\text{-}P}]{} PROD_{P\text{-}P}) \leftarrow\cdots\rightarrow (TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS}) \end{array}$$ (the dotted arrows mean an internal construction).

As we said before, in our approach we would like to make a distinction between a paper-and-pencil environment and a CAS environment. A task in a paper-and-pencil environment ($\mathbf{TASK_{p\text{-}p}}$) provokes the use of a technique ($\mathbf{TECH_{P\text{-}P}}$) that in general is different from that which is used in a CAS environment ($\mathbf{TECH_{CAS}}$), as shown with the example in Fig. 1. The production a student carries out on paper we are denoting as a *semiotic* production in a paper-and-pencil environment ($\mathbf{PROD_{P\text{-}P}}$). The result a student produces using a CAS (on the screen of his/her calculator) we are denoting as a semiotic production in a CAS environment ($\mathbf{PROD_{CAS}}$). The theoretical construction could be different in a paper-and-pencil environment ($\mathbf{THEO_{P\text{-}P}}$) from in a CAS environment ($\mathbf{THEO_{CAS}}$); so we are distinguishing both of them. And, if a theory is constructed as a generalization of the coordination between the two media-based theories, we are denoting this as a theory (**THEO**), one that is understood to be a theory and which is recognized in a social context (by teachers, researchers, mathematicians, etc.).

5 The Sequences of the Designed Activity

When we designed the activity as a set of multitask sequences, a first consideration was the interplay of paper and pencil and CAS. A second consideration was the theorizing that would be elicited by the specific tasks of each sequence. These considerations led to the creation of an activity with the following task sequences.

Table 1 Tasks from the first sequence (remembering factors)

1. (a) Before using your calculator, try to recall the factorization of each algebraic expression listed in the left column of this table	
Factorization using paper and pencil	Verification using FACTOR (show result displayed by the CAS)
$a^2 - b^2 =$	
$a^3 - b^3 =$	
$x^2 - 1 =$	
$x^3 - 1 =$	

5.1 First Sequence (Remembering Factors)

The preliminary task (see Table 1) was related to remembering the factorization of several expressions, in particular, the difference of squares and the difference of cubes in a paper-and-pencil environment and then verifying with the calculator (command FACTOR). The expressions were: $a^2 - b^2$; $a^3 - b^3$; $x^2 - 1$; $x^3 - 1$.

5.2 Second Sequence

THEO$_1$: Given the expression $(x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$ for a specific n, it is equivalent to: $(x^n - 1)$—this theory based on the telescoping technique.

In this particular sequence, we ask the students to carry out a task and to justify their approach. The previous sequence, which was preliminary to this, permits in this second sequence the construction of the telescoping technique. In other words, the first part of the task (see Table 2) was conceived to promote the development of this technical knowledge by expanding $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$ in a **TASK$_{\mathbf{P\text{-}P}}$** and then predicting the product of $(x - 1)(x^3 + x^2 + x + 1)$ without doing any algebraic manipulation. After that, verification was asked for, first with paper and pencil, and then with the calculator. This led to the request to compare expressions: *What do the following expressions have in common:* $(x - 1)(x + 1)$; $(x - 1)(x^2 + x + 1)$ and $(x - 1)(x^3 + x^2 + x + 1)$? Immediately afterward, students were asked to predict the factorization of $x^5 - 1$ and then to explain why

Table 2 Tasks to promote a telescoping technique and its theory (**THEO$_1$**)

1. (b) Perform the indicated operations (using paper and pencil)
   $(x - 1)(x + 1) =$
   $(x - 1)(x^2 + x + 1) =$

2. (a) Without doing any algebraic manipulation, anticipate the result of the following product:
   $(x - 1)(x^3 + x^2 + x + 1) =$

(b) Verify the anticipated result above using paper and pencil, and then using the calculator.

(c) What do the following three expressions have in common? And, also, how do they differ?
$(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$.

(d) How do you explain the fact that the following products result in a binomial: two binomials, a binomial with a trinomial, and a binomial with a quadrinomial?

(e) On the basis of the expressions we have found so far, predict a factorization of the expression $x^5 - 1$.

(f) Explain why the product $(x - 1)(x^{15} + x^{14} + x^{13} + \cdots + x^2 + x + 1)$ gives the result $x^{16} - 1$?

(g) Is your explanation (in (f), above) also valid for the following equality: $(x - 1)(x^{134} + x^{133} + x^{132} + \cdots + x^2 + x + 1) = x^{135} - 1$? Explain:

$(x-1)(x^{14}+x^{13}+\cdots+x+1)$ gives $x^{15}-1$. And finally, we asked if their explanation would also be valid for the following equality:

$$(x-1)\left(x^{134}+x^{133}+x^{132}+\ldots+x+1\right)=x^{135}-1.$$

Our expectation at this stage was that students would analyze relationships, notice structure, and generalize so as to predict the factorization of expressions like $(x^{135}-1)$. That is, they would be constructing a conceptual structure encompassing the telescoping technique and the theoretical notion regarding the factoring of x^n-1 (i.e., the theory that $(x-1)$ and $(x^{n-1}+x^{n-2}+\cdots+x+1)$ for a given integer value of n, are factors of x^n-1—this theoretical fact, supported by the telescoping technique).

5.3 Third Sequence (Promoting an Internal Articulation Among Representations and Techniques)

THEO₂: The equivalence of "$\mathbf{TECH_{P\text{-}P}}$ and $\mathbf{PROD_{P\text{-}P}}$" with "$\mathbf{TECH_{CAS}}$ and $\mathbf{PROD_{CAS}}$".

It is in this sequence that we decided to have students confront the technique developed in the previous sequence when attempting the task ($\mathbf{TASK_{P\text{-}P}}$) related to the telescoping technique, with the technique using the calculator ($\mathbf{TECH_{CAS}}$). This confrontation, absolutely necessary from our theoretical perspective regarding the reconciling of techniques, was intended to encourage change in students' knowledge. That is, in this sequence, the task promotes a confrontation between the technique just learned [i.e., $(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)=x^n-1$] and the results given by the calculator. Indeed, we were promoting in students the construction of a cognitive structure, a theory, related to the treatment of representations and conversion between techniques—one that could generate a "coordination of representations," and in turn, a "coordination between techniques" (or reconciling of techniques). We designed this multitask sequence so as to give the students the opportunity to cope with the results that the calculator provides (i.e., specific factors in a more complete factorization)—an output that hides the general factorization—as was signalled by Lagrange in an earlier quote (2000, p. 19).

First, we asked students to factor an expression using paper and pencil (see Table 3), and then, to use the calculator. Finally, we asked them to reconcile, if appropriate, the two results obtained. Accordingly, our strategy for filling in the table in this sequence was to ask the students to work the task row-by-row, and not column-by-column. The subsequent sequences were designed so as to continue motivating the construction of the cognitive structure described above in the previous paragraph.

Table 3 Promoting a coordination of techniques

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=$		
$x^3-1=$		
$x^4-1=$		
$x^5-1=$		
$x^6-1=$		

5.4 Fourth Sequence

THEO$_3$: The conjecture that $(x^n - 1)$ will have exactly two factors, $(x - 1)$ and $(x^{n-1} + x^{n-2} + \cdots + x + 1)$, when n is odd (a false conjecture).

Taking into account that in the third sequence the students working the multitask sequence as far as $x^6 - 1$ (see Table 3) could have constructed a coordination of techniques, we wanted them to produce a preliminary conjecture (see Table 4) before continuing with the extension of this task. In response to this conjecturing task, we believed that a first conjecture—a false one—might arise: "*If n is odd, $x^n - 1$ contains exactly two factors* $(x - 1)$ *and* $(x^{n-1} + x^{n-2} + \cdots + x + 1)$". We expected pupils to reject eventually this first conjecture—when they would be working on the fifth multitask sequence of the activity. That is, the fourth sequence would lead to the conception of a preliminary conjecture and then the work on the next sequence would provide the opportunity to reject it. This rejection would occur because, when factoring $x^9 - 1$ (see Table 5), students would obtain more than two factors. As the reader can see, in our design, it was very important to pose the question shown in Table 4, which was inserted between the two sequences.

Our expectation for the three questions was that students would answer as follows:

(i) n must be odd ($x^2 - 1$ likely being viewed as a kind of exception),
(ii) n must be even,
(iii) n must be even.

Table 4 Conjectures of the fourth sequence

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
(i) contain exactly two factors?
(ii) contain more than two factors?
(iii) include $(x + 1)$ as a factor?

Table 5 A task promoting a confrontation regarding the pupils' conjecture produced in the fourth sequence

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^7 - 1 =$		
$x^8 - 1 =$		
$x^9 - 1 =$		
$x^{10} - 1 =$		
$x^{11} - 1 =$		
$x^{12} - 1 =$		
$x^{13} - 1 =$		

5.5 Fifth Sequence (Involving Conceptual Change—Rejecting a Prior Conjecture and Generating a New One)

THEO$_4$: Rejecting previous conjectures to produce a new one: $(x^n - 1)$ will have exactly two factors, $(x - 1)$ and $(x^{n-1} + x^{n-2} + \cdots + x + 1)$, when n is a prime number.

If students had in fact been led to generate **THEO$_3$**, we wanted them to confront their thinking by means of the continuation of the task shown in Table 5. With $n = 9$, we wanted to provoke a cognitive conflict in the students—have them to see it as a counterexample and reject their preliminary conjecture that, "*If n is odd,* $x^n - 1$ *contains exactly two factors* $(x - 1)$ *and* $(x^{n-1} + x^{n-2} + \cdots + x + 1)$". Here, in this sequence, students would discover the contradiction and try to cope with it, thus being in a state of cognitive conflict. Neither the researcher nor the teacher would signal the contradiction. It was intended that students would resolve their conflict by rebuilding their conjecture to produce a new one. This task, created with the aim of provoking **THEO$_4$**, is crucial in our design of the whole activity.

With these tasks, we thought that a complex structure (theory) could be constructed by the students. The case of $n = 9$ would function as a counterexample to their conjecture; thus this task could give them a clue to correct their conjecture and to construct a more complex theoretical structure. This cognitive structure, rich in connections, could permit the students to correct the conjecture generated in the fourth sequence. Then, we asked them to express once again these conjectures (a global/meta-level activity) at the end of this part of the activity so as to know whether a conceptual change had occurred [a repeat of the conjectural task of the fourth sequence (Table 4)].

It is important to point out that, if students are expressing a new conjecture regarding the relation of prime numbers as exponents to the number of factors, they are considered as well to be constructing the related theory that composite numbers as exponents yield more than two factors.

5.6 Sixth Sequence (Coordinating Theories or Distinguishing Among Theories)

In order to be able to analyze the new structure constructed by the students, we designed the tasks shown in Table 6 (factorization of $x^{2004} - 1$, $x^{3003} - 1$, $x^{853} - 1$ and specific questions about the factors obtained); the answers to those questions could give us a better idea about the nature of the theoretical knowledge that the students had constructed.

5.7 Seventh Sequence (Justifying a Conjecture—or Deepening of Theory—Through Argumentation)

The task for this sequence was to explain why $(x + 1)$ is always a factor of $x^n - 1$ for even values of n, $n \geq 2$. Taking into account the level of the students, we didn't expect a formal proof, but a kind of deep discussion and arguments to support their conjecture[1]. As we said before, this sequence was designed to be worked in small groups, and discussed in plenary. We are not treating this part of the experimentation in this document (see Kieran et al. 2006, for an account of the whole-class debate on this sequence).

[1] As a perspective for future research, it could be of interest to consider elaborating a task design for older students that promotes discussion regarding the number of factors in a complete factorization of $x^n - 1$, the number of factors depending on the number of divisors of n. If q divides n, then $n = pq$, p being an integer, and $x^n - 1 = x^{pq} - 1 = (x^q)^p - 1$. Thus, the complete factorization of $x^n - 1$ has all the factors of $x^q - 1$.

Table 6 Seeking confirmation about students' conjectures

Without using your calculator, answer the following questions
1. Does $x^{2004} - 1$
(i) contain more than two factors?
(ii) include $(x + 1)$ as a factor?
Please explain:
2. Does $x^{3003} - 1$
(i) contain more than two factors?
(ii) include $(x + 1)$ as a factor?
Please explain:
3. Does $x^{853} - 1$
(i) contain more than two factors?
(ii) include $(x + 1)$ as a factor
Please explain:

6 Analysis of the Interview of a Pair of Students

At this stage of the experimentation, as mentioned earlier, students had worked on five previous activities, regularly using the CAS calculator. In the interview, the students used the calculator without any difficulty as they worked through the task sequences. They were freely using it whenever they wanted to test any conjectures that they had generated.

In the introduction, we pointed out that we were interested in analyzing relationships of the triad TASK–TECHNIQUE–THEORY in a pair-wise interaction of students in a CAS environment. In analyzing **C** and **P**'s performance (notation: **C** and **P** are used for the students and **I** for the interviewer-researcher), we can say that their work was harmonious; thought and action were performed as if by one person. **C** and **P** were used to working together in the classroom. There was only one calculator on the table, placed between the two boys, and it was connected to a panel display that was projected onto a larger screen. So, if **P** was using the calculator, with the projection **C** was able to see what **P** was doing, and vice versa. Both were writing down on the single task-sheet that was provided.

The analysis of the interview indicated several important epistemic moments through which the two students passed during the interview:

- Constructing and verifying a technique (noticing structure, analyzing relationships and predicting), namely the telescoping technique,
- Reconciling paper-and-pencil technique with CAS technique,
- Noticing more structures and generating several conjectures (false conjectures), for example: "The only time it contains two factors is when it is odd…",
- Testing and analyzing their conjecture (arriving at a counterexample: $n = 9$),
- Moving from the fact that disturbed the rule to the Eureka moment.

We now present our analysis of the interview according to these five epistemic moments.

6.1 Constructing and Verifying a Technique (Noticing Structure, Analyzing Relationships and Predicting), Namely the Telescoping Technique (First and Second Sequences)

In this part, **C** was thinking aloud and **P** was using the calculator. When **C** was saying something, **P** was checking it out with the calculator to verify if it worked or not, and the other way around.

Verbatim	Student productions and our interpretation
01:14 **P**: Yeah, Ok. Before using your calculator, try to recall the factorization of each algebraic expression listed in the left column of this table [reads from first page, exercise 1(a) of Sequence 1]	
01:21 **C**: So the eh, first one is a squared minus b squared so it just ends up being	$TASK_{P\text{-}P} \xrightarrow[TECH_{P\text{-}P}]{} PROD_{P\text{-}P}$
01:26 **P**: *a* plus *b*, *a* minus *b*	
01:28 **C**: Right, and then uh, the next one you have a cubed minus *b* cubed [$a^3 - b^3$] which eh, you can just write as *a* plus *b* squared on the outside bracket multiplied by *a* minus *b* [$(a + b)^2\ (a - b)$], 'cause it is a difference of squares but you are left over at a positive *a* plus *b*.	Remembering facts, right and wrong results: $a^2 - b^2 = (a + b)(a - b)$ $a^3 - b^3 = (a + b)^2(a - b)$
01:45 **I**: Ok.	
01:47 **P**: Isn't it *a* plus *b* times *a* squared minus eh, two *ab* plus *b* squared. [$(a + b)\ (a^2 - 2ab + b^2)$]	**P** is remembering correctly. $a^3 - b^3 = (a + b)(a^2 - 2ab + b^2)$
01:53 **C**: Yeah, and then the, eh [points at third expression of exercise 1(a) of Sequence 1].	
01:58 **P**: The third one is kind of like the same thing as the first one.	
02:02 **C**: Yeah, such as *x* minus one; *x* plus one. And *x* cubed minus one [points at fourth expression of exercise 1(a) of Activity 6] is just a difference of cubes again. So, it's just eh, [pause] just showing you different forms of it; sometimes replacing a variable with a natural number. [Pause] And then eh, it says use the calculator to do, [C picks up calculator and P scribbles on paper] so	$x^2 - 1 = (x + 1)(x - 1)$ $x^3 - 1 =$ **C** is remembering about replacing a variable by a number. **C** takes the calculator.
02:32 **C**: So, you just [inaudible] actually you can eh; you just type it in, right [Types into calculator: Factor($a^2 - b^2$); the calculator displayed: $(a + b)(a - b)$], and this comes out to what our answer was and then same thing it was with the cubes. [Types in calculator: Factor($a^3 - b^3$); The calculator displayed: $(a - b)(a^2 + ab + b^2)$] So, here is the answer we ended up coming up with. So, they are all the same, like in that sense which is difference of cubes or difference of squares. [Types in calculator: Factor($x^2 - 1$); The calculator displayed: $(x - 1)(x + 1)$; types in calculator: Factor($x^3 - 1$); The calculator displayed: $(x - 1)(x^2 + x + 1)$] Yeah.	$TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS}$ Factor($a^2 - b^2$). The calculator displayed: $(a + b)(a - b)$. Factor($a^3 - b^3$). The calculator displayed: $(a - b)(a^2 + ab + b^2)$. Here they probably realized they were wrong before. Factor($x^2 - 1$): $(x - 1)(x + 1)$. Factor($x^3 - 1$): $(x - 1)(x^2 + x + 1)$. This is a first confrontation between p/p and CAS techniques. Indeed, **P** and **C** initially had different techniques for calculating $(a^3 - b^3)$.

First of all, we can see that **C** was recalling incorrectly the factorization of $(a^3 - b^3)$ and the calculator gave him the opportunity to verify the alternate technique that **P** was remembering. Here we can note that the answer given by the calculator was taken as right without further verification. After a while the interviewer asks **P** a question.

Verbatim	Interpretation
03:43 **I**: I see. How did you suddenly come upon the fact that you [pause]?	
03:50 **P**: Oh, when I saw the calculator.	**P** expresses that he feels comfortable using the calculator and that he finds it helpful.
03:51 **I**: You saw the calculator.	
03:53 **P**: Yeah it helps.	
03:54 **C**: It helps. [**P** writes in the box for exercise 1(a) of Activity 6] Then eh, so that is it for that section with the cubes you are always going to have a variable that is not squared but that ends up [pause] sometimes confusing.	

From that moment, they were using the calculator consistently, answering what was asked in the multitask sequence. When they arrived at question 2(a), they anticipated the result of the product $(x - 1)(x^3 + x^2 + x + 1)$ without doing any algebraic manipulation. After reading the question, **C** gave a right answer (minute 9:20) but he did not write it down. When **P** took the pen to write down the answer (minute 10:22), he wrote $(x - 1)(x^3 + x^2 + x + 1) = x^4 + x^3 - 1$. The researcher asked **C** if he agreed and **C** said "No…". Working algebraically and step by step, **P** found that the product of $(x - 1)(x^3 + x^2 + x + 1)$ was $x^4 + x^3 + x^2 + x - x^3 - x^2 - x - 1$ and **P** stopped there. What is interesting here is that **C** took the pen and canceled out some terms, arriving at $(x^4 - 1)$, thereby suggesting a prefiguring of the telescoping technique. Immediately **P** picked up the calculator (possibly attempting to coordinate paper-and-pencil productions with CAS productions) when the researcher asked if they agreed on the answer. **C** began to answer, verbalizing what they did and saying "they" were right the first time. It seems that **P** had some doubts.

Verbatim	Interpretation
14:01 **P**: Yeah must have [pause] also	
14:08 **I**: Are you satisfied, with the result you got? [Camera turns towards calculator: Expand((*x* − 1)*(*x*^3 + *x*^2 + *x* + 1)); The calculator displayed: $x^4 - 1$]	
14:13 **P**: Its just that I don't understand how, negative one [pause] when you keep squaring it, it changes the sign but when you have it, difference of cubes, it is still negative one, you have difference of squares it's negative one and when you have *x* to the power of four it's always negative one. Why is it never positive?	Even if **P** did the algebraic manipulations, he's having some doubts about the right answer.
14:34 **C**: Because like it is always a negative there, like that is the meaning of a difference of squares or cubes or, whatever is that, one side is negative one side is positive. So, I think no matter what, you are always going to end up with a negative one, like if you did this thing for power of five it would be the same.	**C** seems more confident and verbalizing the process once again, it seems he's conjecturing a mathematical fact. That is, **THEO$_1$**: The Telescoping Technique.

They continued with the sequence. At about the 22nd minute, they had begun to conceive a technique. Even if **C** was not talking about the number *n* as a general number, when he was saying, "*like whatever you have like, no matter what you have in there*, …", probably he was not thinking about a specific number, even if immediately he said, "*if you had x to the power of 20… it would be like a difference of to the power of twenty-one* [$(x-1)(x^{20}+x^{19}+\cdots+x+1)=(x^{21}-1)$] Because, hmm, everything ends up cancelling out…". It seems he was verbalizing the technique in a general way even if he mentioned some examples.

Verbatim	Interpretation
22:31 **C**: The right bracket I guess is, you can say is always positive, so uh, like if you try, it ends up like, x minus one always ends up canceling, hmm, the terms, because hmm, like whatever you have like, no matter what you have in there, if you had x to the power of twenty, the answer like, like if you had x to the power of twenty is your first variable on the right hand bracket, it would be like a difference of to the power of twenty-one. Because, hmm, everything ends up canceling out. Because, like, the left hand just ends up like making sure everything useless is taken out of it and you are just left with the first term on the right side plus one to the exponent power.	**THEO$_1$**: The Telescoping technique (conjecture) $\left(TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS}\right)$ **-----> THEO$_1$**
24:55 **C**: And then the minus one just cancels everything except itself, it can't cancel.	
25:00 **P**: Yeah.	
25:06 **C**: And then, uh. [pause]. And this only works if it's, like there is no missing terms. Like, you are not allowed to, I don't think, to have a missing term.	Here **C** is convincing himself about the conjecture and pointing out the importance of having all the terms. The *technique* involving "*the missing term*" is important and he will come back to it a few minutes later. They continue testing their theoretical conjecture with the calculator, using some examples that they themselves have generated. $TASK_{CAS} \xrightarrow[TECH_{CAS}]{} PROD_{CAS}$
25:27 **P**: [**P** types into the calculator: Expand(($x-1$)*(x^5 + x^4 + x^3 + x^2 + x + 1); Error display: Missing ")"; P adds ")"; the calculator displayed: x^6-1]	**P** tries with x^6-1 and **C** tries with x^7-1.
26:20 **C**: So, uh, there we just said that the x [**P** types into the calculator: Expand(($x-1$)*(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1); the calculator displayed: x^7-1] in the left hand side make the first exponent too high to be cancelled out and the minus one cancels out all the terms except itself. So, the result is always going to be the leftmost x term in the right hand bracket plus one to its exponent power and then minus one.	
26:50 **I**: Does that fit with your thinking **P**?	
26:51 **P**: Yeah.	

continued

Verbatim	Interpretation
26:52 **I**: May I ask what you were doing on the calculator? [Points at calculator]	
26:54 **P**: I was just checking to make sure that the rule worked for everything.	
26:57 **I**: So, you.	
26:58 **P**: Yeah, you have your right bracket and you add a term and the result, like the first term of the result will always be the first term of the left bracket times the first term of the right bracket.	Here we can see that **P** is trying some examples to convince himself that the conjecture "works for all numbers".
27:16 **I**: So, you are just checking to see.	
27:17 **P**: Yeah, I was just checking to see if it worked.	
27:18 **C**: Like, does it work if one of the variables in the right hand bracket is missing? Like, if you take out the x squared?	**C** is repeating his *technique* involving the missing term and he's proposing to take out x^2 to see what will happen. In Balacheff's sense (1987, p. 148), this kind of argumentation is related to a "preuve"—an important step in arriving at a proof.
27:27 **P**: Yeah, it probably won't work, but I'll check.	
27:30 **I**: Well, could you, before the actual checking, could you think of what might happen?	
27:35 **P**: It would just give you a really [inaudible] answer. It would probably just restate it, one entry. [Erases: x^2 from calculator entry]	**P** understands **C**'s *technique* and he's putting it into practice.

Probably, **C** thought that their approach of verifying their conjecture case by case was not limited to specific cases, and then a new *technique* appeared—erasing one term (x^2) from the expression $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. You can see also that **P** was following **C**'s idea. We think they did this as a general idea, that is, to exclude "any term" in the expression, and asked the calculator to $Expand((x - 1)*(x^6 + x^5 + x^4 + x^3 + x + 1))$ as a means to test and confirm their expectations.

This epistemic moment, one that involved testing the limits of their new technique, reflects an epistemic action that is engaged in by mathematical experts when exploring or constructing new techniques. It is also related to activity involving "preuves" in the sense of Balacheff (1987)[2]. We remind the reader that, at tenth grade, these students do not really know what a proof is. Indeed, it was planned that at the end of the task, the students should work at searching for arguments that could validate their conjectures within a classroom discussion (see Kieran et al. 2006).

It is important to highlight the role of the calculator in this process. When **C** and **P** began with paper and pencil, they produced some errors. The calculator permitted them to

[2] Balacheff (1987, p. 148) has said: "We call a 'preuve' an explanation that is accepted by a given community at a given moment. This decision can be the object of a debate whose significance lies in the need to determine a validation system that is common to the interlocutors" [our translation]. Note that the French word 'preuve' does not translate into the English word 'proof', the French word for 'proof' being 'démonstration'. The French word 'preuve' has at times been translated into English as 'evidence', 'warrant', 'supporting argument', and 'justification'.

become aware of their errors and to notice some structure. But even more important, the calculator provided them with a kind of extension to their minds (see Donald 1991), in the sense that the strategies developed in the process of resolution of a task could be verified immediately to have the feedback that permitted them to go farther.

At this moment, with the result obtained on the screen, they were convinced that they had discovered a rule or a pattern: The telescoping technique and a way of theorizing about it. The investigator requested some elaboration of their finding (see what **C** says at 29:55 below).

Verbatim	Interpretation
29:41 **I**: But the explanation, hmm, is based on what? I mean, if you are going to explain to somebody why that happens.	
29:49 **C**: Because, uh, [pause] well, like we said before.	
29:54 **P**: But it's just the pattern that we've found.	
29:55 **C**: Because the x times the x to the power of fifteen, uh, makes it so that it's a higher exponent number than anything the minus one would produce so, it can't cancel anything out. But, uh so, like the, like everything multiplied by the x would be one exponent higher than it was. The minus one just makes it a negative [inaudible] that exponent what it is, so it will be, like, so the only reason that the first term never gets cancelled out is because it is the highest term. So, if you are doing minus one by everything there, if all the terms are already one higher, you can't reach that one. So, everything else cancels out except that.	Verbalizing the telescoping technique (**THEO$_1$**).

At this moment we can say they showed they knew the telescoping technique according to our *a priori* analysis. The arguments given by **P** and **C** indicate that they had constructed a theoretical justification in Chevallard's sense. The CAS environment permitted them to correct the errors of the paper-and-pencil approach and, more importantly, allowed for the emergence of an articulation between techniques "*Because the x times the x to the power of…*" (see the verbalization above) and theory "$(x-1)(x^{15}+x^{14}+\cdots+x+1)=(x^{16}-1)$". Their strategies and expectations were confirmed by the results of the calculator.

6.2 Reconciling Paper-and-Pencil Technique with CAS Technique

As we mentioned earlier, we put special attention in the multitask sequence on the process of reconciling productions with paper and pencil and with CAS (see Table 3). Even if the tasks of the third sequence seemed more or less easy for the students, for us it was crucial to provide for explicit reconciling of productions involving paper-and-pencil and CAS techniques. In this reconciling, it is important to note that it includes not just an articulation among representations, but much more than that—it includes an articulation among representations and techniques.

Verbatim	Interpretation
33:59 **P**: I noticed there is a pattern, just like we discovered before, so it's, so it's going to be x minus one at the beginning, and then increasing powers of x for the first term [pause] in the right bracket.	
34:10 **C**: But uh, for x to the power of four minus one, the calculator, uh, it takes out one of the x plus ones, and if you (inaudible).	**C** noticed that the calculator is giving a different answer for the factorization of x^4-1.

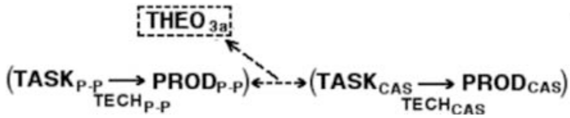
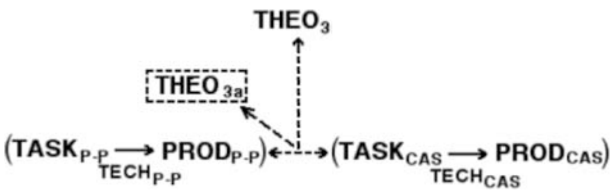
continued

Verbatim	Interpretation
34:23 **P**: *x* to the five minus one.	
34:25 **C**: Yeah, it works (inaudible). Yeah, 'cause this one takes out the *x* plus one, because it thinks it can factor it more, but if you were to multiply the *x* plus one to the *x* squared plus one, you end up with what **P** wrote, which was *x* to the power of three plus *x* squared, plus *x*, plus one.	**C** advanced the idea that the calculator can give, in this case, more factors.
34:44 **I**: Ok. That's for *x* to the?	
34:46 **C**: Power of four.	
34:47 **P**: Minus one.	
34:48 **C**: And then	
34:50 **I**: Ok **P**, are you clear on what **C** just said there?	Here we can say that they are not doubting their results, indeed they are thinking they have equivalent expressions.
34:53 **P**: Yeah, seems that the calculator tried to factor it even more. I think, the answer is still right	

The technique they used to reconcile the productions obtained with a paper-and-pencil technique and a CAS technique was to multiply only the factors that were different within their answers. That is, in this case $(x + 1)(x^2 + 1) = (x^3 + x^2 + x + 1)$. It was **P** who began to write, using this technique, and **C** agreed. At minute 37:12, **C** gave an explanation as to why they have equivalent expressions but different representations: "*But, uh, I think the calculator is factoring it more than we are. 'Cause that's taking it into its simplest possible form, we're taking it into a pattern which we recognize*." The researcher trying to get more information asked them if they could obtain exactly what the calculator provided in the two cases, $x^4 - 1$ and $x^6 - 1$. Here **P** (39:02) answered first, offering another technique to show the equivalence of the expressions: "*Well, we got our result, x minus one, times x cubed, plus x squared, plus x, plus one. And, so, I just factored out the second bracket, 'cause, it can still be factored by taking out x squared plus one, and, so yeah, you get another bracket: which is x plus one*." That is, probably he was thinking about $x^3 + x^2 + x + 1 = x^2(x + 1) + 1(x + 1)$. This answer was justifying the process from paper-and-pencil technique to CAS technique $(\mathbf{TASK}_{P\text{-}P} \xrightarrow[\mathbf{TECH}_{P\text{-}P}]{} \mathbf{PROD}_{P\text{-}P}) \dashrightarrow (\mathbf{TASK}_{CAS} \xrightarrow[\mathbf{TECH}_{CAS}]{} \mathbf{PROD}_{CAS})$. The researcher, noticing the direction of their approach, insisted: "…*Could you have started differently to obtain what the calculator obtained?*" The answer of **C** (39:55) was: "Could you do difference of squares?…" Following this idea, **P** and **C** tried a new technique, leading to **C** (43:17) saying: "*Yeah, because uh, it's really (inaudible). 'Cause what you are showing here is uh, that the x to the power of six minus one, can be split in two terms. But then, one of its terms can be split even more, 'cause it's a difference of cubes.*" As can be seen, they were trying to explain a different process that could have been used by the calculator: $(\mathbf{TASK}_{P\text{-}P} \xrightarrow[\mathbf{TECH}_{P\text{-}P}]{} \mathbf{PROD}_{P\text{-}P}) \dashleftarrow (\mathbf{TASK}_{CAS} \xrightarrow[\mathbf{TECH}_{CAS}]{} \mathbf{PROD}_{CAS})$. This epistemic moment is related to the construction of a coordination between techniques $(\mathbf{TASK}_{P\text{-}P} \xrightarrow[\mathbf{TECH}_{P\text{-}P}]{} \mathbf{PROD}_{P\text{-}P}) \dashleftrightarrow (\mathbf{TASK}_{CAS} \xrightarrow[\mathbf{TECH}_{CAS}]{} \mathbf{PROD}_{CAS})$.

6.3 Noticing More Structures and Generating a Conjecture (a False One): "The Only Time it Contains Two Factors is when it is Odd…" (Third and Fourth Sequences)

In the fourth sequence, students were asked to conjecture about different factors that could be obtained when factoring $x^n - 1$ (see Table 4). To answer these questions, **C** and **P** were obliged to analyze what they had done just before. After reading the questions and analyzing their previous productions, **P** seemed to fix on the second question regarding those cases where they obtained more than two factors, while **C** was focusing on the first question related to obtaining exactly two factors.

Verbatim	Interpretation
45:27 **P**: You, won't after, 'cause' *x*, 'cause' two is not an even number, is not a prime number.	Remembering facts (even if wrongly); **P** says 2 is not a prime number (this is the first time that the term "prime number" is mentioned, but **C** seems not to be paying attention to this). **P** seems to have a conjecture about even numbers (second question). **C** seems to focus rather on the first question. **THEO$_{3a}$**: Conjecture that $x^n - 1$ contains more than two factors when n is even (false statement, they did not exclude $n = 2$).
45:33 **I**: Can you show me what you are pointing out there? [Inaudible]	
45:35 **P**: I guess, it's just for all even numbers. That you will have, uh, they'll contain, they'll contain more than two factors. [Goes to look at their worksheet; see Table 3 of the task sequence]	
45:48 **C**: The only, yeah, the only time it contains two factors is when it is odd. I think which means it can be, [pause] yeah, which means that it can't be uh, [points at paper] like, the uh, our pattern can't be broken down anymore. 'Cause' it always ends up being all positive. And uh, then, because [he looks at first column of their worksheet], it's sort of hard to explain.	It seems that **P** and **C** agreed implicitly with **P**'s proposition and they wrote it down (see Fig. 2). Then, they concentrated on the first question related to odd exponents. **THEO$_3$**: Conjecture that $x^n - 1$ contains exactly two factors when n is odd (false statement). 

What is interesting in this epistemic moment, is that **P** states that, "…two is not an even number, is not a prime number." Even if he was wrong in his comment, it is the first time that one of them mentions "prime number". Also **P** gives a general rule for even numbers (**THEO$_{3a}$**) related to the second and third questions. Even if **C** is advancing a conjecture dealing with odd numbers (**THEO$_3$**), it seems that **C** is disturbed about something, but he is not expressing it clearly at this moment. The arguments about noticing structure (in Kieran's 2004 sense) and conjecturing are promoting the construction of **THEO$_3$**, as we hypothesized in our *a priori* analysis.

6.4 Testing and Analyzing Their Conjecture (Arriving at a Counterexample: $n = 9$) (Fifth and Sixth Sequences)

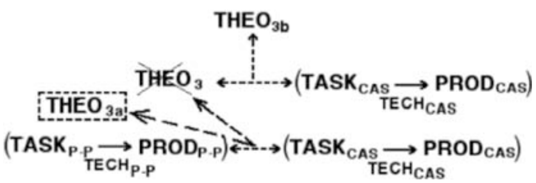
The interviewer asked **C** and **P** to summarize their conjectures thus far. While doing this, they were also moving on to the subsequent expressions of the fifth sequence, for integer values of n from 7 to 13, which allowed them to test further their conjecture regarding odd exponents.

Verbatim	Interpretation
46:35 **C**: Yeah. [Types into calculator: Expand(x^7 − 1); The calculator displayed: $x^7 - 1$] *factor*(x^7 − 1) [the calculator displayed $(x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$] Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] 46:50 **C**: [types into the calculator: Factor(x^9 − 1); the calculator displayed: $(x - 1)(x^2 + x + 1)\ (x^6 + x^3 + 1)$] No ??? 47:02 **I**: What happened there **C**? 47:06 **C**: Hmm, it actually, at a certain point finds uh, that it can be factored more. [types into the calculator: *Factor*(x^11 − 1); The calculator displayed: $(x - 1)(x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$] Uh, the only case there x to the nine, x to the eleven, didn't work. Uh [pause], I think there is like occasional, uh, exceptions to the rule [Types into the calculator: *Factor*(x^13 − 1); The calculator displayed the entire expression, but we use ellipsis dots here $(x^{12} + x^{11} + \cdots + x^3 + x^2 + x + 1)$] Like, of what we were saying about it being, uh [Types into the calculator: *Factor*(x^15 − 1)]	Continuing with further examples to convince themselves of the veracity of their conjecture about **THEO₃**. **C** was astonished by the result, indeed here was an exception to the rule (do we have the right to say counter-example? O.K. Not yet, but this is a big step in their process of learning) **C** makes explicit that there are 'exceptions' to the rule.

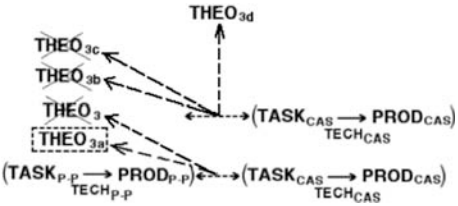
C found a counterexample with $n = 9$, which led him to search for more exceptions to the rule. Our *a priori* prediction worked as designed; **C** tried to resolve the cognitive conflict. This moment of surprise signalled the beginning of a series of epistemic actions whereby **C** would attempt to resolve what was clearly a problem. **C** was visibly *uncomfortable* about the contradiction he was facing. Could we say that that his *uncomfortable feeling* was related to his being sensitive to a contradiction? That is what construction of mathematical knowledge is all about! (See Hitt 2007). We will return to this question in Sect. 6.5.

Verbatim	Interpretation
47:28 **I**: Which is the rule we are debating here? Which, which is the rule that you're finding exceptions to?	Here **C** makes explicit that the conjecture is not a true statement (the exceptions to the rule are becoming counterexamples).

continued

Verbatim	Interpretation
47:32 **C**: We are trying to find, uh, we said that all, odd, all times that the exponent was odd it would only have two, but that wasn't true [looks at calculator; the calculator displayed (partial display shown here): *Factor*(x^15 $-$ 1) equals $(x-1)(x^2+x+1)(x^4+x^3+x^2+x+1)\ldots$], in the few cases now, because uh, it finds what it simplifies. So, but, I'm not sure how you would be able to tell what are the restrictions.	
47:51 **I**: Hmm, well is there anything in particular about the numbers that you found exceptions for? Do they, why don't you look at those?	
48:03 **C**: [Goes over previous results in the calculator] They were x to the power of fifteen, hmm [pause], x to the power of nine, wait if x to the power of twenty-one works, then it may be, uh	
48:16 **I**: **P**, do you see what **C** is doing there?	**THEO$_{3b}$**: Conjecture that $x^n - 1$ contains exactly two factors when the exponent is odd and not divisible by 3 (false statement).  We must say that this conjecture was not considered in our *a priori* analysis.
48:18 **P**: Yeah.	
48:21 **C**: [Types into the calculator: *Factor*(x^21 $-$ 1); The calculator displayed (partial display shown here): $(x-1)(x^2+x+1)(x^6+x^5+x^4+x^3+x^2+x+1)\ldots$] I think, right now, anytime, it's uh, the exponent can be divisible by three like, [Types in calculator *Factor*(x^27 $-$ 1); The calculator displayed: $(x-1)(x^2+x+1)(x^6+x^3+1)(x^{18}+x^9+1)$] Yeah. [Pause] Because, just say an odd number that is divisible by three.	

Here **C** found, with $n = 15$ and $n = 21$, more exceptions to their rule. It is important to say that after trying $n = 15$, **C** tried $n = 99$. You can see that using the calculator, these students have an instrument that permits them to analyze extreme cases; in a paper-and-pencil environment, probably the students would become tired trying out large numbers. **C** and **P** tried out during the minute 48:21 to 49:49: $n = 101$, $n = 11$, $n = 15$, $n = 95$, $n = 5$, $n = 103$!

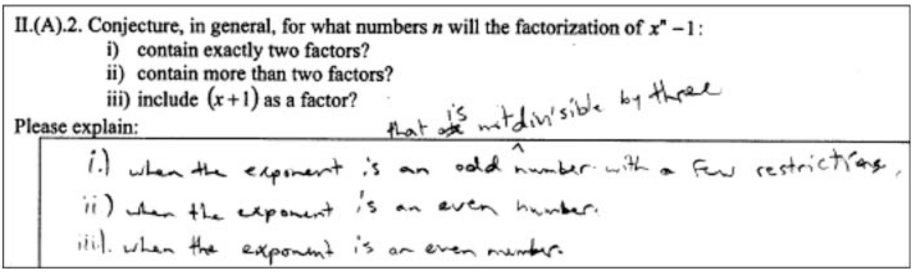
Verbatim	Interpretation
50:19 **P**: When the exponent is an odd number that is not divisible by three it'll contain, most of the time, exactly two factors.	[**THEO$_{3c}$**]: Conjecture that $x^n - 1$ contains exactly two factors when the exponent is not divisible by 3 with a few restrictions (see Fig. 3 for the various restrictions they were to formulate).
50:26 **I**: Most of the time?	
50:27 **P**: Yeah.	
50:28 **C**: 'Cause it can't be definite'. 'Cause that 95 [points at the calculator] that we plugged in (inaudible).	

They decided to write down the reformulated version of the conjecture that they had generated thus far (see Fig. 2).

As we said before, we had considered only that students would conjecture that $x^n - 1$ contains exactly two factors when n is odd (**THEO$_3$**); but as we observed, the students generated a succession of various versions of the conjecture, versions that we have referred to as **THEO$_{3b}$** and **THEO$_{3c}$**.

6.5 Moving from the Fact that Disturbed the Rule to the Eureka Moment!

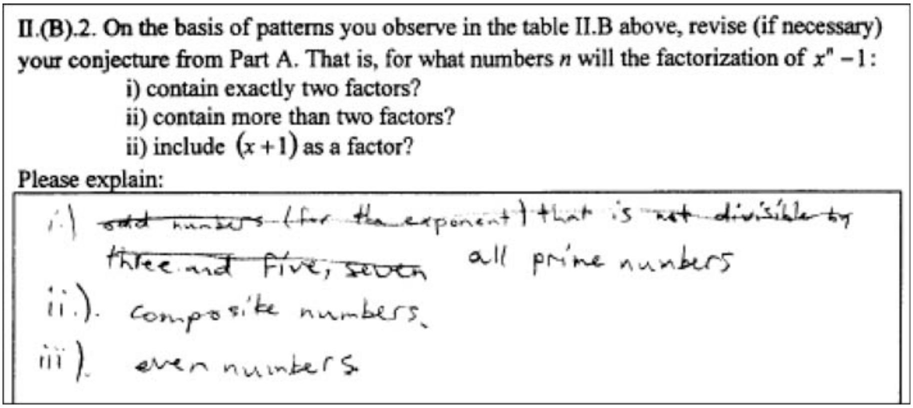
Even if they now had a more refined conjecture, they still had an uncomfortable feeling about the possibility of more exceptions. As was seen above, they had begun to try some numbers such as, $n = 9$, $n = 27$, $n = 99$. At this point, their explanation was that their



II.(A).2. Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x+1)$ as a factor?
Please explain:

i.) when the exponent is an odd number that ~~are~~ is not divisible by three with a few restrictions,
ii) when the exponent is an even number.
iii). when the exponent is an even number.

Fig. 2 Revising their conjecture (still false)


II.(B).2. On the basis of patterns you observe in the table II.B above, revise (if necessary) your conjecture from Part A. That is, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

ii) include $(x+1)$ as a factor?

Please explain:

i.) ~~odd numbers (for the exponent)~~ that is ~~not divisible by three and five, seven~~ all prime numbers

ii.). composite numbers.

iii). even numbers.

Fig. 3 Final conjecture

conjecture worked for some cases but not for others. They knew that 95 was an exception to the "not divisible by 3" conjecture. Then C tried 105, 25, 15, and 55 as exponents. He soon stated that "it cannot be divisible by 5 either." Then he proceeded to try 14, 21, and 49, concluding that "it cannot be divisible by 7 either."

They kept adding amendments to the written version of their conjecture until they arrived at the following formulation, which we refer to as $\mathbf{THEO_{3d}}$: The conjecture that $x^n - 1$ contains exactly two factors for "odd numbers (for the exponent) not divisible by three and five, seven" (see Fig. 3); but their conjecture was still false.

They were still not convinced about their conjecture; **C** stated: "There could be more than these exceptions." Then, something interesting occurred: *They put the calculator to one side* and carried out a holistic analysis of the situation. They went back to some particular cases for n and focused on its divisors.

In our *a priori* analysis, we had not considered this as a possible conjecture, that is, exponents that are odd numbers (excluding prime numbers) not divisible by 3. If we analyze this conjecture, the first counterexample we can construct is with $n = 25$. This number is the first number in the list of odd numbers that is not prime and not divisible by 3. In a paper-and-pencil environment, rejecting their conjecture could be very difficult. As can easily be seen, trying with $n = 25$ in a paper-and-pencil environment could be exhausting for the pupils. On the contrary, with the calculator, they tried numbers not divisible by 3 like $n = 101$, 95, and 103. They were clearly in a situation of *cognitive contradiction*—that is, a contradiction that has not been pointed out by the teacher but rather has arisen from their own experience. The uncomfortable feeling that was created in them could only be removed when they themselves resolved the cognitive contradiction. Sometimes a pupil may think he has resolved a contradiction, even if this is not the case from an expert's point of view. Occasionally, the formal contradiction is resolved, and this is recognized by an expert (see Hitt 2007).

Then, at a certain moment (57:07) **P** said: "Try sixty; sixty is divisible by a lot [**C** types on calculator]" and **C** said: "Yeah, I think it has to do with how many numbers can go into it." Here the interviewer asked: "How many numbers can go into it?" And **C** gave an explanation dealing with the divisors of sixty.

Verbatim	Interpretation
55:53 **C**: But, I think as soon as you get past nine or whatever, you start running into problems…	The exception to the rule was perturbing the students; they tried with other numbers keeping in mind the exception (min. 55 to min. 57).
…	
57:07 **P**: Try sixty; sixty is divisible by a lot [**C** types on calculator]	
Silence…	
57:16 **C**: Yeah, I think it has to do with how many numbers can go into it.	Prelude to a new conjecture, the divisors of a number.
57:19 **I**: How many numbers can go into it?	
56:20 **C**: Like, sixty is divisible by one, it's divisible by two, it's divisible by three, it's divisible by four, five, six.	
57:24 **P**: By four, five, six.	
57:27 **C**: Not seven, [pause] not eight.	
57:31 **P**: Not nine, ten, twelve.	
57:30 **C**: Ten. So, [pause] yeah, but then you are doubling it up.	
57:33 **P**: Yeah.	
57:34 **C**: But, it's just, like uh, [pause] at a certain [pause], prime numbers? [pause] So, a prime number is twenty-three [he types into the calculator] Yeah, prime numbers, that's it. Prime numbers when it is…	
57:51 **I**: and what are prime numbers?	
57:52 **P**: Wait, what about three, five and seven.	
57:53 **C**: Only divisible by itself. Three, five and seven, all work.	$\mathbf{THEO_4}$: Rejecting previous conjectures to produce a new one: $(x^n - 1)$ will have exactly two factors [for] all prime number values of n.
57:56 **P**: They are prime numbers.	
57:56 **C**: Yeah, they all work.	
57:58 **P**: No, but they don't give you exactly two factors.	
58:00 **C**: Yeah, they do. [Types in calculator] That's what I'm doing [pause] three, five, seven	
58:04 **P**: Yeah they do [Looks at screen]	Verifying the conjecture: **P** was not completely convinced about the new conjecture, even if **C** was.
58:06 **C**: Yeah, prime factors. And nine doesn't work because it is not a prime factor. [**P** crosses out the answer that he had written (see Fig. 3) and writes: all prime numbers]	**C** is convinced about the conjecture as a true statement.

As we said before, in our *a priori* analysis of the multitask sequences, we proposed the factoring of $x^9 - 1$ with the aim of provoking a confrontation with students' preliminary conjecture. We wanted them to realize that there was a problem, thereby arriving at a cognitive contradiction. They perceived the failure of their preliminary conjecture, and gave arguments that were not conceived of in our *a priori* analysis. But they concluded as we predicted: That a change in the conjecture was necessary—that there were exceptions as when n is divisible by 3, by 5 or by 7. After formulating these conjectures, they still felt something was wrong. They continued their search for clear counterexamples and finally generated the conjecture **THEO$_4$** regarding prime numbers (see Fig. 3)—thus resolving the state of cognitive contradiction in which they had found themselves.

We would like to stress the importance of the role of the calculator in this part of their work. Indeed, the articulation among representations and among techniques was generated largely because the calculator was used as an instrument. They continued with the rest of the activity (see Table 6) and they used the results they had already obtained, showing confidence and answering correctly.

7 Summary of Theoretical Approach and Discussion

Our results have highlighted the importance of the theoretical triad T$_{\text{ASK}}$–T$_{\text{ECHNIQUE}}$–T$_{\text{HEORY}}$ when designing mathematical tasks. However, in this concluding section, we focus on three aspects in particular. First, we discuss the issue of the way in which students constructed knowledge in our CAS environment. Then, we return to those components of our analysis that related to the question of epistemic actions within the genesis of a technique. Finally, we treat some additional issues associated with designing technological activity within the theoretical T–T–T approach.

7.1 Constructing Knowledge in a CAS Environment

Working solely with paper and pencil, students (in our case **C** and **P**) made some errors. The CAS calculator permitted them to correct these errors and eventually to construct a theory through the telescoping technique ($x^n - 1$ has two factors). Once the theory was constructed, we wanted the students to confront the results obtained using this paper-and-pencil technique for factoring $x^n - 1$, for specific values of n, with the sometimes-different-looking results produced by the calculator. The process of reconciling equivalent expressions that are represented differently promoted an articulation among representations and techniques and, at the same time, allowed students to notice structures and generate conjectures.

Something unexpected was the fact that students generated their own tasks and techniques when trying to convince themselves about their conjectures (see the epistemic moments related to this). And in this part, the calculator played a principal role. This occurred specifically within the part of the task where, as expected by the researchers, the students constructed a conjecture that $x^n - 1$ has exactly two factors when n is odd (this would fail, for example, when $n = 9$). When the students reached that part of the task that involved factoring $x^9 - 1$, the teacher and researchers had agreed not to comment that such a conjecture fails in that case ($n = 9$). The students could perceive the contradiction and it was their responsibility to solve the cognitive conflict. We were surprised that, when the students **C** and **P** saw the contradiction, they tried to resolve it by generating another conjecture, that $x^n - 1$ has exactly two factors when n is odd but not divisible by 3 (this would fail, for example, when $n = 25$). The role of the calculator was crucial at this point; in a paper-and-pencil environment it is difficult to imagine that students might reach the point of generating such a conjecture and, more important, come to reject it by trying large numbers. Then they reformulated the conjecture that '$x^n - 1$ has exactly two factors when n is odd and not divisible by 3' by adding two more restrictions: That the exponent not be divisible by 5, nor by 7. The CAS allowed them to test extreme cases in order to analyse

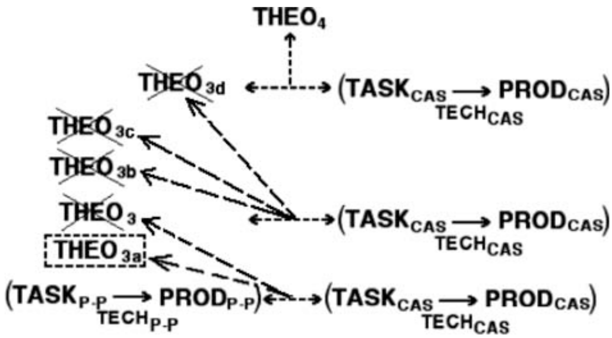


Fig. 4 Students' construction of knowledge in a T–T–T environment involving both paper and pencil and CAS

their conjectures and to generalize and theorize results; as a consequence they constructed knowledge that, in the sense of Kieran (2004), relates to the global meta-level activity of algebra.

The point is that beginning with a task in a paper-and-pencil environment, students improved their performance using the calculator. When asked to reconcile results obtained with the two media (paper-and-pencil and CAS), they began a theorization process that involved constructing an articulation among representations and techniques. In the construction of conjectures, and in their verification, the role of the calculator was crucial and permitted the students to construct an articulation among theories (rejecting old ones and generating new ones). This process is synthesized in Fig. 4.

The experimental results, as much from the point of view of the interviews (as this was the case with the present study) as from the point of view of what happened in the classroom (Kieran et al. 2006), allow us to confirm the importance of the use of technology in the mathematics classroom.

7.2 Epistemic Actions Within the Genesis of a Technique

In our Introduction, we emphasized an important argument made a few years ago by Lagrange (2003, p. 271): "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration." This claim led us to focus part of our analysis on what we perceived to be the key epistemic moments during the process of knowledge construction by **C** and **P**. The technique that was being elaborated within the multitask sequences of the given activity was that of factoring $x^n - 1$ for integral values of n. The process of developing various theoretical ideas associated with this technique involved several epistemic actions on the part of our pair of student interviewees.

We identified, in particular, the following five epistemic actions, according to which we structured the presentation of our results: (1) constructing and verifying a technique, namely the telescoping technique, (2) reconciling paper-and-pencil technique with CAS technique, (3) noticing more structures and generating several conjectures, (4) testing and analyzing their new conjectures, and (5) moving from the fact that disturbed the rule to the Eureka moment. But these were not all the epistemic actions that were engaged in by **C** and **P**. For example, we observed two other, spontaneous, epistemic actions, both of which could be categorized as ‘testing the domain of validity of a new technique’. The first involved testing the limits of their new telescoping technique by dropping the x^2 term from the expression $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ so as to see if, when multiplied by $(x - 1)$, it would still yield the telescoped result $x^7 - 1$. The second was the use of extreme cases (e.g., $n = 99$) in order to test their conjecture regarding the ‘fact’ that odd exponents for $x^n - 1$ always produced two factors.

Clearly some of the above epistemic actions were induced by the wording of our designed tasks—actions such as conjecturing, verifying, and reconciling. Others, it could be argued, were encouraged by the nature and sequencing of the tasks—actions such as generating new conjectures. Still others were made possible by the presence of the CAS technology—actions such as testing new conjectures, even the already-noted generating of new conjectures (as well as the students’ producing of new tasks and techniques whereby these conjectures could be tested). Still other epistemic actions, and these were completely unprompted, were a product of the human-mathematical-mind-assisted-by-digital-technology—actions such as testing the limits of their new telescoping technique by dropping a term from the expression, and testing their ‘odd-exponent conjecture’ by the use of extreme cases. All of these various epistemic actions were seen to play a crucial role in developing the theoretical knowledge underpinning the new technique that was being learned.

Artigue (2002, p. 268) has argued that, “epistemic value is not something that can be defined in an absolute way; it depends on contexts, both cognitive and institutional.” While many of the epistemic actions that we identified can be viewed as quite general mathematical reasoning processes (and thus definable in a certain sense), there is no doubt that the context, that is, the nature of the tasks that we designed, in combination with the technologies that were involved therein, was instrumental in provoking the epistemic actions that we observed. In other words, the epistemic value of technique, when that technique is elaborated within a CAS environment, can be said to depend to a large extent on the epistemic value of the tasks. One cannot be separated from the other.

7.3 Designing Technological Activity with the Theoretical T–T–T approach

The notation we constructed (as illustrated in Fig. 4 just above) reflects a new way to represent an analysis of knowledge construction within technological environments, one that takes into account the theoretical triad Task–Technique–Theory. The analysis of the interview by means of this method within the T–T–T approach allowed us to detect epistemic moments as the students articulated representations and techniques.

This analytical perspective also permitted us to observe the ways in which the cognitive conflict that we had hoped to engender by our task design was resolved by the students, with the help of their CAS calculators. The fact that the *Factor* command of the CAS produces results that are often in a form that is different from that obtained with paper and pencil—results that were found to be both unexpected and surprising for our students—was incorporated into our task design to serve as an epistemic motor for developing their theoretical thinking. In related work, Monaghan and Ozmantar (2006, p. 356) have highlighted the view of Davydov regarding the role played by internal contradiction in the growth of thought: “This progression depends on the ‘disclosure of *contradictions* between the aspects of a relationship that is established in an initial abstraction … It is of theoretical importance to find and designate these contradictions’ (Davydov 1972/1990, p. 291).” Our design of the multitask sequence not only took into account the internal contradictions that could be generated in the students, but also provoked the disclosure of the contradictions. On the other hand, the task sequence also promoted a need for reflection that could in turn help the students to resolve the entire sequence without contradiction.

In view of the findings presented in this paper, we consider that teachers could use this activity in the sociocultural learning setting of the classroom, to promote in their students some rather deep learning on the mathematical concepts involved in the activity—not only factorization, but also important aspects of the development of mathematical thinking that have to do with conjecturing, with the use of counterexamples, the processes of generalization, and argumentation. Indeed, our firm belief is that the T–T–T design of tasks, where an equilibrium exists between paper-and-pencil and technology activities, as we followed in our experimentation, can fill the gap between the practices of the teacher who rejects the use of technology in the classroom and those of the enthusiastic teacher who uses technology in a somewhat naïve way. Our methodology with the T–T–T design promotes a radical change in the use of technology in the classroom.

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