# TAKING ADVANTAGE OF THE 'SURPRISE' FACTOR OF CAS RESPONSES TO SPUR THE GROWTH OF ALGEBRAIC UNDERSTANDING


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# MY APPRECIATION TO RESEARCH COLLEAGUES AND COLLABORATORS:

- Fellow members of the research team: André Boileau, Denis Tanguay, Fernando Hitt, José Guzmán, Luis Saldanha, Armando Solares, Ana Isabel Sacristán
- Our project consultants: Michèle Artigue, Paul Drijvers, & Luc Trouche
- All the participating students and teachers
- And the funding agencies: SSHRC, MRI

# WHAT IS MEANT BY THE 'SURPRISE' FACTOR?

- Any CAS response that is unexpected
  - The unexpected can occur in
    - Exploration within CAS-based work,
    - Verification of paper-and-pencil work,
    - Testing of conjectures, …
- What does a surprising CAS response call for?
  - Making sense of the CAS response,
  - Further testing,
  - Seeking consistency with paper-and-pencil mathematics, …

# WHAT IS THE 'SURPRISE' FACTOR?

- Any CAS response that is not expected by students
  - The unexpected can occur when students are
    - Exploring a new mathematical idea,
    - Verifying paper-and-pencil work,
    - Testing conjectures, …
- Of course, all surprises that a CAS may yield are relative:
    - Relative to students' past mathematical experience and knowledge,
    - Relative to their ability to notice something unexpected in the CAS response, …

- What does a surprising CAS response call for?
  - Trying to make mathematical sense of it,
  - Trying to fit it with one's existing mathematical ideas,
  - Seeking consistency with one's paper-and-pencil techniques,
  - Testing new conjectures arising from the surprise, …

- The traditional paper-and-pencil medium cannot, by its very nature, yield surprises in algebra.
- The paper-and-pencil work that algebra students produce springs from their intentions and from their existing knowledge. Thus, there is no surprise agent.
- Here, CAS technology has something unique to offer.

Artigue (2002) has argued that CAS tasks can capitalize on “the ***surprise*** effect that can occur when one obtains results that do not conform to expectations and that can destabilize erroneous conceptions” (p. 344, my translation)

# STUDENTS' ATTENDING TO CAS SURPRISES: A POWERFUL MOTOR FOR LEARNING!

- But how to create a classroom environment that can capitalize on the 'surprise' factor that CAS can yield?
- At the very least, our research suggests that it requires:
  - Appropriate tasks,
  - Adequate time for students to think about their work,
  - Both students and teacher contributing to the mathematical talk related to the tasks.

# TODAY'S PRESENTATION:

- Two examples of tasks that were designed to create elements of surprise –
  - One where CAS is a tool for exploring patterns,
  - Another where CAS is a tool for verifying

  – but both inviting technical and theoretical learning in algebra.
- Video extracts of a class of Grade 10 students and their teacher, both contributing in their own ways to the mathematical discussions that unfolded during each task activity.

Activity 6

# FACTORING THE $x^n - 1$ FAMILY OF POLYNOMIALS

**Name:** **Date:**

## Activity 6: Factoring

### Part I (Paper & pencil, and CAS): Seeing patterns in factors

1. (a) Before using your calculator, try to recall the factorization of each algebraic expression listed in the left column of this table:

| Factorization using paper and pencil | Verification using FACTOR (show result displayed by the CAS) |
|---|---|
| $a^2 - b^2 =$<br>$a^3 - b^3 =$<br>$x^2 - 1 =$<br>$x^3 - 1 =$ | |

**Classroom discussion of Part I, 1a**

1. (b) Perform the indicated operations (using paper and pencil)

| |
|---|
| $(x-1)(x+1) =$ |

| |
|---|
| $(x-1)(x^2+x+1) =$ |

2. (a) Without doing any algebraic manipulation, anticipate the result of the following product:

| |
|---|
| $(x-1)(x^3+x^2+x+1) =$ |

2. (b) Verify the anticipated result above using paper and pencil (in the box below), and then using the calculator.

| |
|---|
| |

2. (c) What do the following three expressions have in common? And, also, how do they differ?

$$(x-1)(x+1),\ (x-1)(x^2+x+1) \text{ and } (x-1)(x^3+x^2+x+1).$$

2. (d) How do you explain the fact that the following products result in a binomial: two binomials, a binomial with a trinomial, and a binomial with a quadrinomial?

**Classroom discussion following Question 2d**

2. (e) On the basis of the expressions we have found so far, predict a factorization of the expression $x^5-1$.

**Part II: Toward a generalization (activity with paper & pencil and with calculator)**

II(A) 1. In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down.

If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
|---|---|---|
| $x^2 - 1 =$ | | |
| $x^3 - 1 =$ | | |
| $x^4 - 1 =$ | | |
| $x^5 - 1 =$ | | |
| $x^6 - 1 =$ | | |

II.(A).2. Conjecture, in general, for what numbers $n$ will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain:

| |
|---|
| |

**Classroom discussion of Part II A**

# THE FIRST SURPRISE CAME WHEN THE CAS PRODUCED A DIFFERENT FACTORIZATION FOR $X^4 - 1$ THAN WAS BEING INDUCED BY THE GENERAL RULE THAT WAS JUST LEARNED: $X^N - 1 = (X - 1)(X^{N-1} + X^{N-2} + \ldots + 1)$

| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
|---|---|---|
| $x^2 - 1 = (x-1)(x+1)$ | $(x-1)(x+1)$ | N/A |
| $x^3 - 1 = (x-1)(x^2+x+1)$ | $(x-1)(x^2+x+1)$ | N/A |
| $x^4 - 1 = (x-1)(x^3+x^2+x+1)$ | $(x-1)(x+1)(x^2+1)$ | $(x-1)(x+1)(x^2+1)$<br>$(x-1)(x^3+x^2+x+1)$ |

# SO WHAT DID STUDENTS LEARN FROM THIS FIRST CAS SURPRISE REGARDING THE FACTORIZATION OF $X^4 - 1$?

- The factors produced for a given algebraic expression may not be unique – different approaches may yield different factors;

- While an expression may be correctly factored, that factorization may not be complete.

# THE NEXT SURPRISE WAS RELATED TO THE "CONJECTURING" TASK, WHICH WAS PRESENTED AFTER THE "RECONCILING" WORK INVOLVING $X^2 - 1$ THROUGH TO $X^6 - 1$

II.(A).2. Conjecture, in general, for what numbers $n$ will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain:

i, all values & 2

ii) even values

iii) even values

| | |
|---|---|
| Chris | 'Two factors' means two separate sets of brackets, right? |
| Peter | Yeah. |
| Chris | The only time it contains two factors is when it is odd, I think, which means it can be, [pause] like, our pattern can't be broken down anymore. 'Cause it always ends up being all positive. And uh, then, because, it's sort of hard to explain. |
| Peter | When the exponent is [pause], when the exponent is an even number you'll have more than two factors, but when the exponent is not an even number, you'll have exactly two factors all the time. |
| Chris | Yeah. [Types Factor $(x^7 - 1)$ into the CAS] |
| | Yeah, because any time you plug in an odd number as the exponent power, it's uh, the calculator always stays at the most simplified [pause] and [Types in Factor $(x^9 - 1)$; the CAS displays: $(x-1)(x^2+x+1)(x^6+x^3+1)$] |
| | And, no!!! [a look of utter surprise on Chris's face] |

II.(A).2. Conjecture, in general, for what numbers $n$ will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain:

i.) when the exponent is an odd number that is not divisible by three with a few restrictions,

ii.) when the exponent is an even number.

iii.) when the exponent is an even number.

# AND THE 'EUREKA' MOMENT!

Chris: But, I think as soon as you get past nine or whatever, you start running into problems…
Peter: Try sixty; sixty is divisible by a lot [Chris types on calculator]
[Silence…]
Chris: Yeah, I think it has to do with how many numbers can go into it.
Interviewer: How many numbers can go into it?
Chris: Like, sixty is divisible by one, it's divisible by two, it's divisible by three, it's divisible by four, five, six.
Peter: By four, five, six.
Chris: Not seven, [pause] not eight.
Peter: Not nine, ten, twelve.
Chris: But, it's just, like uh, [pause] at a certain [pause], **prime numbers**? [pause] So, a prime number is twenty-three [he types into the calculator] **Yeah, prime numbers, that's it**. Prime numbers when it is…
Interviewer: and what are prime numbers?
Peter: Wait, what about three, five and seven.
Chris: Only divisible by itself. Three, five and seven, all work.
Peter: They are prime numbers.
Chris: Yeah, they all work.
Peter: No, but they don't give you exactly two factors.
Chris: Yeah, they do. [Types in calculator] That's what I'm doing [pause] three, five, seven
Peter: Yeah they do [Looks at screen]
Chris: Yeah, prime factors. And nine doesn't work because it is not a prime factor. [Peter crosses out the answer that he had written and writes: all prime numbers]

# WHICH LED TO A FINAL CHANGE IN THEIR SERIES OF MODIFIED CONJECTURES

II.(B).2. On the basis of patterns you observe in the table II.B above, revise (if necessary) your conjecture from Part A. That is, for what numbers $n$ will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

ii) include $(x+1)$ as a factor?

Please explain:

i.) ~~odd numbers (for the exponent)~~ that is ~~not divisible by three and five, seven~~ all prime numbers

ii.). composite numbers.

iii). even numbers.

# WHAT DID THESE STUDENTS LEARN FROM THE SURPRISE THEY EXPERIENCED WHEN THE CAS FACTORED $X^9 - 1$ INTO MORE THAN TWO FACTORS?

- Eventually, after much conjecture-testing:
  - that $x^n - 1$ will factor completely into exactly two factors when $n$ is prime;
  - when $n$ is a composite number, the number of factors for $x^n - 1$ will be more than two.

- We should note, once again, that what students learn from a given CAS surprise will be relative.

- It will depend on several factors, including the extent to which the task pursues a given issue, as well as the mathematical experience of the pupils.

  - For example, one possible task extension of the conjecture question could aim at developing the concept that, when completely factored under the set of integers, the number of factors of $x^n - 1$, when $n$ is composite, is equal to the number of positive divisors of $n$.

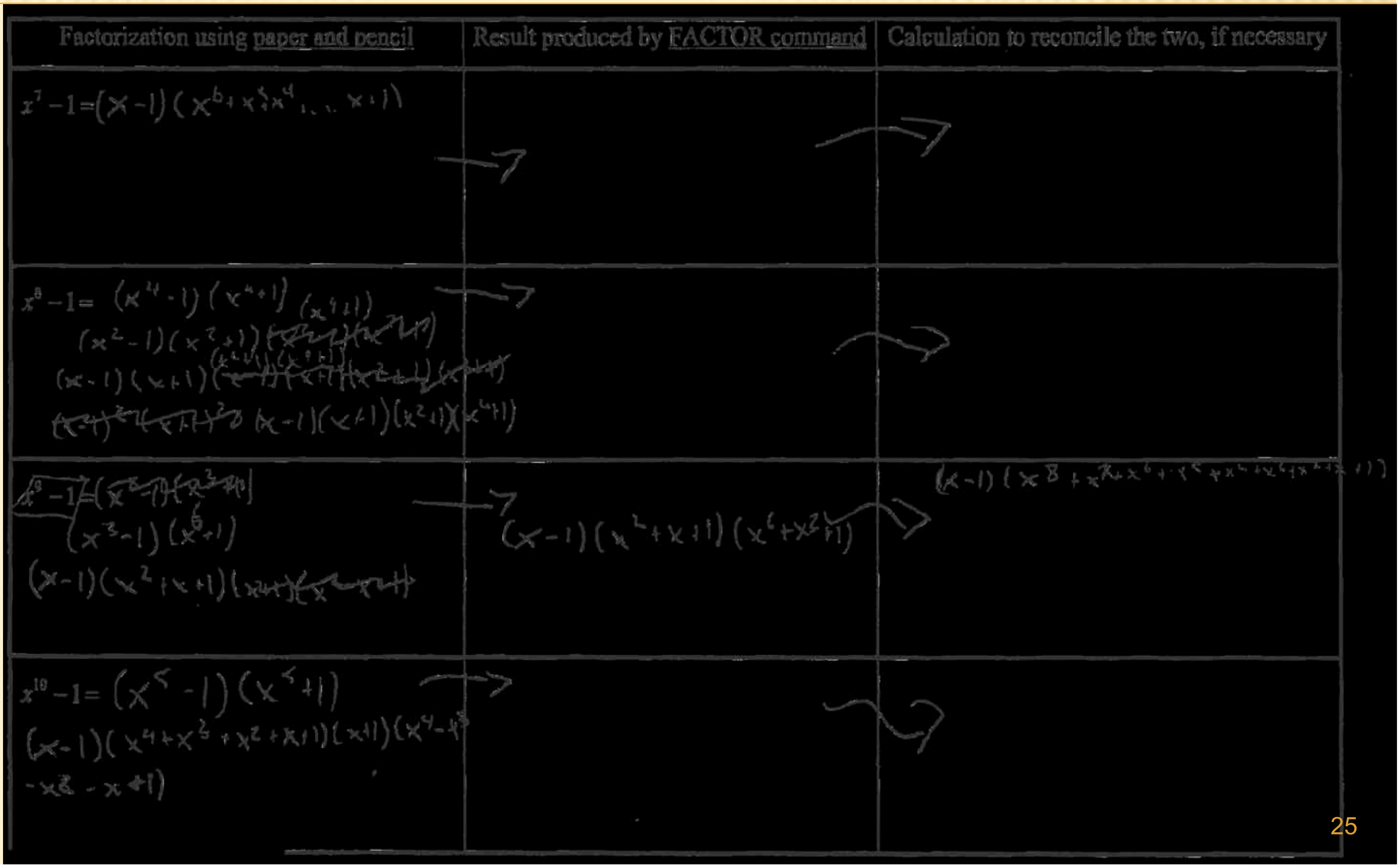
# HOWEVER, THE $X^9 - 1$ FACTORING SURPRISE LED TO MORE THAN REVISING THE "EXACTLY-TWO-FACTORS" CONJECTURE:

- When students in class saw the factors that were produced by the CAS for $x^9 - 1$:

  $(x - 1)(x^2 + x + 1)(x^6 + x^3 + 1)$,

  they wanted to know how they could produce such factors themselves with paper and pencil.

# THIS STUDENT'S WORK ON $X^9 - 1$ HAD LEFT HIM QUITE DISSATISFIED



| Factorization using paper and pencil | Result produced by FACTOR command | Calculation to reconcile the two, if necessary |
|---|---|---|
| $x^7-1=(x-1)(x^6+x^5+x^4 \ldots x+1)$ → | | → |
| $x^8-1=(x^4-1)(x^4+1)$<br>$(x^2-1)(x^2+1)$<br>$(x-1)(x+1)$<br>$(x-1)(x+1)(x^2+1)(x^4+1)$ → | | → |
| $x^9-1=$<br>$(x^3-1)(x^6+1)$<br>$(x-1)(x^2+x+1)$ → | $(x-1)(x^2+x+1)(x^6+x^3+1)$ → | $(x-1)(x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1)$ |
| $x^{10}-1=(x^5-1)(x^5+1)$<br>$(x-1)(x^4+x^3+x^2+x+1)(x+1)(x^4-x^3$<br>$-x^3-x+1)$ → | | → |

# HE FINALLY REQUESTED HELP FROM THE TEACHER.

- He asked: "How do you get those factors?"

- The teacher then suggested to the class that they might try to "see" $x^9$ as $(x^3)^3$ and thus $x^9 - 1$ as $((x^3)^3 - 1)$, which could then be treated as a difference of cubes – which they had already learned how to factor.

# SO WHAT WERE THE STUDENTS IN THIS CLASS BEGINNING TO LEARN FROM THIS CAS SURPRISE REGARDING THE FACTORIZATION OF $X^9 – 1$?

- They were learning to see within composite exponents *n* a way of re-expressing them, at the level of particular cases, so that the expression $x^n – 1$ might be interpreted as $(x^p)^q – 1$, and factored according to a method that they were already familiar with,

  e.g., $x^9 – 1$ as $(x^3)^3 – 1$, $x^{15} – 1$ as $(x^5)^3 - 1$ , $x^8 – 1$ as $(x^4)^2 – 1$, …

# ANOTHER SURPRISE WITHIN THE SAME ACTIVITY: THE CAS FACTORIZATION OF $X^{10}$ - 1

- In the Reconciliation Task, all students initially produced the following paper-and-pencil factorization for $x^{10} - 1$, with the aim of generating a complete factorization:

  $x^{10} - 1$

  $= (x^5 + 1)(x^5 - 1)$

  $= (x^5 + 1)(x - 1)(x^4 + x^3 + x^2 + x + 1)$

## SURPRISE! SURPRISE!
## THE CAS THEN PRODUCED THE FACTORS:
## $(X - 1)(X + 1)(X^4 + X^3 + X^2 + X + 1)(X^4 - X^3 + X^2 - X + 1)$

- This intrigued one of the groups of students.
- They wondered about the pattern in the factors for $x^5 + 1$, with its alternating signs in the "long" factor.
- They noticed that it was the same pattern as for the factoring of $x^3 + 1$, with its alternating signs.
- Then they began to conjecture a general factorization for the expression $x^n + 1$:

  $(x + 1)(x^{n-1} - x^{n-2} + x^{n-3} - \ldots - x + 1)$ and when this might work.

But at that moment, the class moved into the proving segment of the activity. We will hear from this group shortly.

# THE PROVING SEGMENT OF THE ACTIVITY:

- The last task of the activity was: "Prove that ($x$ + 1) is always a factor of $x^n - 1$ for even values of $n$, $n \geq 2$."
- After working individually, or in groups, on this task for about 10 minutes, one student volunteered to present his "proof" at the board:

- Paul: "Ok. So, my theory is that *whenever $x^n$-1 has an even value for n*, if it's greater or equal to 2, that, *one of the factors of that would be $x^2$-1*, and since $x^2$-1 is always a factor of one of those, *a factor of $x^2$-1 is (x+1), so then (x+1) is always a factor*."

- Much animated discussion followed the presenting of this 'proof' (see Kieran & Guzman, 2010).

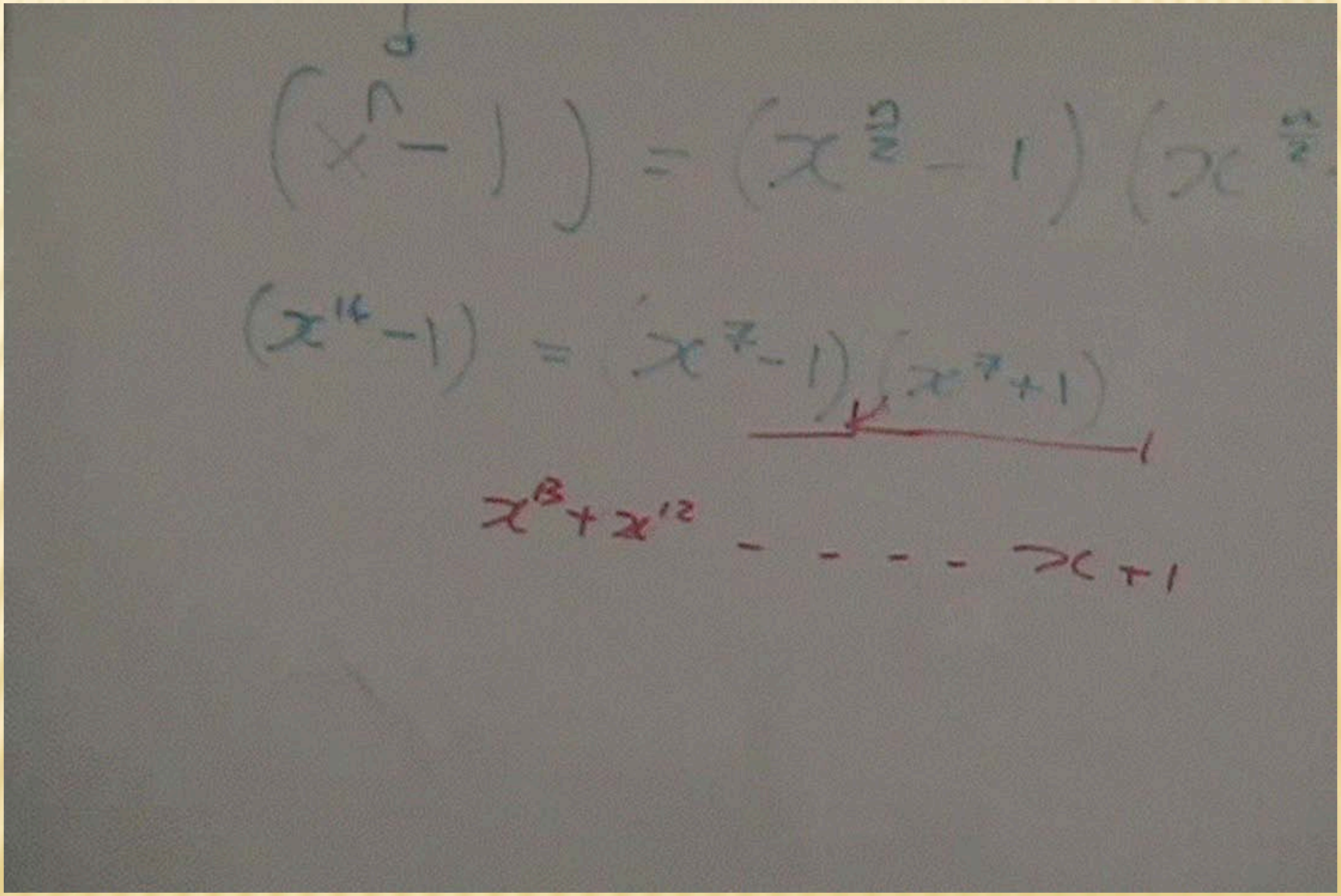
- To provoke the students,

  the teacher offered the following counterexample: "Just out of interest, what would happen if this was $x^{14} – 1$? [he wrote $(x^{14} – 1)$ under Paul's $(x^n – 1)$], to which students answered: "$(x^7-1)$ times $(x^7+1)$."
- So the teacher wrote at the board:

  $(x^{14} – 1) = (x^7-1)(x^7+1)$ and then asked:

  "Where does that leave your proof, Paul?"
- However, rather than leaving the class stymied, this question provided an opening for the group that had been conjecturing something new:

# THE GROUP'S NEW CONJECTURE ON $X^N + 1$:



# SO WHAT DID ANDREW AND HIS GROUP LEARN FROM THIS CAS SURPRISE REGARDING THE FACTORIZATION OF $X^{10} – 1$?

- Because they noticed the way in which the $x^5 + 1$ factor of $x^{10} – 1$ had been refactored by the CAS, and linked this with their previous factoring of the sum of cubes, $x^3 + 1$, they were able to generate a new conjecture regarding a general rule for factoring $x^n + 1$, for odd *n*s:

  $x^n + 1 = (x + 1)(x^{n-1} - x^{n-2} + x^{n-3} - \ldots - x + 1)$, for *n* odd

- This allowed them to explain, at least partially, why for the given proving task – including the counterexample of $x^{14} – 1 = (x^7 - 1)(x^7 + 1)$, which had been put forward regarding Paul's 'proof' [i.e., that $(x + 1)$ is always a factor of $x^n – 1$ for even values of $n$, $n \geq 2$]:
  - $x^n-1=(x^{n/2}-1)(x^{n/2}+1)$ for even *n*s;
  - so if $n/2$ is odd, then $(x^{n/2}+1)$ will have $(x+1)$ as one of its factors.

- The way in which the surprise factoring of $x^{10} - 1$ led to this novel conjecture by Andrew and his group was made possible by a classroom environment that encouraged such mathematical exploration and that used CAS tools.
- However this kind of generative work by students in response to CAS surprises can be easily thwarted by well-intentioned teachers who prepare too much of the terrain for students and end up giving away the punch-line.

# THIS OCCURRED IN ANOTHER CLASS WITH THE SAME $X^{10} – 1$ FACTORING TASK

- While students were still working on the Reconciliation Task, the teacher rapidly wrote $(x^{10} – 1) = (x^5 – 1)( x^5 + 1)$ on the board.

  He then blurted out: "The one that may give you some trouble is the $x$ to the 10*th*. I will explain why."
- He proceeded to explain at the board the factorization $x^5+1 = (x+1)( x^4–x^3+x^2–x+1)$, with much hand-waving regarding the signs, and did not let the students notice this pattern for themselves.

Activity 7

# USING FACTORING TO SOLVE EQUATIONS THAT CONTAIN EXPRESSIONS WITH RADICALS

**Name:**
**Date:**

## Activity 7: Factoring and Solving Equations Involving Expressions with Radicals

*Note to student*: The primary objective of this activity is that you come to view and employ factoring (taking out a common factor) as a tool for solving equations, particularly when used in conjunction with the "zero product theorem."

1. Suppose you were asked to solve this equation:

$$5\left(\sqrt{x-4}\right)^3 + 11\sqrt{x-4} = (2x+1)\sqrt{x-4} \quad (*)$$

a) How would you proceed when faced with such a "monster"? (Don't actually solve the equation, just state your general approach.)

1.b) Using paper and pencil, see whether you can first solve the following equation that is somewhat analogous to the above monster:

$$(y-2)^3 - 10(y-2) = y(y-2) \qquad (**)$$

*Hint*: Factoring (taking out a common factor) might be useful here.

1.c) Compare your solution to equation (**) with that obtained using the calculator's SOLVE command. If the solutions obtained are different, verify your paper and pencil algebraic work. If the calculator produced an additional solution to the ones you found, determine which among the paper and pencil algebraic manipulations you used led to the loss of this additional solution. Please show all your work in the space below.

**Classroom discussion of Questions 1a, b, & c**

# ISSUE: RETHINKING THE PROPERTY OF CARRYING OUT THE SAME OPERATION ON BOTH SIDES OF AN EQUATION

1.b) Using paper and pencil, see whether you can first solve the following equation that is somewhat analogous to the above monster:

$$(y-2)^{3} - 10(y-2) = y(y-2) \qquad (**)$$

*Hint*: Factoring (taking out a common factor) might be useful here.

$$(y-2)^2 - 10 = y$$
$$(y-2)(y-2) - 10 = y$$
$$y^2 - 4y + 4 - 10 = y$$
$$y^2 - 4y - 6 = y$$
$$y^2 - 5y - 6 = 0$$
$$(y-6)(y+1) = 0$$
$$y_1 = 6$$
$$y_2 = -1$$

# WHAT DID THE SURPRISING CAS RESPONSE CONSISTING OF THREE SOLUTIONS PROVOKE WITHIN STUDENTS?

- It had made sense to students to divide both sides by the common factor of ($y$ – 2).
- The teacher's focusing on the issue of "not dividing by zero" – rather than just on the zero-product property – exposed some limitations in students' use of a particular equation-solving technique that they had never before questioned.
- As Mikey commented: "Then we can never divide when there's a variable because it's always going to be a solution."

# TRYING TO MAKE SENSE OF THE CAS RESPONSE; TRYING TO FIT IT WITH THEIR EXISTING MATHEMATICAL IDEAS

- So this CAS surprise led to students' questioning of when the rule of carrying out the same operation on both sides of an equation is valid.
- The students had never before had occasion to question this rule – one that had always yielded equivalent equations and the sought-for equation solutions.

# CONCLUDING REMARKS

# IN ANALYZING THE DATA OF OUR STUDY, WE PAID CAREFUL ATTENTION TO THE DEVELOPMENT OF BOTH TECHNIQUE AND THEORY IN STUDENTS

- We used as a framework the Task-Technique-Theory model developed by researchers in France in the late 1990s and early 2000s (Chevallard, 1999; Artigue, 2002; Lagrange, 2002, 2003).

- Since the mid-1990s, in France, when CAS technology started to make its appearance in secondary school mathematics classes, these researchers noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of the technical work in algebra learning.

- However, this emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection.

- From their observations, the research team of Artigue and her collaborators came to think of techniques as a link between tasks and conceptual reflection, in other words, that the learning of techniques was vital to related conceptual thinking.

- Thus, in the examples I have used today to illustrate the nature of CAS surprises that led to student learning, all of the conceptual thought that they generated was in the context of technical work:

  - Factoring $x^4 - 1$ more completely,
  - Factoring several odd-exponent examples so as to establish when $x^n - 1$ will have exactly two factors,
  - Learning to view a composite exponent in terms of its divisors so as to "see" more easily how it might be factored [e.g., the case of $x^9 - 1$ being viewed as $(x^3)^3 - 1$],
  - Working with common factors (without losing a solution), performing the same operation on both sides, and using the zero-product rule to solve the equation: $(y-2)^3 - 10(y-2) = y(y-2)$

# THE NATURE OF THE LEARNING THAT OCCURS

- As we have reported in some of our past publications (e.g., Kieran & Drijvers, 2006), technique and theory co-emerge in mutual interaction. With appropriate tasks and a suitable classroom environment, technical work can give rise to theoretical thinking; and the other way around, theoretical reflections lead students to develop and use techniques.

- In another study, reported in Kieran and Damboise (2007), which was a comparative study of a CAS class and non-CAS class involving the same tasks in both classes, the CAS class improved much more than the non-CAS class in both technique and theory, but especially in theory; and the sequence of lessons was one where the technical component was clearly to the forefront.

# BUT THE USE OF TASKS THAT LEAD TO CAS SURPRISES MAY NOT BE ENOUGH!

- Our research observations suggest that an additional component is a teaching practice that is oriented toward assisting students' in becoming aware of the conceptual aspects of their technical work in algebra within a CAS environment:

  Orchestrating classroom discussion in such a way as to draw out students' thinking regarding the mathematics of the task at hand, by asking for their conjectures, their observations, their elaborations, and their justifications.

# IN CLOSING, I WOULD LIKE TO QUOTE ATTILA KOVÁCS, A HUNGARIAN COMPUTER SCIENTIST

- *"I see the main role of symbolic algebra systems as that of helping to formulate hypotheses, search for examples and counterexamples, and in general explore ramifications of mathematical models.*
- *In other words, the main role of these systems is to obtain mathematical insight."* (Kovács, 1999, p. 43)

# KOVÁCS:

- *"Computer algebra systems often can cause 'pleasant surprises'. … Due to computer algebra systems, problems are being considered from new points of view. … These lead us to new discoveries about the world of miraculous mathematical structures."* (p. 52)

- Even if Kovács's comments were directed to an audience of computer scientists, they still have a great deal of relevance for our thinking about CAS use with high school algebra students.

*Don't you agree?*

# THANK YOU

- Website for access to the activities we have designed:

http://www.math.uqam.ca/apte/indexA.html

**References, and additional publications by our research group**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