ORCHESTRATING WHOLE-CLASS DISCUSSIONS IN ALGEBRA WITH THE AID OF CAS TECHNOLOGY

Carolyn Kieran	J. Guzmán	A. Boileau	D. Tanguay	P. Drijvers
UQAM	CINVESTAV-IPN	UQAM	UQAM	Utrecht Univ.
Montreal	Mexico City	Montreal	Montreal	Utrecht

This paper describes the practice of a 10th grade algebra teacher during whole-class discussions of equivalence of algebraic expressions some of which contain restrictions. The transcripts of the whole-class discussions were structured into a series of content-process cycles, with each shift in mathematical content signaling the start of a new cycle. Each cycle is also characterized by a particular role played by the CAS technology. The ways in which the teacher involved his students in discussing the mathematics of the task illustrates how algebra can be problematized with the aid of technology in whole-class discussions and thereby potentially lead to students' richer understanding of the mathematical content. The paper concludes with a synthesis of the strategies the teacher used to make the class discussions work.

INTRODUCTION AND RELATED LITERATURE

Since the advent of reform-based approaches to the teaching of school mathematics (NCTM, 1989), classroom discussion has been considered central to students' mathematical learning. However, teachers have admitted to finding it difficult to encourage discussion in algebra lessons, especially when the content involves the literal-symbolic and associated concepts and techniques (Sherin, 2002). Research interest in teaching practice, and the ways in which classroom discussion might be orchestrated so as to induce greater involvement and deeper mathematical learning on the part of students, is reflected in a few studies that have recently been carried out. These studies have focused on the role of, for example, teacher talk in the classroom (Boaler, 2003), teachers' revoicing of students' ideas within the context of classroom discussion (O'Connor, 2001), and norms that encourage mathematically-productive participation in classroom discussion (Yackel & Cobb, 1996).

Some of these studies have involved technology, in particular graphing technology (e.g., Doerr & Zangor, 2000; Huntley et al., 2000). Goos et al. (2003) noted that, when calculators and computers are permitted to become a part of face-to-face discussions, they facilitate communication and sharing of knowledge. One aspect of technology use that has been found to stimulate mathematical discussion in the classroom is the fact that the technology often surprises with unexpected representations or output (e.g., Hershkowitz & Kieran, 2001). However, none of this research has inquired into the particular combination of teaching practice, whole-class discussion, and CAS technology. In this regard, the research question that motivates the analysis presented in this paper is the following: *What is the nature of teaching practice that builds on the power of Computer Algebra System (CAS)*

In O. Figueras, J.-L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the joint 32nd PME Conference and 30th PME-NA Conference* (Vol. 3, pp. 249-256). Morelia, Mexico: PME et PME-NA.

technology in order to problematize the mathematics that is discussed in the algebra classroom? Problematizing the mathematics means making it open to discussion, that is, creating a mathematical arena in which one poses questions and tries to think deeply about the mathematics, including what might appear to be inconsistencies or contradictions and, in fact, using dilemmas provoked by the technology as a means to move one's thinking forward. This paper describes the practice of a 10^{th} grade algebra teacher during whole-class discussions of equivalence of algebraic expressions some of which contain restrictions. The ways in which he involved his students in discussing the mathematics of the task, and in coordinating this with the related outputs provided by the CAS technology, illustrates how algebra can be problematized with the aid of technology in whole-class discussions and thereby potentially lead to students' richer understanding of the mathematical content.

THE STUDY

This study is part of an ongoing program of research. The previous phase of our research, from which the data for this analysis were drawn, involved six 10^{th} grade classes (15-year-olds), each of which was observed and videotaped over a five-month period. While student learning was the focus of the previous research (see, e.g., Kieran & Drijvers, 2006), it is teaching practice that is the current emphasis. The teacher whose practice is analyzed in this paper is one of the six initial teachers. We decided to start our analyses with this particular teacher because his interactions with the students were always supportive of their thinking; also he was a teacher whose eye remained on the mathematical horizon (Ball, 1993). He, a teacher of mathematics for five years, believed that it was important for students to struggle a little with mathematical tasks. He also encouraged his students to talk about their mathematics in class; he liked to take the time needed to elicit their thinking, rather than quickly give them the answers.

As a technique for structuring and analyzing our data in terms of teaching practice – practice aimed at encouraging whole-class mathematical discussion in CAS-supported algebra classes – we decided to use an approach that we adapted from Sherin (2002): We structured the transcripts of the whole-class discussions into a series of content-process cycles, with each shift in mathematical content signaling the start of a new cycle. Each cycle is also characterized by a particular role played by the CAS agent (Boaler, 2003).

ANALYSIS OF CYCLES OF WHOLE-CLASS DISCUSSIONS

For the previous study, the research team had created several sets of activities that aimed at supporting the co-emergence of technique and theory. One of these sets of activities provides the context for this paper – equivalence of algebraic expressions and the role of restrictions in determining admissible values for the equivalence. At the start of the teaching sequence, numerical evaluation of expressions by use of the CAS served as the entry point. One of the main tasks here was the Numerical Substitution Task (Figure 1), where two numbers to be substituted were given and students were to choose two others. It aimed at students' noticing that some pairs of expressions seemed *always* to end up with equal results. The task was followed by two reflection questions.

The task involved the following definition of equivalence of expressions:

> We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.

The stress on the set of admissible numbers was made deliberately by the designers, so as to lead students to become aware of the attention that one has to pay to considering possible restrictions on the equivalence of expressions. Expression 5 in Figure 1 was a first example of this.

For x =	1/3	-5		
Expression	Result	Result	Result	Result
1. $(x-3)(4x-3)$				
2. $(x^2+x-20)(3x^2+2x-1)$				
3. $(3x-1)(x^2-x-2)(x+5)$				
4. $(-x+3)^2+x(3x-9)$				
5. $\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{x+2}$				

Figure 1. Numerical Substitution Task

The two reflection questions that followed were:

> Question 1B: *Compare the results obtained for the various expressions in the table above. Record what you observe in the box below.*
>
> Question 1C: *Based on your observations with regard to the table above, what do you conjecture would happen if you extended the table to include other values of x?*

After the students had written up their answers to these two questions, the following whole-class discussions ensued.

Cycle 1: Venturing into Equivalence of Algebraic Expressions – CAS as a Calculating Agent

To initiate the discussion, the teacher posed an open question to the entire class as to what they had observed while filling in the table:

> L43. Teacher: So, 1B, "compare the results obtained" (as he reads part of the task question); what results did you obtain? Anyone?

Notice that he started immediately with Q.1B on students' interpretations. His question aimed at uncovering the regularities that the students might have noticed as they filled the table with the values obtained by the CAS substitution operator (Exp | *x*=…). One student responded:

L44. Susan: Expressions 3 and 5 end up having the same answers. So [teacher wrote on the board: #3 = #5].
L45. Teacher: For all of the ones you put in, they ended up having the same answer?
L46. Susan: Yes, and 1 and 4 also.
L47. Teacher: 1 and which one?
L48. Susan: 1 and 4 [teacher wrote on the board: #1 = #4].

We note a particular kind of "notational revoicing" that the teacher has just engaged in: He translated "having the same answers" to the equality #3 = #5. The inference is that, if two algebraic expressions yield the same results when one substitutes a value for x, then they are equal. Clearly, the equality of two algebraic expressions for certain values of x does not imply that the expressions are equivalent. The latter requires consideration of the domain – that is, whether the expressions are equal for all, or almost all, real values versus being equal for only some real values of x. Furthermore, the equivalence of Expressions 3 and 5 is constrained by a restriction.

Up to now, Susan's observations, with which the rest of the class seemed to agree, had centered on the equality of the numerical results that had been obtained. However, at this particular moment, another student wished to add an idea to the discussion – one that brought the talk from a numerical to an algebraic level:

L49. Ken: The expressions are the same thing as the other ones, just in a different form.
L50. Teacher: So you're saying that these pairs of expressions (points to #3 = #5 and #1 = #4) are exactly the same?
L51. Ken: Equivalent representations of the same thing.

This interesting comment on the part of the student immediately led the teacher to assume his whole-class-discussion stance: He sat on the corner of an empty desk near the front left-hand side of the class. This suggested that a discussion would ensue – a discussion that could take some time and some thinking. Thus, he sent a signal to the class to listen to and question what was in the process of being discussed. The teacher then invited the student, Ken, to elaborate further:

L52. Teacher: Equivalent representations? Did anybody not get that? So what are we saying? What did you mean by what you said, Ken?
L53. Ken: They represent the same thing, they give you…like if you substitute in x, like it will come out to the same answer.
L54. Teacher: But why is that the case?
L55. Ken: Because they're just a different form, like they're an unfactored form of a, uh, multiplication of two binomials, or something like that.

The student's difficulties in expressing his mathematical idea in a clear way led to a further question by the teacher, this time directed to the entire class:

L56. Teacher: Does everyone follow what he is saying?
L57. Class: Uh, huh (some students) … no (other students)
L58. Teacher: No?
L59. Linda: I don't understand what he is saying.
L60. Teacher: Then stop me. So another way to talk about it, I guess, Ken, would be that they could each be represented in a common form.

Here the teacher (L60) revoiced Ken's prior response, but then asked the class to explain (L62) what they thought his revoicing meant.

L61. Ken: Yeah
L62. Teacher (to the class): Yes? What do I mean by a common form?
L63. Linda: Simplified?
L64. Teacher: Well, sort of
L65. Linda: Factored
L66. Sara: Expanded …

The teacher next decided to pull together the last few contributions to the discussion, and in the process added a couple of technical points:

L69a. Teacher: So, in order to be in common form you may have to expand, you may have to factor, you may have to do a combination of the two. You may have to stop half way to get a common form.

Résumé of Cycle 1. In this first cycle, the technology played a role in the whole-class discussion, but one behind the scenes – the CAS had permitted the students to rapidly and correctly evaluate the five given algebraic expressions. So, while the CAS was not mentioned explicitly, as agent of calculation it provided the basis for the mathematical discussion that ensued. The content of this cycle focused on the fact that some pairs of expressions, when evaluated numerically, produce the same numerical values. This was linked to algebraic ideas of common form and discourse such as, "equivalent representations of the same thing," which opened up to the issue of restrictions in the next cycle. The orchestration of the whole-class discussion was highlighted right from the start with the teacher's inquiring into the students' thinking regarding the mathematics of the task at hand. He did this by asking for their observations, their elaborations, and their clarifications.

Cycle 2: Refining the Concept of Equivalent Expressions to Include Consideration of Restrictions – CAS as a Provoking Agent

In this cycle, which began immediately after the previous one ended, the teacher wanted to dig more deeply into students' conjectures as to what would happen if they extended the table to include certain values of x. The issue was that Expressions 3 and 5 were equivalent under a restricted domain that excluded -2 from the set of admissible values. Even though an open question related to this issue had been posed above in L45, no student had brought forth the idea of restrictions.

L69b. Teacher: Does anyone not agree with these two statements (i.e., #3 = #5, #1 = #4) for any value that they put in? Is it true for all values? [pause] It's true for all values in both pairs of expressions?
L70. Yannick: It's the exact same equation [he means *expression*]. If you factor it out, they turn out to be exactly the same.

It was not clear (L70) whether Yannick had, in fact, used the CAS to factor Expressions 3 and 5. If he had, he would have observed – as he had said – the same factored (and simplified) expression for both. However, the CAS would not have alerted him to the issue of a restriction for Expression 5 because the CAS we used

(TI-92 Plus) did not display restrictions. Not obtaining any disagreement from the class regarding this issue, the teacher continued with a less open formulation (L75):

L75 Teacher: It would always be the same? So whatever you put in for number 3 will always give you the same as for number 5? [pause] There's no exception to that rule?
L76. Bob: Yeah, there is.
L77. Yannick: It's the exact same equation [i.e., expression]. It's always equal.
L78. Bob: Well I did negative two and it didn't work.
L79. Art: If you put in negative two in the fifth one, then the expression's undefined.

Although Bob and Art had both used the CAS to evaluate for $x = -2$, they had not stated why it did not work. This led the teacher to ask them to justify their claims:

L80. Teacher: Why?
L81. Art: Because it will be divide by zero.
L82. Bob: Ok, because it's a restriction.

But the teacher felt that the students had not yet linked this restriction to the issue of the equivalence of the two expressions. Thus, he encouraged further discussion. The voice of the CAS emerged via Matt who had just tried out the following with CAS:

L86. Matt: If you do what Art said [L79], and instead factor number 5, and then put in negative two as a substitute for x, it will give you the same answer as number 3.

Matt had proposed that they transform Expression 5, by using the CAS factor command, and only then do the substitution of x = -2. A potential conflict had just arisen here: evaluating at x = -2 before, or after simplifying the given expression, yielded two different answers. In the former case, it produced "undefined" and, in the latter, -84. The numerical output of -84 was the same as that obtained when Expression 3 was evaluated at x = -2. So the teacher confronted the class:

L89. Teacher: So, which is right and which is wrong?
L90. Yannick: One just isn't formatted properly.
L91. Teacher: What's the answer if you put -2 in?
L92. Matt: Undefined. Well, -84. That's what it should be.
L93. Linda: What?
L94. Teacher: It *should* be? (with an emphasis on *should*).
L95. Matt: When you factor it and you put in negative two it will give you negative eighty-four as the answer.
L96. Teacher: But are you missing something there?
L97. Matt: The restriction.

However, the class did not quite see yet that they should remove –2 from the set of admissible values for Expressions 3 and 5, as was suggested by the conversation that followed. As the discussion continued to unfold, it became clear that this issue was not going to be easily resolved:

L98. Teacher: What is the restriction, what does it mean?
L99. Matt: x can't equal negative two.
L100. Teacher: What does it mean, why is that a restriction?
L101. Matt: Because you can't divide by zero.
L102. Teacher: So should it be negative eighty-four or should it be undefined?
L103. Matt: Undefined.
L104. Yannick: But if you factor it out?
L105. Teacher: You need to leave the, you need to be aware of that restriction.

The teacher realized that the class was at an impasse with respect to the mathematics at stake and decided to leave aside for the time being the discussion on restrictions. Students were not linking the concept of restrictions with that of equivalence. But, the teacher knew that there would be other tasks coming up that involved new CAS commands and more work on equivalence; thus, he would be able to pursue in a later discussion the relation of restrictions to equivalent expressions.

Résumé of Cycle 2. This cycle began with a shift toward the issue of restrictions, which the teacher orchestrated by returning to, and questioning, an assertion made earlier by one of the students. However, the issue deepened when another student shared his CAS explorations with the class – explorations that had allowed him to "remove the restriction" by factoring and simplifying it away. The class was thus faced with a mathematical dilemma: two different evaluations of the same expression, depending on the sequence preceding its evaluation. The teacher's orchestration included persisting with his initial query, asking students to be more complete in their responses, and even confronting them with the question as to which was right and which was wrong. Eventually, he reminded them that they should not lose sight of the restriction, but realized that they needed more time and additional mathematical activity to adequately think about relating restrictions to equivalence.

DISCUSSION AND CONCLUDING REMARKS

In closing, we revisit the issue of the documented difficulties experienced by teachers (e.g., Sherin, 2002) in generating and maintaining whole-class discussions in literal-symbolic algebra lessons, and the potential of CAS technology to reduce such difficulties. But, first, two caveats. One is a design issue and concerns the tasks. It should be said that the tasks in our study included reflection-type questions that were related to specific output from the CAS and that asked students to think about what these outputs meant. The second touches upon the fit between such tasks and the teacher's view of how best to bring out the mathematics inherent in them. The teacher in our study considered class discussions to be of crucial importance in this regard.

Even if the mathematics in these tasks involved the letter-symbolic – an area known to be difficult for engaging students in whole-class discussion – the teacher made these discussions work. He employed several strategies, such as:

- phrasing open-ended questions that stimulated, queried as to meaning, asked whether there was disagreement, and sought precision or clarification;
- encouraging student ideas, reflection, and discussion, and signaling the latter by a change in his posture that suggested that they were about to engage in some thinking that could take time;
- revoicing students' formulation of ideas; and
- elaborating students' ideas, but only after trying repeatedly to have these elaborations emerge spontaneously from them.

The role that the CAS played was central to the quality of these whole-class discussions in that it was the technology that underpinned both the students' contributions to the discussions and the teacher's inviting of these contributions. As the calculating agent behind the scenes in the first cycle of discussion, the CAS had provided the evaluations that permitted students to talk about their observations and conjectures. As the provoking agent in the second cycle of discussion – an agent whose role had also included being available for students' generating examples and their testing and verifying conjectures – the CAS permitted students to question, within a single discussion, the issues of restrictions, division by zero, and the pseudo-removal of restrictions when an expression is factored and simplified. Even if the questions that had been raised were not all resolved by the end of the discussion, the CAS had clearly played a role not only in adding to the texture of the discussion but also in helping students begin to realize that they had to specify the admissible domain for the equivalence while the expressions were in their original form.

The analysis of the algebra teaching practice that was presented in this paper illustrates how CAS technology can be used as a basis for orchestrating whole-class discussion – discussion that problematizes mathematics with the help of technology.

Acknowledgments

This research was made possible by grants from the Social Sciences and Humanities Research Council of Canada – INE Grant # 501-2002-0132 and Grant # 417-2007-1485, Ministère des Relations Internationales (Québec), and CONACYT (Mexico) – Grant # U49788-H. We express our appreciation as well to the teacher and his class of students who participated.

References

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*, 373-397.

Boaler, J. (2003). Studying and capturing the complexity of practice: The case of the dance of agency. In N.A. Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of PME* (Vol. 1, pp. 3-16). Honolulu, HI: PME.

Doerr, H. M., & Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics, 41*, 143-163.

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2003). Perspectives on technology mediated learning in secondary school mathematics classrooms. *Journal of Math. Behavior, 22*, 73-89.

Hershkowitz, R., & Kieran, C. (2001). Algorithmic and meaningful ways of joining together representatives within the same mathematical activity: An experience with graphing calculators. In M. van Heuvel-Panhuizen (Ed.), *Proc. 25th PME* (V.1, pp. 96-107). Utrecht, The Netherlands.

Huntley, M. A., et al. (2000). Effects of Standards-based mathematics education: A study of the Core-Plus Mathematics Project Algebra and Functions Strand. *Journal for Research in Mathematics Education, 31*, 328-361.

Kieran, C., & Drijvers, P., with Boileau, A., Hitt, F., Tanguay, D., Saldanha, L., & Guzmán, J. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: a study of CAS use in secondary school algebra. *Inter. Jour. of Computers for Mathematical Learning* 11: 205-263.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educ. Studies in Maths., 46*(1-3), 143-185.

Sherin, M. G. (2002). A balancing act: developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education* 5: 205-233.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Jour. for Research Maths. Education, 27*, 458-477.