ATTENDING TO STRUCTURE AND FORM IN ALGEBRA: CHALLENGES IN DESIGNING CAS-CENTERED INSTRUCTION THAT SUPPORTS CONSTRUING PATTERNS AND RELATIONSHIPS AMONG ALGEBRAIC EXPRESSIONS (1)

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We report on a study with adult algebra learners that employed an activity to explore patterns and relationships between expressions that might suggest the sum/difference of cubes identity. Our report describes the activity and discusses aspects of expressions to which students attended. Our results to point to a need for refinements to this activity that might better support re-invention of the identity.

Although it has been shown that technology can be used to support student understanding of calculus (Heid, 1988), there continues to be debate about the role that technology can and should play in algebra learning (NCTM, 1999). We report on a study with adult learners that employed an activity involving a computer algebra system (CAS) to explore patterns and relationships between expressions that might suggest the sum/difference of cubes identity $(a + b)(a^2 – ab + b^2) = a^3 + b^3$. This activity was inspired by the work of Goldenberg (2003), who outlined a similar activity designed to support students in re-inventing the difference of squares identity $(a – b)(a + b) = a^2 – b^2$.

In an effort to understand the sources of their difficulties in deriving the sum/difference of cubes identity, our report explores aspects of expressions to which students attended. We detail the instructional activity and its outcomes, and we suggest refinements to the activity that might better support this population of students in achieving the intended learning goals.

Setting and Participants

Our study was conducted in the context of an intact intermediate-level algebra course offered at a large urban university in the Pacific northwestern United States. The course is designed for students who have not had Algebra II or who require a review of elementary algebra concepts and techniques. The students in this class have a wide variety of dispositions and experiences related to mathematics, a variety of educational backgrounds and interests, and a wide spectrum of ages. The 15 students who participated in our study reflect this diversity.

We initiated our study during the third week of classes, after students had developed some experience in multiplying polynomial expressions and factoring perfect square trinomials and differences of squares. The data we report on are drawn from students' individual written work on the first two parts of the activity. We examined their work for evidence of patterns they might have construed from the sum/difference of cubes expressions generated by the CAS and for connections they drew between those expressions and their factored forms.

Lamberg, T., & Wiest, L. R. (Eds.). (2007). Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Summary Description of the Task

The Cubes activity was designed to support students in deriving the identities for factoring sums and differences of cubes. The activity began by having students explore patterns arising from the multiplication of a given sequence of a binomial and a trinomial whose product results in a sum/difference of cubes. This was followed by having students use those patterns to generate the sum/difference of cubes identity. Specifically, students were presented with factored forms of five sums and differences of cubes, each of which they were asked to multiply out using the EXPAND command on their TI-92 calculator. Students recorded the results of these expansions in the appropriate cells of a table (Figure 1), and they were directed to investigate whether the "indicated multiplication of factors produces interesting results".

Factored form	Expanded form displayed by the calculator
1. $(x + 2)(x^2 – 2x + 4)$	

Figure 1. A portion of a table that students completed in the first part of the activity.

The intent in having students use the CAS to expand the products in this part of the activity, rather than requiring them to do so by hand, was to focus their attention on the form of each final expression, and to reduce the potential of diverting their attention to the mechanics of algebraic manipulations. The next part of the activity aimed to orient students' attention towards patterns in the expanded expressions recorded in their table. To that end, the students were directed to take note of the form of each expanded result produced by the CAS and to "describe how this form relates to that of the corresponding factors".

Analyses and Results

Our analysis of the data unfolded in a sequence of interrelated phases. Following Saldanha & Kieran (2005), we began by listing relationships of form and structure among expressions to which the students might be expected to attend. We then examined students' responses for evidence of explicit or implicit attention to these aspects. On the basis of this first examination we then revised and refined our initial list of aspects to arrive at a comprehensive list of them. The authors then each re-examined the data independently, coding it according to the dimensions in this list and documenting the frequency of their occurrence in the students' responses across task questions. In a final phase, the authors compared their code assignments and resolved their few differences through a process of negotiation that involved re-examining relevant parts of the data whenever necessary. This process converged to a 100% agreement in the coding of the data.

The table below displays a sampling of our results; it describes some of the aspects of the expressions to which we hoped the students would attend and it gives the frequency with which students actually attended to each of them. The aspects are listed in descending order, from the most to the least salient for students, as indicated by our analysis.

Aspect of expressions	**Students who attended to aspect** #	%
Expanded version of each expression is shorter than the factored version	12	80%
The first term of the expanded version is the product of the first term in the binomial and the first term in the trinomial $(\boldsymbol{a} + b)(\boldsymbol{a^2} - ab + b^2) = \boldsymbol{a^3} + b^3$	10	67%
The first term of the expanded version is a cube $(a + b)(a^2 - ab + b^2) = \boldsymbol{a^3} + b^3$	8	53%
The second term in the expanded version of the expression is a cube $(a + b)(a^2 - ab + b^2) = a^3 + \boldsymbol{b^3}$	2	13%
The last term of the trinomial is the square of the last term in the binomial $(a + \boldsymbol{b})(a^2 - ab + \boldsymbol{b^2}) = a^3 + b^3$	2	13%
The middle term of the trinomial is the product of the two terms in the binomial (sign withstanding) $(\boldsymbol{a} + \boldsymbol{b})(a^2 - \boldsymbol{ab} + b^2) = a^3 + b^3$	1	7%
The first term of the trinomial is the square of the first term in the binomial $(\boldsymbol{a} + b)(\boldsymbol{a^2} - ab + b^2) = a^3 + b^3$	0	0%

Discussion and Conclusion

The results suggest that, in general, the students did not attend to most of the aspects of the expressions that we deemed important. Due to space constraints, we only discuss a few of the aspects and results in this paper. For instance, of the twelve aspects we targeted, only four were noted by more than 50% of the students. Only one aspect—that the expanded forms of the expressions are shorter than their factored forms—was noted by more than 75% of the students. It is important to note that this modal aspect is arguably the least useful one in terms of helping students generate the intended identity. Moreover, the characteristics apparently noted by the fewest students centered on relationships between the terms in the factored form of the expressions, which are arguably the most useful for generating the identity.

These findings suggest that the expanded versions of the expressions were much more salient for students than were the factors themselves or relationships among the factors. This state of affairs is consistent with, and might therefore be attributed to, students' being inattentive to how the expanded forms of the expressions arose from the factored forms. This hypothesis, in turn, raises questions about whether our activity prompts adequately provoked students to reflect on *why* the expanded forms of the expressions are shorter than their corresponding factored forms. Indeed, since the students did not attend to aspects that could form the basis of a derivation of the sum/difference of cubes identity, our activity sequence stands to benefit from prompts that orient students to reflect on why only certain terms appear in the expanded form of the product of expressions. Changes such as these might better support this population of students in attending to crucial aspects of these expressions and structural relationships among them.

Endnote

1. The research reported in this paper was funded by the College of Arts and Sciences of Portland State University.

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