"HOW CAN WE DESCRIBE THE RELATION BETWEEN THE FACTORED FORM AND THE EXPANDED FORM OF THESE TRINOMIALS? – WE DON'T EVEN KNOW IF OUR PAPER-AND-PENCIL FACTORIZATIONS ARE RIGHT": THE CASE FOR COMPUTER ALGEBRA SYSTEMS (CAS) WITH WEAKER ALGEBRA STUDENTS[1]

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A small comparative study was carried out with two classes of 10th grade students in need of remedial help in algebra – one class being provided with CAS technology and the other class not. Two sets of parallel tasks were designed with the main difference between the two being the use of the CAS tool. Both classes were taught by the same teacher over the course of one month. Results indicate that the CAS class improved much more than the non-CAS class with respect to both technique and theory. The CAS technology played three roles that were instrumental in increasing students' motivation and confidence: generator of exact answers, verifier of students' written work, and instigator of classroom discussion. These findings suggest that the algebra learning of weaker students can benefit greatly from the integration of CAS technology.

PAST RESEARCH IN THIS AREA

While research evidence is beginning to accumulate regarding the positive roles that Computer Algebra Systems (CAS) can play in the learning of school algebra by academically oriented pupils (e.g., Kieran & Drijvers, 2006; Thomas, Monaghan, & Pierce, 2004; Zbiek, 2003), considerably fewer CAS studies have been specifically identified as being carried out with weaker students. Thus, little is known of the benefits of CAS technology for weak algebra students. Even though Heid and Edwards (2001) have proposed that "computer symbolic algebra utilities may encourage weak students to examine algebraic expressions from a more conceptual point of view" (p. 131), they did not refer to specific studies that could support this claim. However, Lagrange (2003) has emphasized from the research his group carried out with precalculus students that easier symbolic manipulation did not automatically enhance student reflection and understanding. In contrast, Jakucyn and Kerr (2002) have pointed out that precalculus students who lacked certain procedural skills could apply their conceptual understanding of the same procedures toward the solving of related problem situations, when provided with CAS technology. Similarly, in a study involving low-ability grade 12 students, who were using CAS in a unit on differentiation, McCrae, Asp, and Kendal (1999) noted that CAS technology led to improved strategy choice for solving calculus problems. In addition, Shaw, Jean, and Peck (1997) found that college students who were enrolled in a

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developmental, CAS-based, intermediate algebra course not only seemed to develop some of the skills that they had not mastered from previous mathematics courses, but also performed better in a follow-up mathematics course than those students who took the traditional intermediate algebra course.

Heid (2002), in a review of arguments against CAS use in the secondary algebra classroom, including the idea that they lead to a loss of by-hand skills, argued for the opposite view, that is, that CAS enhances students' understanding of the symbolic aspects of algebra rather than supplanting such skills. However, as Driver (2001) pointed out, students who are weak in algebra continue to be barred from access to CAS due to concerns that such students may be "unable to benefit from the use of an algebraic calculator or become over-reliant on it and not develop the necessary knowledge and procedures required by the course" (p. 229). Thus, while the evidence is extremely scanty with respect to weaker algebra students, the main issue appears to be whether the use of CAS permits these students to develop a stronger symbol sense than would otherwise be the case in a paper-and-pencil environment – a symbol sense that can in fact lead to improved by-hand skills. To adequately address this issue, a comparative study involving two comparable classes of weak algebra students was designed, one class having access to CAS technology and the other class not. The construction of the tasks and instructional sequences to be used in the study was underpinned by a theoretical framework based on the instrumental approach to tool use: the Task-Technique-Theory framework.

THEORETICAL FRAMEWORK OF THE STUDY

The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). The instrumental approach has been recognized by French mathematics education researchers (e.g., Artigue, 2002; Lagrange, 2002; Guin & Trouche, 2002) as a potentially powerful framework in the context of using CAS in mathematics education. As Monaghan (2005) has pointed out, however, one can distinguish two directions within the instrumental approach. In line with the cognitive ergonomic framework, some researchers (e.g., Trouche, 2000) see the development of schemes as the heart of instrumental genesis. More in line with the anthropological approach, other researchers (e.g., Artigue, 2002; Lagrange, 2002) focus on techniques that students develop while using technological tools and in social interaction. The advantage of this focus is that instrumented techniques are visible and can be observed more easily than mental schemes. Still, it is acknowledged that techniques encompass theoretical notions. In this regard, Lagrange (2003, p. 271) has argued that: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is this epistemic role played by techniques that is essential to understanding our perspective on CAS use, that is, the notion that students' mathematical theorizing develops as their techniques evolve within technological environments. However, the nature of the tasks presented to students – tasks that include a focus on the theoretical while the technical aspects are developing – is crucial. Thus, the triad Task-Technique-Theory served as our framework not only for gathering data during the teaching sequences and for analyzing the resulting data, but also for constructing the tasks and tests of this study.

METHODOLOGICAL ASPECTS OF THE STUDY

Research Questions

The central research questions of this study were the following: Do students who are weak in algebraic technique and theory benefit more from CAS-based instruction in algebra than from comparable non-CAS-based instruction? If so, what are the specific benefits, and what roles does the CAS play that can account for these benefits?

Participants

The participants were two classes of weak Grade 10 algebra students (15 to 16 years of age) who were required by the school to take one month of supplementary algebra classes in May 2005 (50 minutes per day, every 2^{nd} day). The teacher of these two classes (the second author of this report) was enrolled in a master's program at the first author's university and so arranged that her master's research project would involve the students of these two classes. One class had access to CAS technology (TI-92 Plus calculators) during the month-long teaching sequence on algebra and the other class did not.

Task and Test Design

A set of parallel activities was developed for the two classes – focusing mainly on factoring and expanding, an area where these students were particularly weak. Every effort was made to have identical tasks for the two classes, except that where one class would use paper-and-pencil only, the other class would use CAS or a combination of CAS and paper-and-pencil. Some of the task questions were technique-oriented, while others were theory-oriented. Tasks that asked students to interpret their work, whether it was CAS-based or paper-and-pencil-based, aimed to focus students on structural aspects of algebraic expressions and to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom. An example of one of these task activities is presented in the following section on the analysis of student work. Each pupil was provided with activity sheets containing the task questions, where he/she either gave answers to the technical questions or offered interpretations, explanations, and reflections for the theoretically-oriented questions.

In addition to generating two parallel sets of task activities, we also constructed one pretest and one posttest. The questions of these two tests focused primarily on factoring and expanding algebraic expressions, on describing the reasoning involved in carrying out these procedures, on describing the structural features of factored and expanded forms, and on explaining the relation between them. Test questions were divided for purposes of analysis into two types: technical and theoretical; students' tests were scored according to these two dimensions.

Unfolding of the Study

Both classes were administered the paper-and-pencil pretest at the start of the study. There was no significant statistical difference between the pretest scores of the two classes on either the technical or theoretical dimensions. However, the class that had the marginally weaker technical score was the class that was designated the CAS class. Because the students of the CAS class had not had any prior experience with symbol-manipulation technology, a few periods were then spent in initiating them to this technology, in particular to the commands that would be used during the teaching sequence. Each student was provided with a CAS calculator for the duration of the study. The same teacher taught both classes. She had not had any prior experience with using CAS technology in her algebra teaching. She taught both classes in a similar manner: introducing the topic of the day at the blackboard; describing briefly the content of the given worksheet; circulating and answering questions while students engaged with the tasks of the worksheets; and bringing all the students together during the last 15 minutes of class in order to discuss the material that they had been working on during that period. Students in the CAS class were sometimes encouraged to use the view-screen to present their work during the discussion period. At the close of the month-long instructional sequence, both classes wrote the paper-and-pencil posttest, which was an alternate version of the pretest. Neither class had access to CAS technology for the writing of the posttest.

Data Sources

The data sources, which permitted a combination of qualitative and quantitative analyses, included: (a) all the task worksheets of each student from the two classes; (b) the pretest and posttest of each student; (c) the daily summaries in the teacher's logbook, which she entered at the close of each class; here she kept track of the discussions that had occurred, and also recorded individual students' comments, concerns, difficulties, high and low points of the classroom activities, and any other items worthy of note.

ANALYSIS OF STUDENT ACTIVITY AND WRITTEN WORK

Analysis of Pretest and Posttest

An analysis of the pretest and posttest scores of the two classes of students was first carried out (see Table 1).

	Pretest Technique	Posttest Technique	Pretest Theory	Posttest Theory
CAS class	74.9%	91.2%	19%	39%
non-CAS class	75.9%	85.6%	15.2%	23.8%

Table 1: Mean percentage scores for the technical and theoretical components of the pretest and posttest by the CAS and non-CAS classes.

The wide discrepancy in the pretest scores between the technical and theoretical components is attributable to the fact that neither class had had experience with theoretically-oriented questions in their algebra classes prior to the unfolding of this study. (Furthermore, while the pretest-technique scores may appear to be quite strong, they were considered weak in a school where mastery learning was the goal.) In any case, it is clear that the posttest improvement in the CAS class on the Theory dimension was considerably greater than was the case for the non-CAS class. With respect to the Technique dimension, again both classes improved as a result of the teaching sequence that occurred between pretest and posttest, but the CAS class improved more. While this was a small study involving only two classes of students, the results of this first analysis indicate that the CAS class benefited more from the remedial instructional sequence than did the non-CAS class (see Damboise, 2006, for a detailed analysis of student responses to the two tests). To try to find explanations that could account for the greater improvement in the CAS class, we then analyzed the teacher's logbook entries and students' worksheets.

Analysis of Teacher's Logbook Entries and Students' Worksheets

The analysis of the entries in the teacher's logbook led to several conjectures regarding the mechanisms at play in the CAS class – mechanisms that could account for the superior performance of the CAS class on the posttest. These conjectures were supported by the analysis of students' technical and theoretical responses to the worksheet questions. In brief, the technology was found to play several roles in the CAS class: it provoked discussion; it generated exact answers that could be scrutinized for structure and form; it helped students to verify their conjectures, as well as their paper-and-pencil responses; it motivated the checking of answers; and it created a sense of confidence and thus led to increased interest in algebraic activity. As space constraints do not permit the presenting of data to support each of these results, we will confine ourselves to what we believe is one of our most important findings with regard to the role that CAS can play in helping weaker algebra students.

The CAS generates exact answers that can be scrutinized for structure and form. Of all the roles that the CAS played in this study, this was found to be the most crucial to the success of the weaker algebra student. It proved to be the main mechanism underlying the evolution in the CAS students' algebraic thinking. Ironically, the crucial nature of this role was first made apparent to us by the voicing of a frustration by one of the students in the non-CAS class – a frustration that we will share shortly. First, we present the CAS version (see Figure 1), then the non-CAS version (Fig. 2) of the task that led to this finding.

Activity 3 (CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Use the calculator in completing the table below.

Given trinomial (in "dissected" form)	Factored form using FACTOR	Expanded form using EXPAND
(a) $x^2 + (3+4)x + 3 \bullet 4$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why did the calculator not factor the trinomial expressions of 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 1: A task drawn from Activity 3 (CAS version).

Activity 3 (non-CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Complete the table below by following the example at the beginning of the table.

Given trinomial (in "dissected" form)	Factored form	Expanded form
Example: $x^2 + (3+4)x + 3 \bullet 4$	$x^2 + (3+4)x + 3 \bullet 4$ $= x^2 + 3x + 4x + 3 \bullet 4$ $= x(x+3) + 4(x+3)$ $= (x+3)(x+4)$	$x^2 + 7x + 12$
(a) $x^2 + (5+6)x + 5 \bullet 6$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why could you not factor the trinomial expressions in 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

Figure 2: The non-CAS version of the same task that was presented in Figure 1.

Note that, in the CAS version of Question 1, students are asked to enter onto their worksheet the output produced by the CAS, while in the non-CAS version they are to record their paper-and-pencil factorizations and expansions. (N.B.: The "dissected" form of the first column was one that both classes were quite familiar with by the time they met this Activity.) The problematic nature of this task, and the potential of the CAS for assisting with such tasks, showed up when the students in the non-CAS class tried to tackle Questions 2c and 2d.

Students in the non-CAS class were at a loss to answer these explanation-oriented questions. They stated emphatically that they were not sure of their answers to Question 1, and could hardly use these as a basis for answering, say, Question 2d. As one student put it so forcefully: *"How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our factorizations and expansions from Question 1 are right."* In contrast, the students in the CAS class had at their disposal a set of factored and expanded expressions that had been generated by the calculator. They thus had confidence in these responses and could begin to examine them for elements related to structure and form.

CONCLUDING REMARKS

This study analyzed the improvements of two classes of weak algebra students in both *technique* (being able to do) and *theory* (i.e., being able to explain why and to note some structural aspects), in the context of tasks that invited technical and theoretical development. One of the two participating classes had access to CAS technology for the study. At the outset, both the CAS class and the non-CAS class scored at the same levels in a pretest that included technical and theoretical components. However, the CAS class improved more than the non-CAS class on both components, but especially on the theoretical component.

This is an interesting finding for several reasons. Many teachers insist that students learn to do algebraic work with paper-and-pencil first and only later use CAS – and then simply to verify the paper-and-pencil work. However, we found that the students' paper-and-pencil technical work actually benefited from the interaction with CAS. The CAS provided insights that transferred to their paper-and-pencil algebraic work and enhanced their learning. Secondly, and this is quite an exciting finding: Being able to generate exact answers with the CAS allowed students to examine their CAS work and to see patterns among answers that they were sure were correct. This kind of assurance, which led the CAS students to theorize, was found to be lacking in the uniquely paper-and-pencil environment where students made few theoretical observations. The theoretical observations made by CAS students worked hand-in-hand with improving their technical ability. Last but not least, the CAS increased students' confidence in their algebra. This confidence boosted their interest and motivation. These findings suggest that the algebra learning of weaker students can benefit greatly from the integration of CAS technology.

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REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245-274.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*, 221-266.

Damboise, C. (2006). *Rôle d'un logiciel de manipulation symbolique dans l'apprentissage de l'algèbre au secondaire* (Master of Mathematics thesis). Université du Québec à Montréal.

Driver, D. (2001). The trial of algebraic calculators in senior mathematics by distance education. In H. Chick et al. (Eds.), *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra* (Vol. 1, pp. 228-237). Melbourne, Australia.

Guin, D. & Trouche, L. (Eds.). (2002). *Calculatrices symboliques. Transformer un outil en un instrument du travail mathématique: un problème didactique.* Grenoble: Pensée Sauvage.

Heid, M. K. (2002). Computer algebra systems in secondary mathematics classes: the time to act is now! *Mathematics Teacher, 95*(9), 662-667.

Heid, M. K., & Edwards, M. T. (2001). Computer algebra systems: Revolution or retrofit for today's mathematics classrooms? *Theory into practice, 40*(2), 128-136.

Jakucyn, N., & Kerr, K. E. (2002). Getting started with a CAS: Our story. *Mathematics Teacher, 95*(8), 628-633.

Kieran, C., & Drijvers, P., with Boileau, A., Hitt, F., Tanguay, D., Saldanha, L., & Guzman, J. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning, 11,* 205-263.

Lagrange, J.-B. (2002). Étudier les mathématiques avec les calculatrices symboliques. Quelle place pour les techniques? In D. Guin & L. Trouche (Eds.), *Calculatrices symboliques* (pp. 151-185). Grenoble, France: La Pensée Sauvage.

Lagrange, J.-B. (2003). Learning techniques and concepts using CAS: A practical and theoretical reflection. In J. T. Fey et al. (Eds.), *Computer Algebra Systems in Secondary School Mathematics Education* (pp. 269-283). Reston, VA: NCTM.

McCrae, B., Asp, G., & Kendal, M. (August, 1999). *Teaching calculus with CAS.* Paper presented at 4th International Conf. on Technology in Math. Teaching. Plymouth, England.

Monaghan, J. (2005). *Computer algebra, instrumentation and the anthropological approach.* Paper presented at the 4th CAME conference, October 2005. Retrieved on Dec. 20, 2005 from http://www.lonklab.ac.uk/came/events/CAME4/index.html

Shaw, N., Jean, B., & Peck, R. (1997). A statistical analysis on the effectiveness of using a computer algebra system in a developmental algebra course. *Journal of Mathematical Behavior, 16,* 175-180.

Thomas, M. O. J., Monaghan, J., & Pierce, R. (2004). Computer algebra systems and algebra: Curriculum, assessment, teaching, and learning. In K. Stacey et al. (Eds.), *The future of the teaching and learning of algebra* (pp. 155-186). Dordrecht, The Netherlands: Kluwer.

Trouche, L. (2000). La parabole du gaucher et de la casserole à bec verseur: étude des processus d'apprentissage dans un environnement de calculatrices symboliques. *Educational Studies in Mathematics, 41*, 239–264.

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*, 77-103.

Zbiek, R. M. (2003). Using research to influence teaching and learning with computer algebra systems. In J. T. Fey et al. (Eds.), *Computer algebra systems in secondary school mathematics education.* (pp. 197-216). Reston, VA: NCTM.