Making Model Theory Coordinate Free

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Goal: to express adequately the idea that a structure M gives rise to an entire mathematical world wM.

Example An algebraically closed field K gives rise to "algebraic geometry over K" (very rich, but well "below Gödel")

Structures will be many-sorted, and starting with M we obtain wM in several steps:

$$M \longrightarrow cM \longrightarrow CM \longrightarrow wM$$

M denotes an S-sorted structure, S = set of sorts,
so M is given by a family $(M_s)_{s\in S}$ of sets,
and for each $\vec{s} = (s_1, \dots, s_n) \in S^n$ we are given a
boolean algebra $\mathcal{D}_{\vec{s}}$ of subsets of $M_{\vec{s}} := M_{s_1} \times \dots \times M_{s_n}$,
the algebra of <u>definable</u> subsets of $M_{\vec{s}}$. This system
of boolean algebras must be closed under:

- diagonals
- cartesian products: $A \in \mathcal{D}_{\vec{s}}$, $B \in \mathcal{D}_{\vec{t}} \Rightarrow A \times B \in \mathcal{D}_{\vec{s}\vec{t}}$
- taking <u>images</u> under projection maps

M gives rise to a <u>small</u> category cM:
— its <u>objects</u> are the definable sets, always labeled as elements of a specific $\mathcal{D}_{\vec{s}}$
— its <u>morphisms</u> are the definable maps between these.

cM has some nice category-theoretic properties, especially, it has (finite) products, given by cartesian products. But it may not have finite sums, or quotients.

We want to be able to glue definable sets together:

Def. A finite M-atlas on a set X is a finite set $\{f_i : U_i \to f_i(U_i) \mid i \in I\}$ of bijections between sets $U_i \subseteq X$ and $f_i(U_i) \subseteq M_{\vec{s}_i}$ such that $X = \bigcup_i U_i$ and for all $i, j \in I$ $f_i(U_i \cap U_j) \subseteq M_{\vec{s}_i}$ is definable, and the transition map $f_{ij} = f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ is definable.

Two finite M-atlases on X are equivalent if their union is a finite M-atlas on X.

Def A finitary M-set is a set X equipped with an equivalence class of finite M-atlases on X.

Let us fix a finitary M-set X, given by a finite M-atlas $\{f_i : U_i \to f_i(U_i) \mid i \in I\}$ as above.

1. Call a set $E \subseteq X$ <u>definable</u> if each $f_i(U_i \cap E) \subseteq M_{\vec{s}_i}$ is definable. (Then E itself is finitary M-set with atlas $\{f_i|_{U_i \cap E} : i \in I\}$)

2. Given a second finitary M-set Y with atlas $\{g_j : V_j \to g_j(V_j) | j \in J\}$ we make $X \times Y$ into a finitary M-set by taking as finite M-atlas $\{f_i \times g_j : U_i \times V_j \to f_i(U_i) \times g_j(V_j) | (i,j) \in I \times J\}$.

3. A map $F : X \longrightarrow Y$ is a <u>finitary M-map</u> if its graph $\Gamma(F) \subseteq X \times Y$ is a definable subset of the finitary M-set $X \times Y$.

4. $CM :=$ category with <u>objects</u> the finitary M-sets, and as <u>morphisms</u> the finitary M-maps.

5. CM contains cM as a <u>full</u> subcategory, by viewing each definable set $X \subseteq M_{\vec{s}}$ as a finitary M-set with $\{id_X : X \to X\}$ as finite M-atlas.

6. CM has products and sums

7. If M has EI (elimination of imaginaries) then each object of CM is isomorphic to an object of cM, so CM and cM are <u>equivalent</u> categories

8. Any family $N = (N_t)_{t \in T}$ of finitary M-sets "is" a T-sorted structure in an obvious way

<u>Locally definable objects</u>

CM is still too small for many purposes, for example in algebraic geometry we have besides varieties also <u>large</u> auxiliary objects associated to them: their coordinate rings, sheaves, groups of cycles. While these objects are <u>too</u> large to live in CM, they are typically functorial in nature, and can be used to introduce invariants such as (co)homology groups that are "definable", i.e. in CM.

We extend CM to a ~~category~~ category in which these "large" objects live.

<u>Def</u>. An <u>M-atlas on a set</u> X is defined like a <u>finite</u> M-atlas on X, by leaving out "finite" in the definition.

Two M-atlases $\{f_i : U_i \to f_i(U_i) \mid i \in I\}$ and $\{g_j : V_j \to g_j(V_j) \mid j \in J\}$ on X are <u>equivalent</u> if their union is an M-atlas on X, each U_i is covered by finitely many V_j, and each V_j is covered by finitely many U_i.

<u>Def</u>. An <u>M-set</u> is a set X equipped with an equivalence class of M-atlases.

<u>Example</u> Suppose M is a field.
Then $M[X_1, \dots, X_n] = \bigcup_{d \in \mathbb{N}} U_d$
with $U_d := \{P \in M[X_1, \dots, X_d] : \text{degree}(P) \leq d\}$.
For each d, let $\lambda_d : U_d \to M^{N(d)}$ be an M-linear bijection.
Then $\{\lambda_d : U_d \to M^{N(d)} \mid d \in \mathbb{N}\}$ is an M-atlas on
[illegible] $M[X, \dots, X]$ into an M-set

Fix M-set X given by M-atlas $\{f_i : U_i \to f_i(U_i) \mid i \in I\}$

$$f_i(U_i) \subseteq M_{\vec{s}_i}$$

Call a set $E \subseteq X$ <u>locally definable</u> if each set $f_i(E \cap U_i) \subseteq M_{\vec{s}_i}$ is definable.

If Y is a second M-set given by M-atlas $\{g_j : V_j \to g_j V_j \mid j \in J\}$, then we make $X \times Y$ into an M-set by taking $\{f_i \times g_j : U_i \times V_j \to f_i U_i \times g_j V_j \mid {}^{i \in I}_{j \in J}\}$ as M-atlas.

3. Given an M-set Y, a map $F : X \to Y$ is an <u>M-map</u> if its graph $\Gamma(F) \subseteq X \times Y$ is locally definable <u>and</u> each $F(U_i)$ is covered by finitely many V_j.

4. <u>Def</u> wM is the category whose objects are the M-sets and whose morphisms are the M-maps.

5. Each finitary M-set is an M-set in a natural way. CM is a <u>full</u> subcategory of wM

An M-*group* is a group in the category wM, that is, an M-set G with a distinguished element $e \in G$ and endowed with M-maps $p : G \times G \to G$ and $i : G \to G$ such that (G, e, p, i) is a group.

Similarly one defines M-rings, and M-fields (for which "multiplicative inverse" is required to be an M-map).

Examples Suppose M is a field. Then $M[X_1, \cdots, X_n]$ is naturally an M-ring.

The algebraic closure $\tilde{M}$ of M is an M-field: for each finite degree subextension $L \supseteq M$ of $\tilde{M} \supseteq M$ we take an M-linear bijection $L \simeq M^d$ as a chart for $\tilde{M}$.

Suppose M has EI

Then wM has quotients:

If X is an M-set and E an equivalence relation on X, locally definable in $X \times X$, then we can make the quotient set X/E into an M-set such that the projection map $X \longrightarrow X/E$ is an M-map, and each M-map $X \longrightarrow Y$ that is constant on E-classes induces an M-map $X/E \longrightarrow Y$.

For example, if R is an M-ring and I a two-sided locally definable ideal in R, then the equivalence relation $x \equiv y \bmod I$ is locally definable, and the residue ring R/I is easily seen to be an M-ring

Remark If M is a field, then each ideal I of $M[X_1, \ldots, X_n]$ is locally definable.

Suppose M has EI

Then for any finitary M-sets X and Y, the set $[X \to Y]$ of finitary M-maps with parameters $X \to Y$ can be made into an M-set such that

$$(F, x) \mapsto F(x) : [X \to Y] \times X \longrightarrow Y$$

is an M-map.

Also, for any finitary M-sets A, X and Y, any finitary M-map $A \times X \to Y$ induces an M-map $A \longrightarrow [X \to Y]$.

In the special case $Y = \{0,1\}$ we obtain that for any finitary M-set X the collection of subsets of X that are definable with parameters is an M-set in a natural way.