Design Researchers' Documentational Genesis in a Study on Equivalence of Algebraic Expressions

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Abstract: While documentation work is at the core of teachers' professional activity and professional development, this article argues for and illustrates the way in which documentation work is also central to the professional activity of design researchers. It has the double aim of contributing both to the documentational approach of didactics as well as to the literature on the teaching and learning of equivalence of algebraic expressions. The construct of documentational genesis, a component of the documentational approach of didactics as framed by Gueudet and Trouche (2009), is extended to the processes of documentation work by design researchers and the production of three documents within the present study. The initial document is designed for secondary school mathematics on the topic of equivalence of algebraic expressions using CAS tools. Subsequent analysis of the classroom work involving the use of the designed activity sequence led to the generation of a follow-up complementary document on the domain-related theoretical underpinnings of algebraic equivalence. The third document is this research paper itself, which offers a description of the documentational genesis processes as well as the products thereby generated.

Key words: documentational genesis of design researchers, mathematical documentation work, equivalence of algebraic expressions, domain-related issues in equivalence of algebraic expressions, CAS tools in algebra learning, documentational resources for algebra learning and teaching

1. Introduction

Equivalence of algebraic expressions is a key concept in high school mathematics – key in the crucial role it plays in expression simplification and equation solving, but even more broadly as one of the central "big ideas" of algebra. However, the treatment accorded to the concept in much of the teaching of school algebra is, in Canada at least, cursory in nature, rarely going beyond students' everyday meaning for the term *equivalence*, that is, "the same as" or "identical in value" (as in the Wiktionary definition of equivalence). Although a certain amount of resource literature exists with respect to the mathematical concept of equivalence relation and its reflexive, symmetric, and transitive properties (see, e.g., Asghari 2005), comparatively little can be found regarding elaborations of the concept of equivalence applicable to the various types of expressions and functions encountered in high school mathematics.

As a design research team[1], we began to think about these limitations regarding equivalence-related resources at the time that we were developing a research project involving the use of Computer Algebra System (CAS) technology in the teaching and learning of school algebra. We came to decide that this was an area where we could attempt to fill an obvious shortfall in the available resources. We also opted to keep some traces of the generative process in a journal – for our own archival reasons. At the time, we could not foresee that our documentational process would be shared with others. This volume on the documentational approach of didactics provided us, however, with an opportunity to present not only the products of our research work, but also the processes underlying their genesis. Note that we do not view these products and processes as two separable aspects of our research, but rather as co-emergent phenomena. We emphasize as well that mathematical considerations were central to the intertwining process and product aspects of our research. This research paper is thus a story of design researchers' mathematically-oriented documentation work.

2. The Documentational Approach of Didactics and Documentational Genesis

2.1 The Basic Notions of Documentational Genesis

The documentational approach of didactics is a relatively new way of thinking about various aspects of the development and use of documents and resources in the educational field, an approach that was initially framed by Gueudet and Trouche (2009), and further elaborated in two successive volumes (Gueudet and Trouche 2010, Gueudet et al. 2012). Within this framework, Gueudet and Trouche (2009) describe *documentation work* as including all facets of activity in which teachers interact with resources, and where *resources* are defined as comprising a variety of artifacts such as, "a textbook, a piece of software, a student's sheet, a discussion with a colleague" (p. 205).

One of the pivotal constructs of the documentational approach of didactics is *documentational genesis*. Gueudet and Trouche emphasize that, when teachers draw upon resources for their documentation work, a process of genesis takes place, producing what they call a *document*. The document, in turn, gives birth to a new resource that may be

[1] Note that the composition of the design research team has varied over time, but that its makeup during the latter phase of documentational work and the development of this article consisted of three UQAM colleagues with a tradition of working together and one visiting researcher.

combined with other resources in a later cycle of documentational genesis. The process of genesis, which involves building or adapting *schemes of utilization* for sets of resources, is represented by Gueudet and Trouche (2009) in terms of the relation: *Document = Resources + Scheme of Utilization*. They describe schemes of utilization as follows:

> A scheme of utilization of a set of resources entails both an observable part and invisible aspects. The invisible aspects are the *operational invariants*, the cognitive structure guiding the action. The observable part corresponds to the regularities in the teacher's action for the same class of situations through different contexts. This part is what we call *usages*. (Gueudet and Trouche 2009, p. 208)

The relational formula representing the process of documentational genesis is then re-expressed more precisely by Gueudet and Trouche as: *Document = Resources + Usages + Operational Invariants*. With respect to operational invariants, they note additionally that these must usually be inferred from the observation of regularities in teachers' behavior, that is, their *usages*, which the researchers refer to more particularly as *action rules*. In summarizing the documentational product generated by one of the teachers of their research studies, they write: "The document produced [for the class of situations: 'organize mental arithmetic sessions in class on the product of decimal numbers'] includes the *resources* selected [such as the computer and projection device]; *rules of action* like 'prepare a precise schedule for the slides', 'propose a task on deducing decimal number products from the results of the corresponding integer products'; and *operational invariants* like 'computing in limited time enhances mental arithmetic procedures' and 'students must be able to recognize and use the property: if $a \times b = c$, then $m \times a \times n \times b = m \times n \times c$" (Gueudet and Trouche 2009, p. 210). The inferred operational invariants are, according to Gueudet and Trouche, part of the set of beliefs and knowledge of the teacher and are both driving forces and outcomes of the teacher's activity.

Gueudet and Trouche (2009) consider that documentation work is at the core of teachers' professional activity and professional development; however, we would argue that documentation work is also at the core of design researchers' professional activity (see Kelly et al. 2008, for the nature of design research and the activity of design researchers). With respect to the frame of the documentational approach of didactics, Gueudet and Trouche (2012) themselves have recently commented that "complementary studies are needed; the theoretical aspects of the approach have to be refined" (p. 39). We offer here a new direction in response to their comment. It comprises an extension and elaboration of the construct of documentational genesis to render it usable as a framework for describing and analyzing the professional activity of design researchers in their document-generation work, work that includes collaboration on document-use with practitioners in the classroom. We propose to do this in two steps, first sketching its broad lines in a general manner in this section and then, with the aid of exemplification provided in succeeding sections, returning with a more detailed treatment in the last section of the article.

2.2 Our Extending of the Construct of Documentational Genesis

We take as our starting point Gueudet and Trouche's second relational formula for the process of documentational genesis, *Document = Resources + Usages + Operational Invariants,* and adopt their term *Action Rules* to reflect the main component of *Usages.* Examples of *documents* resulting from the documentational genesis of design researchers might include hypothetical learning trajectories (HLTs) (Simon 1995), sequences of learning activities to support HLTs, teacher guides to accompany the learning sequences, any of a variety of designed products that teachers might consider using for their classroom teaching, software environments designed for learning, research articles written for teachers and/or for researchers, and so on. Note that, according to this theoretical framework, a *document* relates directly to the cognitive structures of those who have been involved in its design; for potential users, a document is materialized in the form of a *resource*. To avoid confusion, we shall use the term *document* when we are referring to the resources that we have designed.

Typically, design research consists of three phases (Cobb and Gravemeijer 2008): (i) initial preparation of an HLT and related sequence of learning activities, (ii) experimentation of the sequence with actual learners, and (iii) analysis of the data obtained during the second phase. These three phases can be repeated over several cycles, with each cycle building upon the insights derived from the previous. Using the lens of documentational genesis, we attempt to cast a different light upon the professional activity of design researchers. As an aside, we note that design research is much broader than *research design*, which is typically considered akin to *research methodology* in the design literature (Kelly and Lesh 2000). Consequently, and in line with Gueudet and Trouche (2009), we suggest that a first phase of documentation work by design researchers (perhaps in collective engagement with teachers) can include consideration of various existing *resources* such as text-books, digital tools, research articles on the curricular topic at stake, results of prior studies on students' activity with teaching sequences designed around the same topic, features of related software-based learning environments, conversations with teachers and researchers, as well as the bringing to bear of their own creative ideas – with the aim of developing an envisaged learning trajectory and designing materials or an activity sequence that will allow the researchers to better understand and to support student learning. This phase of the design process, which results in the generation of a document, is also built upon *operational invariants* and *action rules.*

We emphasize here that it is in the following elaboration of operational invariants and action rules for the case of design researchers that we move beyond what exists already in

Gueudet and Trouche's framework of documentational genesis – an elaboration that is based on a retrospective analysis of our own design processes. Just as the *operational invariants* can be quite varied for the case of teachers, we suggest that the operational invariants (OIs) for design researchers can also be of several types and will reflect the principles that for them underlie the design of certain classes of situations. For the class of situations, 'design activity sequences to support the mathematical learning of such-and-such a topic,' the OIs could include the following (see also the design research articles of, e.g., Cobb & Gravemeijer 2008, Holmqvist et al. 2008, Middleton et al. 2008):

(a) The activity sequence must be embodied within a *theoretical* framework;
(b) The activity sequence must be embodied within an approach to *teaching* where the teacher encourages students to think and to explain their thinking;
(c) The activity sequence must be embodied within an approach to *learning* where classroom discussion of mathematical ideas is considered central to the development of individual learning;
(d) The activity sequence must have a clear *mathematical* goal and engage the students deeply in the mathematics related to that goal;
(e) The activity sequence must be based on *what the students already know* about the mathematics in question; and
(f) The activity sequence must make use of digital *technology* resources, if appropriate, and in such a way that they can be used as thinking tools.

The *action rules* (ARs) that researchers employ in their design are evidenced by the explicit decisions they make in operationalizing their design principles within the process of design. Depending on the specificity of the underlying OIs, there could be one AR per OI or several, as in the case of quite general OIs. For example, an underlying OI as general as "the activity sequence must have a clear *mathematical* goal and engage the students deeply in the mathematics related to that goal" (that is, OI-d) can give rise to several ARs for the design of an activity sequence on the equivalence of algebraic expressions, such as:

* Include in the sequence both numerical and algebraic approaches to equivalence;
* Make the expressions in the tasks complex enough that equivalence cannot be determined on a purely visual basis;
* Include in the tasks: equivalent polynomial and rational expressions, non-equivalent expressions, expressions of equality involving both equivalent and non-equivalent expressions;
* Highlight in some of the tasks the issue of "restrictions" (i.e., the values for which the expressions are undefined); and
* Introduce a variety of techniques to determine equivalence: factoring (both paper-and-pencil and CAS), expanding (both paper-and-pencil and CAS), and the CAS test of equivalence.

This brief sketch of our extension of the basic notions of the documentational genesis frame, with a few illustrations relating to its application within the professional work of design researchers, will now be further exemplified by details from a study on the equivalence of algebraic expressions – a study that through the documentational processes engaged in by the researchers yielded the design of three documents: a first document on the activity sequence to support the learning of equivalence of algebraic expressions, which we refer to as the Activity Sequence Document; a second document on domain considerations related to algebraic equivalence, which we refer to as the Complementary Theoretical Document; and a third document that describes the unfolding of the design researchers' documentational geneses, which we refer to as the Research Paper Document.

3. Literature Study on Equivalence in High School Algebra: A First Step in the Team's Documentational Genesis

A review of the mathematics education literature attests to the scarcity of conceptually-elaborated papers on the topic of equivalence that are relevant for the high school level. While studies of students' difficulties with the concept of equivalence are ample enough (e.g., Kieran 1984, Steinberg et al. 1990, Sackur et al. 1997, Knuth et al. 2011), including research on the role that technology environments can play in both fostering and enriching students' thinking about equivalence of algebraic expressions (e.g., Ball et al. 2003, Nicaud et al. 2004), few reports address explicitly the mathematical concept of equivalence of algebraic expressions beyond stating that, for example, "understanding two algebraic expressions to be equivalent entails knowing that they denote the same numerical value for a given common replacement value and realizing that the usual algebraic transformations performed on them conserve this denotation" (Sackur et al. 1997, p. 47) or that "two algebraic expressions are said to be equivalent if and only if it is possible to transform one into the other (or both into a third one) by means of the axioms" (Cerulli 2004, p. 89). As well, it is fairly standard practice in both textbooks and in research on algebra learning at the high school level to restrict the treatment of equivalence to polynomial expressions and to consider, often implicitly, the domain to be that of the real numbers $\Re$.

Mindful of these limitations emerging from the available resources, the activity sequence we were to design – as will be seen in Section 4 – included, for example, expressions whose domain was other than the real numbers. However, the mathematical deliberations that underpinned the design of the activity sequence were later found to be wanting in certain respects. The period during which the activity sequence was used in algebra classes, which is described in Section 5, was to disclose that our thinking about the underlying mathematics had to move much deeper. We were gradually to come to realize that the notion of *domain* was a key component of the concept of equivalence. But first we elaborate on the documentational process underlying the genesis of the Activity Sequence Document.

4. Document 1: The Activity Sequence Document

4.1 Resources and the Process of Documentational Genesis

In developing the Activity Sequence Document on equivalence, as was the case with all of the activity sequences we designed for our project, we relied upon many different resources: (a) the multi-facetted experience of the various members of the research team, with each having special strengths in particular areas, such as mathematics, informatics, didactics, psychology, and design research; (b) the availability of Computer Algebra System (CAS) technology and its potential as a thinking tool in the learning of algebra; (c) the collaboration of the project consultants whose combined mathematical and didactical expertise and experience with CAS technology were especially relevant; (d) the participation of post-doctoral fellows, which yielded additional levels of didactical experience that derived from research-oriented degrees from other universities; (e) the collaboration of 10^{th}-grade school teachers who provided feedback on early versions of the activity sequences and who were pivotal to the study in that they used the activity sequences in their classes; (f) textbooks and related curricular documents used by the participating teachers; (g) the results of a written pretest administered to the students of the participating teachers; and for the particular case of the activity sequence on equivalence of expressions: (h) the past experience of certain members of the team in prior research on algebraic equivalence and a familiarity with the professional and research literature in that area. Lastly, there were (i) the initial drafts of the designed activity sequence that were generated throughout the development process.

These resources were of two types: material (b, f, g, i) and human (a, c, d, e, h) (see also Lampert et al. 2011). However, it was the collective interaction between the human and the material that characterized the process of documentational genesis. For example, the initial idea for the theme of the activity was proposed by certain members of the research team, who then worked on developing a first draft. This initial draft drew upon the background knowledge, experience, and beliefs of its crafters (i.e., upon their taken-as-shared OIs), in conjunction with other available material resources. It was operationalized by specific action rules and constituted the first of several versions of material resource "i". The draft was then put to the team during their regular research meetings (the project teachers participated in one of these meetings, contributing their own OIs and related ARs to the discussion – in particular, those related to issues of timing, potential student difficulties, additional tasks to bridge these difficulties, and overall presentation of the activity sequence[2]). The collective team discussions involved engaging in a back-and-forth flow of ideas where all team members explained and justified their thinking. In this way, team members both learned from the interactions around the given draft resource (i.e., the *instrumentation* component of documentational genesis) and also contributed their own knowledge- and experience-based suggestions regarding the draft (i.e., the *instrumentalization* component). The documentational process, which was stimulated by comments related to the material resource, was therefore one that both constituted and was constituted by the thinking of the participants. Each round of the process encouraged the sharing of individual OIs (and associated ARs), so that eventually the final version of the activity sequence document came to be based on a shared set of OIs, those that had been verbalized, discussed, and refined during the design research meetings and thus brought to a level of general awareness among members of the team.

4.2. Operational Invariants and Action Rules Reflected in the Designed Activity Sequence on Algebraic Equivalence

In Section 2, we listed six operational invariants (OIs) that could potentially undergird researchers' design of activity sequences to support the mathematical learning of various topics. All six of these principles were foundational to our design of the activity sequence on equivalence and were discussed extensively during the collective research meetings that led to the final version – even if considerations related to the OI on mathematical underpinnings tended to dominate at times. Here we sketch the main features of the activity sequence document and refer briefly to the underlying OIs and their operationalizations.

Because the participating students had had no prior, explicit experience with equivalence, but were already quite skilled in certain basic algebraic manipulations – as was disclosed by a pretest we designed and administered to them (see OI-e) – we considered that an appropriate instructional goal was an emphasis on the semantic (i.e., numeric) and its articulation with the syntactic (i.e., algebraic transformations). Because of the students' previous work with transformations such as factoring, expanding, grouping, and simplifying, we conjectured that they had likely already developed the beginnings of a spontaneous notion of equivalence – one involving the linking of expressions by means of algebraic transformations. Thus, we aimed at transitioning this spontaneous notion toward a conception of equivalence that was associated more strongly with the semantic/numeric (see OI-d). Furthermore, we also wished to include both polynomial and rational expressions in the task set and so we eventually decided on the following

[2] For an elaboration of the ways in which the teachers, as a result of their participation in this project, evolved professionally, see Kieran and Guzman (2010).

definition of equivalence for the activity sequence: *We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined); if, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.*

This definition of equivalence constituted the central mathematical orientation of the activity sequence. The sequence began with a focus on numeric evaluation by CAS and comparison of the resultant values for the given expressions (see Figure 1 for the expressions used in several of the tasks).

Expressions
1. $(x-3)(4x-3)$
2. $(x^2+x-20)(3x^2+2x-1)$
3. $(3x-1)(x^2-x-2)(x+5)$
4. $(-x+3)^2+x(3x-9)$
5. $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$

Figure 1. The expressions used throughout several of the equivalence tasks

The next part of the activity sequence drew upon the use of the CAS techniques, factor and expand, to determine equivalence based on the search for "common forms," which were made possible by these algebraic transformations. The CAS technology the students were using (the TI-92 Plus hand-held calculator) neglected referring in any way to the presence of possible restrictions in rational expressions – a constraint that the research design team intended would serve as a basis for reflection and classroom discussion (see OI-f).

The activity sequence continued with tasks that required, for example, determining the largest set of admissible values for a group of equivalent rational expressions, constructing an equation from a pair of non-equivalent rational expressions and then determining the set of admissible values for the solution, and finding solutions to various types of equations with the CAS tool so as to be able to respond to questions such as: "What does the nature of an equation's solution(s) indicate about the equivalence or non-equivalence of the expressions that form the equation?"

Tasks were structured into blocks, with each block to be followed by a classroom discussion – focused classroom discussion being considered crucial to student learning of mathematics (see OI-c). Each block of tasks was headed by a descriptive phrase that suggested the content of the block and reflected the *action rule* (AR) that had underpinned the generation of that block (e.g., the descriptive phrases, "Compare expressions by numerical evaluation" and "Compare expressions by algebraic manipulation," designated the content of AR blocks that were designed for operationalizing the generally-stated OI-d that the activity sequence must have a clear mathematical goal). Two versions of the Activity Sequence Document were generated, one for the students and one for the teacher[3]. The teacher version included all the tasks of the student version, as well as supplementary information, such as the kinds of thinking students might engage in, possible erroneous approaches they might use, entry points for whole class discussion of particular issues associated with equivalence, and encouragement to engage the students in explaining their thinking during classroom discussions (see OI-b).

Lastly, the design of the activity sequence was embedded within the theoretical framework (see OI-a) of the Anthropological Theory of Didactics (ATD) (Chevallard 1999), a framework that soon after its development came to be integrated within the instrumental approach to tool use (Artigue 2002). A central feature of this framework is the interplay between the conceptual and the technical, which was reflected in several of the action rules underlying our task design (Kieran and Drijvers 2006).

The generation of the Activity Sequence Document did not, however, bring the process of documentational genesis to an end. As will be seen shortly, the classroom experience of teachers and students actually working with the activity sequence was to constitute a new and crucial resource for the team, one that disclosed underdeveloped aspects on equivalence not just in the activity sequence itself but also in the team's mathematical discussions that had culminated in the activity sequence. And thus began the next phase of documentational genesis for the design research team.

5. The Use of the Activity Sequence in the Classroom: A New Resource in the Documentational Process

Once the activity sequence on equivalence of algebraic expressions was generated, the teachers integrated it into their 10th grade classroom teaching of mathematics over a

[3] For both teacher and student versions of the Activity Sequence Document, which are available in three languages, see Activities 1, 2, and 3 on the project web site: http://www.math.uqam.ca/~apte/TachesA.html

period of about two weeks. Despite evidence of the positive manner in which the designed activity supported student learning (Kieran and Drijvers 2006), our observations and videotape analyses of the classroom work also revealed that the designed activity sequence was underdeveloped in a certain respect. The following illustrative extracts of classroom discourse on "restrictions" suggest the nature of the gap – one that revolved around issues related to the notion of domain.

Restrictions that "disappear"

Early classroom discussions on the equivalence of Expressions 3 and 5 soon brought to the fore the restriction involving Expression 5 (see Figure 1). However, a complication arose for the students when that rational expression was evaluated with its restricted value, both before and after simplification:

Matthew:	When you factor it and you put in negative two, it will give you negative eighty-four as the answer [i.e., the value of the expression].
Teacher:	But are you missing something there?
Matthew:	The restriction.
Teacher:	What is the restriction, what does it mean?
Matthew:	x can't equal negative two.
Teacher:	What does it mean, why is that a restriction?
Matthew:	Because you can't divide by zero.
Teacher:	So should it be negative eighty-four or should it be undefined?
Matthew:	Undefined.
Paul:	If you factor it out?
Teacher:	You need to be aware of that restriction.

When the teacher remarked that "you need to be aware of that restriction", but did not expand further, we wondered if he might have said more – some link with the domain of definition for the two given expressions. However, the teacher version of the document had not been very detailed in this regard, offering only the suggestion that the teacher pose the following question: *What is the domain of definition for each of the given expressions?*, which he did not do.

Restriction: "Does it automatically apply to the other side of the equation?"

A second domain-related issue arose in the context of using the *CAS Equivalence Test* when Expression 5 was the right-hand member of an equation and Expression 3 the left-hand member: $(3x-1)(x^2-x-2)(x+5)=\dfrac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$. The issue concerned the application of the right-hand restriction to the entire equation:

Emile:	If one side has a restriction at negative two, doesn't the other side automatically have a restriction at negative two also?
Teacher:	No, because they're two different expressions, aren't they. The expressions aren't the same.
Emile:	But … if you can't put in negative two on one side, then that means you can't put negative two in the other either.

With respect to this particular equation, the set of admissible values for the entire equation is clearly the set $\Re$ excluding -2. However, the discussion around this issue focused exclusively on the restrictions of the component expressions rather than the equation as a whole.

Restrictions: "All the numbers for which two expressions are not equal"

Yet another ambiguity related to restrictions and equivalence revealed itself during one of the classroom discussions on the relation between non-equivalent expressions and equation solutions:

Ron:	I'd define it [equivalent expressions] as an equation where values of *x* exist that will make both sides equal to each other.
Teacher:	How many values of *x*?
Ron:	At least one, one or more (for which Ron provided as an example the two expressions $x+2$ and $x/2$).
Teacher:	$x+2$ and $x/2$ are equivalent?
Ron:	Could be.

Ron's comment that the two members of an equation can be said to be equivalent for only certain values of *x* was a surprise both for the researchers and for the teacher. Where a *restriction* had for most students meant that evaluating both members of the equation by that restricted number would not yield the same numerical result on both sides (on the side consisting of the rational expression, the result would be "undefined"), Ron had taken the term *restriction* in its most literal sense and had thereby included as restrictions all the numbers for which the right- and left-hand members of the equation would not be equal.

Relating restrictions to transitivity of equivalence

In another task, students were faced with four expressions, three of them polynomials and one a rational expression:

1. $4(x-1)^2 - (x+1)^2$
2. $(2x+5)(x-3) - (x-3)^2$
3. $(x-3)(3x-1)$
4. $\dfrac{(3x-1)(x^2-x-6)}{(x+2)}$.

They were asked: (a) *Use your CAS to determine which of these expressions are equivalent;* and (b) *Which are the equivalent expressions (don't forget to specify the set of admissible values for x)? Please explain your decisions about equivalence.*

The following two separate fragments of conversations were recorded while students were working on this task (note that Expressions 1, 3, and 4 are equivalent with the restriction that x cannot be -2):

> *Matthew saying to Jake*: So these are equivalent all the time, 1 and 3. Just listen, 1 and 3 are equivalent all the time, right, 3 and 4 are equivalent except for when x equals negative two.
>
> *Andrew saying to Peter*: If expression 1 is equivalent to expressions 3 and 4, but not to expression 2, then expressions 3 and 4 won't be equivalent to expression 2 either – because they're equivalent to expression 1, which isn't equivalent to expression 2 [he says nothing about the restrictions].

These two extracts highlight issues related to conciliating transitivity with values where the expressions are not defined, as well as allowing for an equivalence that involves a restriction applicable to more than two expressions.

As suggested by the above four extracts, certain aspects related to domain had not been adequately considered and planned for in the design of the Activity Sequence Document.

6. Document 2: The Complementary Theoretical Document

In the ensuing discussions of the research team, it became clear that we needed to return to the question of the nature of the mathematics underpinning equivalence for secondary school algebra. The classroom discourse resource, and the insights it yielded, had provoked us into beginning a deeper reflection on mathematical considerations that had not been fully elaborated in the earlier discussions related to the Activity Sequence Document. Nevertheless, the process engaged in was very much like that described for the generation of the earlier document – a process of documentational genesis that involved both the shaping of collective ideas as well as the collective's being shaped by them, that is, an interactive process of document production and collective awareness-building on the part of the design research team. The document whose genesis was constituted by this second phase of collective discussions treated much more precisely the mathematics of *domain of definition* and *transitivity* as they relate to equivalence of algebraic expressions. This theoretical-mathematical document, and its corresponding resource, was intended to serve as a complement to its predecessor on the activity sequence, thus creating a combined set of potential resources for teachers as well as, possibly, for other researchers on the topic of equivalence of algebraic expressions.

We include here only the key features and some illustrative examples included in the Complementary Theoretical Document[4] – features that reflect the way in which the design team's collective OI related to the mathematical underpinnings of equivalence of algebraic expressions was deepened and refined as a result of the documentational genesis process. All of the text that follows in this section is extracted verbatim from the Complementary Theoretical Document.

There are two definitions for the equivalence of two expressions $f(x)$ and $g(x)$, equivalence for which we will use the usual notation $f(x) \equiv g(x)$ – note that we restrict ourselves here to single-variable expressions:

- A *syntactic definition*: $f(x)$ and $g(x)$ are equivalent if and only if we can establish their equality by symbol manipulation, using rules recognized as true for the set $\mathcal{E}$.
- A *semantic definition*: $f(x)$ and $g(x)$ are equivalent if and only if for every element a in $\mathcal{E}$ we have an equality between $f(a)$ and $g(a)$ (we shall refer to this particular definition as **Semantic Definition of Equivalence, Version 1**).

There is some difficulty involved in making the syntactic definition more precise, as this would require an exhaustive enumeration of all recognized rules. We shall not pursue this direction further. Instead, we choose to consider the semantic definition, which seems less problematic. This definition poses no problem in the case where the expressions $f(x)$ and $g(x)$ are polynomials; but we will see that the situation gets more complex if we accept, within our expressions, operations such as division, roots, or other functions like trigonometric functions.

Let's consider, for example, the following "equivalences":

$$\frac{x-1}{x-1} \equiv 1,\ \sqrt{4x} \equiv 2\sqrt{x}, \text{ and } \cos(x) \times \tan(x) \equiv \sin(x).$$

If we apply Version 1 of our Semantic Definition of Equivalence, these "equivalences" are all false, because we can find a counterexample in each case:

$$\frac{1-1}{1-1} \neq 1,\ \sqrt{4(-1)} \neq 2\sqrt{(-1)}, \text{ and } \cos(90°) \times \tan(90°) \neq \sin(90°).$$

Indeed, in each case[5], at least one of the two expressions is undefined.

[4] For the entire text, see *A Complementary Theoretical Resource on Equivalence of Algebraic Expressions* on the web site: http://www.math.uqam.ca/~apte/TachesA.html

[5] For the second case, we assume that we are working with the real numbers. Also note that we consider that asserting the truth of $f(a) = g(a)$ presupposes that $f(a)$ and $g(a)$ are both defined.

Note that, in the usual practice of symbol manipulation, rules corresponding to these three "equivalences" are used, sometimes subject to certain "precautions". We intend that this practice be reflected in our theory, which brings us to look for a definition of equivalence with a more general reach.

Semantic Definition of Equivalence, Version 2:
$f(x) \equiv g(x)$ if and only if

- For every element a in $\mathcal{E}$**,** we have: $f(a)$ is defined iff $g(a)$ is defined;
- For every element a in $\mathcal{E}$ for which $f(a)$ and $g(a)$ are defined, we have $f(a) = g(a)$.

We see at once that this new definition settles the case $\sqrt{4x} \equiv 2\sqrt{x}$, but not the cases of $\frac{x-1}{x-1} \equiv 1$ and of $\cos(x) \times \tan(x) \equiv \sin(x)$. In fact, for these last two cases, the right-hand expression is everywhere defined, but not the left-hand one. We hope to improve the situation by proposing a new definition.

Semantic Definition of Equivalence, Version 3:
$f(x) \equiv g(x)$ iff for every element a of $\mathcal{E}$ for which $f(a)$ and $g(a)$ are both defined, we have $f(a) = g(a)$.

One can easily check that the three above-mentioned examples are in fact equivalences according to this new definition. But we still have some problems: Contrary to the preceding definitions, this new definition leads to a non-transitive equivalence, as shown by the following example:

$$|x| \equiv \left(\sqrt{x}\right)^2 \text{ and } \left(\sqrt{x}\right)^2 \equiv x, \text{ but it is not the case that } |x| \equiv x.$$

In fact, in this last example, the first two equivalences are verified (because both sides are defined and equal over the non-negative numbers), while the third equivalence is not verified (because both sides are defined, but not equal, over the negative numbers).

Why is it so important that equivalence be transitive? One crucial reason is that transitivity constitutes an essential part of proofs by the syntactic approach. Let's simply consider the following example:

$$\begin{aligned}(x+1)^2 &= (x+1)(x+1) = (x+1)x + (x+1)1 = (x+1)x + x + 1 \\ &= xx + x + x + 1 = x^2 + x + x + 1 = x^2 + 2x + 1.\end{aligned}$$

Each rule used allows us to be certain that each expression is equivalent to the next one. But how can we conclude that the first expression is equivalent to the last one? Precisely because of transitivity!

We next try to formulate a definition conciliating transitivity and the presence of values where expressions are not defined. The aim is to restrict ourselves to a subset $\mathcal{D}$ of $\mathcal{E}$ where both expressions are defined.

Semantic Definition of Equivalence, Version 4:
Let $\mathcal{D}$ be a subset of $\mathcal{E}$**.** We will say that $f(x) \equiv_{\mathcal{D}} g(x)$ ("$f(x)$ is equivalent to $g(x)$ on $\mathcal{D}$") iff for every element a in $\mathcal{D}$, $f(a)$ and $g(a)$ *are both defined and equal.*

This new definition satisfies a restricted form of transitivity:

$$f(x) \equiv_A g(x) \text{ and } g(x) \equiv_B h(x) \text{ implies } f(x) \equiv_{A \cap B} h(x).$$

Let's see how this can be used in practice. Just cast a new glance at the preceding example:

We have $|x| \equiv \left(\sqrt{x}\right)^2$ on the positive numbers and $\left(\sqrt{x}\right)^2 \equiv x$ on the positive numbers; thus we shall have $|x| \equiv x$ on the positive numbers.

Let's now look at another example:

We have $\dfrac{x-1}{x-1} \equiv 1$ everywhere except in 1,

and also $1 \equiv \dfrac{x-2}{x-2}$ everywhere except in 2.

Thus we will have $\dfrac{x-1}{x-1} \equiv \dfrac{x-2}{x-2}$ everywhere except in 1 and 2.

Note that we used, in this last example, a variant of Version 4 of our definition, where the emphasis is put not on a set $\mathcal{D}$ of numbers where everything is fine, but rather on a set $\mathcal{N}$ of numbers where problems (i.e., restrictions) are present. One can imagine, in the following version, that $\mathcal{N} = \mathcal{E} \setminus \mathcal{D}$.

Semantic Definition of Equivalence, Version 5:
Let $\mathcal{N}$ be a subset of $\mathcal{E}$**.** We will say that $f(x) \equiv g(x)$ except on $\mathcal{N}$ ("$f(x)$ is equivalent to $g(x)$ except on $\mathcal{N}$") iff for every element a in $\mathcal{E}$ but not in $\mathcal{N}$, $f(a)$ and $g(a)$ *are both defined and equal.*

Here is the form taken by transitivity with this Version 5 of our definition:
$f(x) \equiv g(x)$ except on A and $g(x) \equiv h(x)$ except on B implies $f(x) \equiv h(x)$ except on $A \cup B$.

Even if set $\mathcal{N}$ could be any subset of $\mathcal{E}$, we would want it to be as small as possible. But this will not always be possible: Sometimes, $\mathcal{N}$ will have to be finite, countable, even co-finite, as shown by the following examples:

- $\frac{x-1}{x-1} \equiv \frac{x-2}{x-2}$ except in 1 and 2
- $\cos(x) \times \tan(x) \equiv \sin(x)$ except when x = 90º + k•180º, where k is an integer
- $\left(\sqrt{x}\right)^2 \equiv x$ except when x is negative
- $\sqrt{x} \equiv \sqrt{-x}$ except when x is nonzero.

So, Version 5 of our definition seems necessary and best suited in situations where expressions are not polynomials (a case where Version 1 is sufficient) or rational functions (a case where Version 3 is sufficient [6]). But we must acknowledge that it seems rather too complex for secondary school students and thus may need to be somewhat transposed for the school environment.

7. Reconsidering the Teaching of Equivalence in Secondary School Algebra in the Light of Document 2

The documentational genesis that led to the generation of the Complementary Theoretical Document, with its formulation of several successive versions of a semantic definition of equivalence, has allowed for more clearly identifying, albeit a posteriori, those aspects that had not been sufficiently developed within the activity sequence. We synthesize those aspects as follows:

- The *domain-restriction* duality: the values where the expression is defined forms the *domain* of the expression; its complement within the reference set – which is $\Re$ by default – has as its elements the *restrictions*.
- The equivalence between two expressions can only be considered on the intersection of the domains. The dual point of view involves accumulating the restrictions (the union of the complements of the domains in $\Re$).
- In order to say that two or more expressions are equivalent, in particular, when invoking the transitivity of equivalence (which supposes at least three expressions), it is necessary in principle to consider the intersection of all the relevant domains (the accumulation of all the restrictions).

[6] In fact, if $f(x)$ and $g(x)$ are quotients of polynomials that take the same values for an infinite number of elements of $\mathcal{E}$, then they take the same values for every element in the intersection of their domains.

- But to apply transitivity is to “jump over” an intermediate expression and thus lose sight of the restrictions brought by this intermediate expression, while a rigorous approach would necessitate keeping track of *all* the restrictions. In fact, the strict application of this rule sometimes obliges the inclusion of restrictions that are later seen not to be true restrictions with respect to the starting and ending expressions. It is therefore necessary, if possible, to reconsider the restrictions one by one.
- To consider the restrictions one by one is possible when working with quotients of polynomials in one variable, which give rise to only a finite number of restrictions. But when considering expressions that involve roots or functions (transcendental) such as sin, cos, tan, log, etc., one might not have any other choice but to determine the intersection of domains (or the union of the sets of the excluded values), and to verify afterward whether one has added or removed too many values.

These considerations might seem artificial when one is dealing only with equivalence of expressions, especially if the expressions are just polynomials or quotients of polynomials (i.e., rational expressions): One could always sufficiently restrain the domain so that none of the involved expressions would cause any problems. But the situation can be otherwise when equivalence of expressions serves as a tool to establish the equivalence of certain *equations*.

From the start, we wished to separate the two notions of equivalence: *equivalence of expressions* and *equivalence of equations*. Here too, the domain of definition poses an obstacle to what could otherwise be a simple syntactic link between the two equivalences, such as, “replacing one of the members of an equation by an equivalent expression yields an equivalent equation.” Thus, in the Activity Sequence Document, we had proposed, within the later part of the sequence, the beginnings of work on the links between equivalent expressions and equivalent equations by offering a semantic definition of the equivalence of equations. It was the usual definition: “Two equations are equivalent if and only if they have exactly the same solutions.” As was the case with the earlier classroom work in the activity sequence, the later work suggested that the activity sequence had not gone far enough. This was made abundantly clear by the comment of the student Ron, presented in one of the above extracts, who took the term *restrictions* literally to mean all the numbers that were not solutions to an equation (i.e., all the numbers for which the right- and left-hand members of an equation do not produce equal values). It had never occurred to the research group that one could speak about equivalence over a domain $\mathcal{D}$ when $\mathcal{D}$ is reduced to a finite or discrete subset of $\Re$. Ron’s remark provided evidence for the difficulty involved in coordinating the notions of domain, restrictions, admissible values, expressions, equations, equation solutions, and so on.

From the perspective of algebra teaching and learning, what emerges from the classroom data of this study, which proved to be a crucial resource in the second phase of documentational genesis of the design team, is that the notion of *domain* is absolutely key. The notion of *domain* is common to different "zones" of work related to algebra: algebraic syntax and the direct manipulation of expressions, equation solving, systems of equations, inequalities, the solving of word problems (with modeling by equations), and lastly the vast zone circumscribed by the study of functions. But to place the question of domain at the center requires considering from the start a wider variety of domains than would be touched upon with the study of polynomials and polynomial quotients, where the question reduces to the study of a finite number of restrictions. This poses a problem for teaching: the elaboration of significant algebra activities that would necessitate considering domains that are more complex than those of polynomial quotients. One could think of activities with roots and radicals (an example is that of Activity 7, which involves equations, roots, and CAS, and which can be found on the project web site; it has been discussed in Kieran et al. 2012, pp. 203-207). Certainly, such activities can involve quite difficult algebraic manipulations for students, but in integrating symbolic calculators (CAS) for some of the manipulation work, various possibilities open up – possibilities for a greater variety of activities that allow students to engage in more sophisticated reflection on the question of domain. Such possibilities, however, remain relatively unexplored by research and few resources specifically dedicated to them are to be found.

8. The Design Researchers' Documentational Genesis: A Retrospective

In this closing section we reflect on our team's design research on equivalence from a documentational genesis perspective. More particularly, if the construct of documentational genesis is usable as a framework for describing and analyzing teachers' professional activity, we set out to inquire whether its application to the professional activity of design researchers offers a fruitful perspective as well.

As in Section 2, the starting point for this retrospective is the relational formula, *Document = Resources + Usages + Operational Invariants*, where *Action Rules* are considered the main component of *Usages*. However, the Operational Invariants (OIs) that were listed in that section were stated in a rather general manner and the Action Rules (ARs) that were sketched related to only one OI. Here we will be more specific as we return to the ARs and OIs for each of the three main documents and the corresponding resources of our design research team's work: (i) the Activity Sequence Document, (ii) the Complementary Theoretical Document, and (iii) the Research Paper that you are actually reading.

For the *Activity Sequence Document*, the design of the student and teacher resources was guided by several ARs and corresponding OIs. Without claiming to be exhaustive, we focus on the following four pairs. As a first OI, we notice that we felt the need to embed our design in a theoretical framework, in this case Chevallard's (1999) Anthropological Theory of Didactics, and the notion of the co-emergence of technique and theory/concept in task-based activity in particular. As a concrete AR, we wanted to include tasks that ask students to compare the results obtained from several different techniques for testing equivalences and have them attempt to draw conceptual conclusions/conjectures from this. As a second OI, one that characterizes the research team as having an agenda on the use of digital technology in mathematics education, we wanted to structure the teaching sequence so as to include the use of digital technology (CAS) as a thinking tool. This was elaborated into an AR of including tasks that involve reconciling the result of a CAS technique with the result of a paper-and-pencil technique, as well as tasks where the use of the CAS tool might engender an unexpected or surprising result that might be capitalized on for deepening reflection on a given concept. A third OI was the wish to create a sequence within a theory of learning where classroom discussion, augmented by teacher input, supports the development of individual knowledge. This was reflected in the AR of including within the teacher guide indications of those moments when it would be advisable to have a whole-class discussion on the topic at hand. A fourth OI was based on the belief that the mathematical underpinnings of school algebra in general, and equivalence in particular, ought to involve both the syntactic and the semantic, and not just the syntactic as is so often the case. This resulted in the AR of trying to reintroduce a semantic perspective by including tasks that call for numerical evaluation of expressions and attention to those values for which the expressions are not defined, as well as the articulation of the numeric/semantic with the syntactic. Overall, operational invariants in this Activity Sequence Document can be seen as global design principles or heuristics, whereas action rules are operationalizations of these principles for the concrete design on equivalence.

As shown in Section 5, the process of learning about equivalence of expressions exposed a certain fragility regarding issues of domain, as well as the way in which the concept of domain actually comes into play when one uses the property of transitivity. The observations also disclosed that the designed resource did not go far enough in supporting these two aspects of the learning of equivalence. This dual insight led us to revisit, as a first measure, our design principle related to the mathematical underpinnings of equivalence and its operationalization into specific action rules – a revisiting that would take into account the gaps in our previous action rules. The decision was made to generate a complementary theoretical resource. This resource would draw attention to those issues of domain and transitivity that had been neglected in the mathematical underpinnings of the activity sequence. This led to the *Complementary Theoretical Document* (see Section 6). A main OI involved in this document was the belief that a deep mathematical underpinning of the topic at stake is indispensable in the design of resources for mathematics education, which was a sharpening of the mathematically-oriented OI of the previous document. As ARs, the text in Section 6 discusses various domain-related definitions for equivalence so as to draw out the importance of considering such issues, and illustrates the manner by which transitivity can be violated in certain definitions of equivalence. The goal was to arrive at a definition of equivalence that is adequate for secondary school algebra and which renders precise both the domain constraints associated with expressions being compared for equivalence and the requirements for transitivity of equivalence. As the resulting text reflects the mathematical thinking by the research team, it also reveals the underlying operational invariant that the design researchers' process is part of the research findings, and that it is important to report about this process candidly.

Besides these two documents, a third document needs to be considered: the *Research Paper* that you are actually reading. For us as authors, writing, reading, discussing, and rewriting this article is not just a matter of publishing results; it is a means to capture our thinking, to summarize our findings, and to reflect on their theoretical impact. As such, it is an important type of documentational work included in the study. Again, different operational invariants can be identified. A first OI is the team's stance that their experience with conjecturing a teaching sequence for the learning of equivalence of expressions and designing specific resources to support that learning should be shared with the mathematics education community. This results in the AR that the article provide the details of the process engaged in by the researchers so as to be shared with and traced by a wider community. A second OI is the vision that theory – in this case the frame of documentational genesis – is important for advancement in our field. As an AR, we are now trying to be explicit about the operational invariants and action rules with respect to the content of each of the resources, as well as the factors involved in their geneses. Third, this article reflects the OI that mathematics and its foundations are important considerations in research on mathematics education. As a corresponding AR, we try to be explicit about the mathematical learning aims of the teaching sequence, the mathematical considerations that guided its design, as well as the way in which this operational invariant was deepened and refined during the later phase of the documentational genesis process.

Altogether, the above retrospective suggests that the frame of documentational genesis can be applied to design researchers' professional practices and that the notions of operational invariants and action rules may offer insights into the – sometimes implicit – design heuristics and research guidelines that play a role. As such, a further exploration of these types of applications is recommended.

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