

PAUL DRIJVERS

## STUDENTS ENCOUNTERING OBSTACLES USING A CAS \*

**ABSTRACT.** The paper describes a pilot study on the use of computer algebra at upper secondary level. A symbolic calculator was introduced in a pre-examination class studying for advanced pre-university mathematics. With the theoretical framework of Realistic Mathematics Education and Developmental Research as a background, the study focused on the identification of obstacles that students encountered while using computer algebra. Five obstacles were identified that have both a technical and a mathematical character. It is the author's belief that taking these barriers seriously is important in developing useful pedagogical strategies.

**KEY WORDS:** computer algebra, optimization, Realistic Mathematics Education, student behaviour, symbolic calculators

## INTRODUCTION

During the last decade the availability of computer algebra environments has increased dramatically, not least because of the development of hand-held symbolic calculators to which students now have access. A further diffusion of Computer Algebra Systems (CAS) is to be expected.

Educational researchers and teachers are concerned with fundamental questions that arise as soon as computer algebra is integrated into the teaching and learning of mathematics. How can the use of a CAS improve conceptual understanding? How might a CAS affect the curriculum? What is the role of paper-and-pencil skills in a computer algebra environment? What prerequisite knowledge and skills are required for students to benefit from the availability of computer algebra? A more elaborated list of such questions can be found in Drijvers (1997).

Many studies have been undertaken to answer (parts of) these questions, as indicated by the overview provided by Mayes (1997). In the Netherlands, the Freudenthal Institute carried out a pilot study on the role of the symbolic calculator in 1998. This project was a natural follow-up of research on the integration of graphing calculators in mathematics

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education (Drijvers and Doorman, 1997). This paper presents some of the findings of the study, whose main result is the identification of obstacles that students encounter while working with a CAS.

This paper first summarizes some findings from previous research that are relevant to the present study. Then, the relation between the theory of Realistic Mathematics Education and computer algebra is discussed. The research question, the methodology and the instructional units are briefly presented. The longest section of the paper contains exemplary classroom observations that will lead to the identification of obstacles that students encountered while working with the symbolic calculator. A discussion and a conclusion complete the article.

### PREVIOUS RESEARCH

In this section some studies that are of particular interest for the scope of this paper are briefly presented.

For some years the white-box/black-box issue dominated the discussion about the pedagogy of computer algebra. Buchberger (1989) suggested that students should use computer algebra only for tasks that they are able to readily perform by hand. It is not until a new subject has been mastered manually that computer algebra can be used to carry out the (now trivial) work that has to be done. Computer algebra, then, is used as a black box, that can be opened by the students, if they would like to do so, because they know 'what's behind' it. This white-box/black-box sequence, however, can be inverted. Proponents of the black-box/white-box approach use a computer algebra system as a generator of examples and as an exploratory tool that may elicit curiosity and can lead to interesting discoveries (see Drijvers, 1995, for an overview of this discussion).

As one of the first studies in this domain, Heid (1988) showed how a CAS can be used to facilitate the development of mathematical concepts. Subjects of the experiment were first year university students enrolled in a calculus course. The experimental condition consisted of the use of a CAS to build up the concept of the derivative by means of using graphs, combining representations etc. Techniques of differentiation were not taught until the end of the course. The students showed a good conceptual insight of the derivative and performed not worse than the control group on the technical part of the post-test. The results of this 'concept-first' course indicate that the development of concepts can precede the learning of techniques. Heid suggests that the use of a CAS might provoke a resequencing of concepts and skills in mathematics courses.

As soon as plotting devices such as graphing calculators entered education, it became clear that students may have difficulties in interpreting graphical representations as they appear on a screen (e.g. see Goldenberg, 1987; Hillel et al., 1992). In the same line but more generally, Guin and Trouche (1999) point out the confusion that students can experience when they are not able to distinguish between a mathematical object (not restricted to graphs) and its calculator representations. They provide a detailed description of the process of instrumentation – the so-called ‘instrumental genesis’ – of a computer algebra tool that students need to go through. The growing awareness of the constraints and the potentials of the CAS tool are important parts of this process.

Related to the work of Guin and Trouche are the studies of Lagrange (1999a, 1999b). Using the terminology of Verillon and Rabardel, Lagrange stresses the importance of instrument utilisation schemes that are built up by both individual and social genesis. He gives a detailed example of such a scheme that students develop for finding the variation of a function. He suggests that the technical aspect of doing mathematics does not disappear in a technology environment: paper-and-pencil techniques may lose importance but machine techniques are becoming more important instead.

The research reviewed above serves as a frame of reference for the present study, and will be referred to when appropriate.

## REALISTIC MATHEMATICS EDUCATION AND COMPUTER ALGEBRA

The domain specific theory of Realistic Mathematics Education (RME) forms the theoretical background of this study. This instruction theory has acquired considerable impact in The Netherlands during recent decades. According to this theory, mathematics is considered as a human activity (Freudenthal, 1991). Van Reeuwijk (1995) provides the following characteristics of Realistic Mathematics Education: ‘real’ world, free productions and constructions, mathematization, interaction and integrated learning strands. These points may need some explanation.

- *‘Real’ world*

Learning of mathematics starts from problem situations that students perceive as real or realistic. These can be real life contexts, but they can also arise from mathematical situations that are meaningful and natural to the students. The word ‘real’ thus refers to ‘experientially real’ rather than to ‘real world’. The didactical phenomenology of the

topic provides adequate contexts that serve as a start of the learning process.

- *Free productions and constructions*  
Students should have the opportunity to develop their own informal problem solving strategies that can lead to the construction of solution procedures. The models that they develop will gradually turn into generic models for a class of situations. This ‘bottom-up’ reinvention process is guided by the teacher and the instructional materials. The concept of guided reinvention is essential in RME.
- *Mathematization*  
Organising phenomena by means of progressive mathematization is important in the learning of mathematics. Usually two types of mathematization are distinguished: horizontal mathematization which refers to modelling the problem situation into mathematics and vice versa, and vertical mathematization, which refers to the process of reaching a higher level of mathematical abstraction.
- *Interaction*  
Interaction among students and between students and the teacher is important in RME, because discussion and co-operation enhance the reflection that is essential for the mathematization process.
- *Integrated learning strands*  
In the philosophy of RME, different mathematical topics should be integrated in one curriculum. The student should develop an integrated view of mathematics, as well as the flexibility to connect the different sub-domains. Note that in real life, phenomena are integrated in a context as well.

This list shows that the theory of RME is more than just saying ‘use real life contexts in mathematics education’. Although a ‘real’ life context can be an important starting point for mathematization, the main points here are the processes of reinvention and mathematization. An extensive discussion of the theory of realistic mathematics education can be found in Freudenthal (1991), De Lange (1987) and Treffers (1987).

Now what are the implications of these general principles of realistic mathematics education for the use of a computer algebra device in class? On the one hand, one might hope that the availability of a CAS could help the ideas of RME become a reality in the classroom. The theory of RME on the other hand can, *a priori*, point out some risks in using a CAS in the learning process. This twofold relationship between RME and CAS needs a closer look.

The idea of technology as a catalyst for the realization of RME is not new. Drijvers and Doorman (1997) described the potential use of the

graphics calculator for more realistic contexts, for concentrating on the process of mathematization, for exploration, integration, flexibility and dynamics. Based on the theory of RME and on classroom experience, I extrapolated these ideas to the computer algebra environment. This led to the following three conjectures on the possible benefits of using computer algebra to realise the goals of RME.

#### *Horizontal mathematization*

Using a computer algebra device enables the students' attention to shift from purely algorithmic operations to the translation of realistic problems into mathematical models and to the interpretation of the results with respect to the context. Freeing the student from technical work may open the way for this two-directional horizontal mathematization. This holds to a further extent with CASs than with other technological tools because of the algebraic facilities that enable solution strategies to match more closely to what students are used to.

#### *Exploration*

Because of its direct feedback, the computer algebra tool offers opportunities for exploratory activities. Discovery and classification tasks can lead to findings, which then, through reflection and generalization, result in the reinvention of properties or theorems. The CAS can evoke vertical mathematization in a way similar to other technological devices that can serve as investigation instruments but, again, the algebra inside the system provides new and possibly powerful opportunities to include also algebraic techniques.

#### *Flexible integration of different representations*

Using a CAS enables the student to switch easily between mathematical representations such as graphs, tables and formulae. This can lead to a more integrated and flexible use of these representations that will be perceived as different but related faces of the same die. The sophisticated way of representing and editing formulas are not exclusive to a CAS, but are often weak points in other technological tools.

Now the opposite perspective: what possible conflicts between the theory of RME and the phenomenon of computer algebra can be expected to arise? A priori, the most important dangers seem to be threefold.

#### *Top-down tool*

Because a CAS contains so much mathematical expertise, a risk of using it for an educational purpose is that results are obtained in a top-down

manner. Everything ‘is already there’, is already invented. This may frustrate a student’s motivation for construction and reinvention, unless adequate didactical measures are taken.

#### *Black box*

Usually, a CAS does not give insight into the way its results are obtained. The software is a black box that does not show the methods it uses. These methods are often, even in simple problems, far more sophisticated than the methods students would use themselves. This may become manifest in unexpected outcomes or representations of results. Clearly, the CAS does not support elementary or informal strategies.

#### *Idiosyncrasy*

As a tool for the apprentice user, a CAS is not very flexible. Input requires strict syntax and output can be presented in unfamiliar ways. The ‘CAS-language’ is different from mathematical and natural language, and the system does not allow informal language. Each CAS has its rules, constraints and habits. Thus, students may perceive a CAS as an idiosyncratic tool instead of a flexible and natural instrument that can deal with their own informal notations and strategies. This makes the ‘instrumental genesis’ (Guin and Trouche, 1999; Lagrange, 1999a) a difficult process.

## RESEARCH QUESTION

The integration of computer algebra in a RME educational setting seems to have its complications. The black-box aspect, already mentioned by Buchberger, may be in conflict with the reinvention principle. Furthermore, the problems with the instrumentation process, as reported by Lagrange and Guin and Trouche, are factors that have to be taken into account. On the other hand, studies such as Heid’s research give reason to hope that there are some benefits to using a CAS.

This paper takes the ‘negative’ perspective and concentrates on the obstacles of using a CAS in a RME setting. This does not mean that I do not believe in the possible benefits of CAS use. Rather, it is necessary to be conscious of the difficulties. Recognition of students’ obstacles is a first step to finding ways to deal with them, and can lead to the identification of prerequisite knowledge and skills for meaningful use of computer algebra. Identifying obstacles is useful in determining to what extent the dangers really exist. The central research question, therefore, is:

What obstacles do students experience while working with computer algebra?

First, what is considered to be an obstacle? In the classroom experiment that is reported below, students perform horizontal mathematization and shift between 'real' life situations and mathematical translations. The CAS can do the procedural mathematical work. This requires going through an instrumentation process. An obstacle, now, is a barrier provided by the CAS that prevents the student from carrying out the utilisation scheme that s/he has in mind. As a result, the obstacle stops the process of shifting between the 'pure' mathematics and the problem situation. Such obstacles can be technical, but they often have a mathematical or conceptual component. Language too can be involved: changing vocabulary or notation can be (part of) an obstacle.

### METHODOLOGY AND DESIGN

As a research paradigm, the developmental research method was used (see Gravemeijer, 1994). This methodology has similar characteristics to the theory of RME: In interaction with the 'real-life' classroom situation, the researcher tries to 'reinvent' the theory by means of constructing and developing thought experiments and educational experiments. This involves a cyclic process of consideration and testing, an alternation of thought experiment and educational experiment. Developmental research design typically makes use of qualitative, close-to-the-students, observations. Another characteristic of developmental research is that students' erroneous behaviour is seen as a source for further development of the theory and the educational experiment. The latter characteristic is well suited to the research question on obstacles.

The educational experiment in this study took place in a pre-examination class studying advanced pre-university level mathematics. The class consisted of 22 students – 8 female, 14 male – all about 17 years old. The computer algebra platform they used was the symbolic calculator TI-92. The main reason for that was the practical advantage of not having to go to a computer lab, where PC's usually dominate the educational setting. The fact that the students already owned a TI-83 graphing calculator for more than a year was a factor in making this choice: it was believed that the similarities between the interfaces of the two machines would facilitate the students' introduction to the TI-92.

The students received a TI-92 for a four week period. There were four 50-minute mathematics lessons each week. During these lessons, students

worked in pairs for a significant part of the time. Bearing in mind the relevance of interaction in the theory of RME, the partners were stimulated to work together and to communicate on what each of them was doing with their 'personal' machine. Every lesson, one pair used a TI-92 that was connected to a viewscreen in order to have the screen video-taped. These pairs were alternated throughout the experiment. A side-effect of this was that the other students could also see what "today's victims" were doing. During classroom discussions – also very important in the light of interaction and reflection! – the teacher often asked this pair to do the calculations, so that all students could see it. This is what Guin and Trouche recommend as the 'Sherpa-student' role (Guin and Trouche, 1999). The teacher himself did not use the machine or the viewscreen during the lessons.

Data was gathered by means of participating classroom observations and interviews. Complementary to this qualitative data were the results of pre- and post-tests and a questionnaire. Qualitative data were interpreted and scored by means of classification in different categories, with reliability checked by the three observers.

#### DEVELOPING INSTRUCTIONAL UNITS

Two instructional units were developed for the classroom trial: 'Introduction to the TI-92' and 'Optimization using a symbolic calculator'. The purpose of the first unit was learning how to perform the most important calculations on the TI-92. In the meantime, some problems focused on specific aspects that one encounters when working with a CAS, such as numerical versus exact calculations, equivalence of expressions, rewriting expressions, substitution and finding general solutions. An investigation task formed the end point of the unit. This reflects the RME-idea that students need to have room for exploration and for construction in order to build up their own theory. The investigation task resulted in written reports that were presented to the class.

The second unit cannot be understood without some comments on the prerequisite knowledge the students had when entering the experiment. At the start of the school year, the students had worked through a unit called 'Sum and difference, distance and speed' (Kindt, 1997). This unit is about the principles of differentiation and integration that are developed simultaneously. The concept of the derivative is introduced using the rate of change: the context of speed in a time-distance graph gradually develops into a more generic model for the concept of the derivative. In a similar manner, the integral is introduced as the distance travelled in a time-speed graph. After that, the students learned how the derivative can be used to



find extreme values of functions. The only functions they could differentiate manually, however, were power functions. No derivatives of rational or trigonometric functions, nor any rules for differentiation were in the students' repertoire yet.

Several ideas from the theory of RME guided the development of the unit 'Optimization using a symbolic calculator', a revision of an existing unit 'Optimization using a graphing calculator'. A central concept is the modelling of '*Real life*' situations into optimization problems. This involves horizontal *mathematization*. A second concept is the relation between the extreme value of a function and the zeros of the derivative. Because the symbolic calculator does the technical part of the work, the student can concentrate on this concept and on the *construction* of a problem solving strategy. Optimization problems often can be solved in various ways: numerically, with graphs, with algebra/calculus and with geometry. By means of mixing up all these methods, the unit aims at *integration* of the approaches and increasing the flexibility of the student. Technology (i.e. hand held computer algebra) can support flexibility in problem solving methods, because it takes over a great part of the manipulative work.

The concept of the derivative as a 'rate of change' has been taught to the students, but they do not yet know how to apply the rules for differentiation. They are forced to leave the derivation of the functions that model the optimization problems to the symbolic calculator. Computer algebra thus serves as a 'black box' that may motivate the students to learn the rules after the experiment is finished. The inversion of the usual order of the course was inspired by Heid's resequencing. In this case, however, the work is not preparing for a conceptual understanding of the techniques of applying rules for differentiation. The aim is, on the one hand, to allow students to concentrate on modelling and on problem solving strategy, and on the other to investigate students' reactions to this black box approach.

Figure 1 provides an overview of the unit with brief descriptions of the core of each chapter. In the first chapter the students work on the optimization of the area of a rectangle with a given perimeter. Equal division of the perimeter turns out to provide the optimal area. The second chapter concentrates on the problem of the railway station that is presented later in this paper where graphical, numerical, analytical and geometrical methods are used. In chapter 3, students work on several variations of the problem from chapter 1, where again different methods can be used. Generalisation takes place to problems with parameters instead of numbers, and the railway station problem is extended to the situation of the refraction of light entering a different medium. As in the first unit, investigation tasks

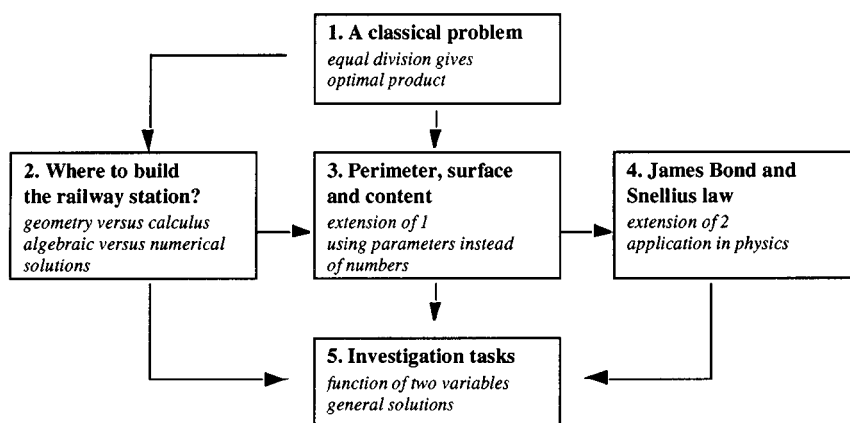


Figure 1. Overview of the unit 'Optimization using a symbolic calculator'.

conclude this unit, one of which is the problem of the pipe in the corridor that is presented later.

### EPISODES OF STUDENT BEHAVIOUR

This section contains five episodes that describe how students behaved in the classroom. They are prototypical in as much as similar behaviour was observed regularly. The significance of these observations is that they give insight into obstacles that students might experience while using a CAS. Each episode is used to formulate such an obstacle.

#### *What Is a Simple Algebraic Representation?*

Figure 2 illustrates a problem situation. A railway station  $S$  is to be situated on the railroad  $CD$  so that the total distance from the station to the two cities  $A$  and  $B$  will be minimal. Where should this railway station be built?

Students typically reacted like this:

$$\begin{aligned}
 DS &= x \\
 BS &= \sqrt{10^2 + x^2} \\
 AS &= \sqrt{5^2 + (12 - x)^2}
 \end{aligned}$$

The total distance is equal to  $AS + BS$ . Students type in the expressions and differentiate the total distance function with respect to  $x$  (see Figure 3).

The last part of the unit contains the solutions to the exercises. In order to link the algebraic form to the 'map' in Figure 2 and to the way 'experts'

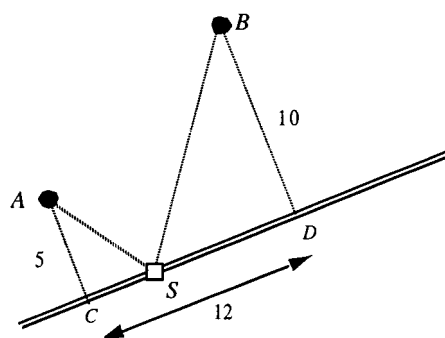


Figure 2. Where to build the railway station?

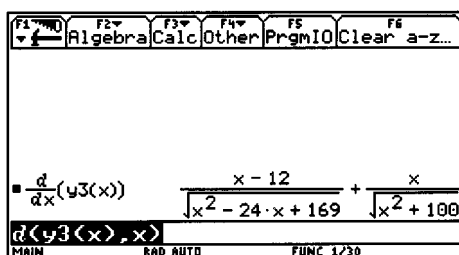


Figure 3. TI-92 screen for the railway station problem.

differentiate  $AS$  by hand, the left part of the derivative of  $y_3$  is represented in the solutions as:

$$-\frac{(12-x)}{\sqrt{5^2 + (12-x)^2}}$$

One of the students noticed the difference and reacted during the classroom discussion:

Irene: In the back of the booklet it says something else, there is a minus in front.

Teacher: (changes  $(x-12)$  in  $-(12-x)$  at the blackboard) Is that the same?

Irene: mmmno

Teacher: Is  $5 - 3/4$  the same as  $5 + -3/4$ ?

Irene: Yes . . . Oh yes. But that is clumsy, to put the minus in front, isn't it?

Teacher: Then I'll put a + again for you.

Even after this explanation, some students still have difficulties with this.

Teacher: Is  $12-x$  the same as  $x-12$ ?

Dennis: No.

Teacher: What is it then?

Dennis: I don't know.

Teacher: If  $12-x=3$ , what is then  $x-12$ ?

Dennis: minus 3,  $9 - 12$

Teacher: Could you see that immediately, without substituting the 9?

Dennis: Oh yeah, it is the opposite.

The students clearly had difficulties in understanding that the two formulas were equivalent. (By the way, the denominator looked different as well.) The way the CAS represented the solution was different from the representation that the students considered as the most simple, as it was the result of their horizontal mathematization of the problem situation. In order to solve this, students were not taught to check the equivalence by substitution of numbers or by simplification, but to try to ‘see’ the equivalence. This, however, was difficult for them.

Generally speaking, the computer algebra routines have their own ‘context free’ rules for simplifying expressions, that may not result in what the student considers to be the most simple representation in a specific situation. Coping with this requires mathematical expertise. For instance, students should know how to check whether two expressions are equivalent or not. Furthermore, they should develop ‘an eye’ for the way a CAS operates while determining the algebraic representation, so that some representations become ‘logical’ to the student. This episode suggests a first obstacle:

Obstacle 1 is the difference between the algebraic representations provided by the CAS and those students expect and conceive as ‘simple’.

#### *Exact and Decimal Numbers*

Continuing the problem of the railway station shown in the previous section, students wanted to calculate the zeros of the derivative. The ‘old’ TI-92 that these students had – without ‘Plus-module’ – solves the equation in AUTO-mode to  $x = 8$ . (see Figure 4). Here the students usually did not notice the point behind the 8, an idiosyncratic representation indicating an approximate result. In EXACT-mode, the machine returned an empty solution set. The current TI-92 does solve this equation in exact mode.

The following dialog took place:

Niels: The textbook says I should get an approximated result. Why don’t I have that?

Teacher: You have it. Look: the result is 8.; that means 8.00000000

Niels: But . . . how about the zeros?

Teacher: It just doesn’t write them.

Clearly, the way the TI-92 presents the result is confusing. But there is more. To some students, using a calculator means obtaining a numerical

$$\text{zeros}\left(\frac{x-12}{\sqrt{x^2-24x+169}} + \frac{x}{\sqrt{x^2+100}}, x\right)$$

$$\text{zeros}\left(\frac{x-12}{\sqrt{x^2-24x+169}} + \frac{x}{\sqrt{x^2+100}}, x\right)$$

Warning: Solve might specify more zeros

Figure 4. The approximate solution.

$$\text{solve}\left(\frac{(x-12)^2}{x^2-24x+169} = \frac{x^2}{x^2+100}, x\right)$$

$$x = 24 \text{ or } x = 8$$

$$\dots 2-24*x+169)=x^2/(\sqrt{x^2+100}),x)$$

Figure 5. Helping the machine to find an exact solution.

result. Some students even put their machine in APPROXIMATE-mode, as they were frustrated by the fractions they got while performing calculations during physics and chemistry lessons. Even returning to AUTO-mode after realizing the disadvantages of APPROXIMATE for the mathematics course, the students had difficulties in dealing with the different ways of calculating.

Sometimes a CAS performs exact, algebraic calculations and sometimes it makes approximations. A single command can thus evoke two conceptually different methods. In order to understand the ‘status’ of a result, students have to be conscious of this. They should be able to classify the CAS output in this respect, know how to influence this and how to choose between the two approaches. This suggests a second obstacle:

Obstacle 2 is the difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.

### *Helping the Machine*

Again continuing the railway station example, students still wanted to find the exact zeros of the derivative, but they did not know how to achieve this. In order to ‘help the machine’, it was suggested to square the two parts of the derivative. This yields two exact solutions (see Figure 5), one of which is due to the squaring.

Students often did not know what to do when the symbolic calculator did not give an answer. Finding out how to ‘help the machine’ to overcome its limitations was not an easy thing for them. They did not have the feeling that they might try to help the machine by specifying a domain, by squaring to get rid of roots, by choosing another precision mode and so on.

For optimal help, the user should have an idea of why the machine does not do what it is meant to, and of a step that might take them in the right direction. This requires an understanding of algebraic strategies and suggests a third obstacle:

Obstacle 3 is caused by the limitations of the CAS and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.

#### *What Can Computer Algebra Do for You?*

Related to the third obstacle is the fact that students are not always aware of the algebraic power of the tool that they have in their hands. Taking advantage of this is not as obvious as it may seem, as is shown in the following observation.

In the first investigation task factorial numbers are considered. The question is how one can find out how many zeros there are at the end of a factorial number without having to calculate it. This task was found in Trouche (1998) who gives a detailed description. Two boys, Dennis and Niels, were developing a (beautiful!) procedure that would calculate the number of zeros at the end of  $x!$  They discovered that one has to divide  $x$  by the subsequent powers of 5, and then add up the integer parts of the outcomes. They entered:

$$y1(x) = \sum_{k=1} \text{int} \left( \frac{x}{5^k} \right)$$

They did not know what to fill in as the upper boundary of the summation. First, they tried infinity, “certainly enough”, as they said. Unfortunately, the TI-92 did not accept this. Then they took 20 as an upper limit, which worked “as long as  $x$  is not too big”. Too big, they realized, meant exceeding  $5^{20}$ . Thinking about this, they found out that the upper bound should be the biggest  $n$ , so that  $5n$  does not exceed  $x$ . However, they were unable to solve  $5n = x$  for  $n$  by hand.

Dennis: We want to choose  $n$  so that  $x$  exceeds  $5n$ . Then you have to do something with logarithms?

[Dennis enters  $\log(x)$  and the result is  $\ln(x)/\ln(10)$ .] Dennis: What is  $\ln$ ?

Teacher: Do you know what the number  $e$  is?

Dennis: Yes, isn't that the thing that makes decades?

He probably thought of E in the scientific notation that the TI-92 uses.

They tried  $x^{1/5}$ , and this seemed to work, although this was not the solution of the equation, as they knew. The point here is, that in spite of (or maybe because of?) the quite sophisticated work they were doing, they did not realize that the TI-92 would easily solve this equation for them!

On the other hand, students can expect too much. In order to find the maximum value of a function, a student entered  $y1(\text{solve}(d(y1(x),x)=0,x))$ . Unfortunately, the TI-92 does not support this short-cut.

Generally speaking, students do not always realize what computer algebra can do for them. This observation provides an example of not using the solve-command to isolate a specific variable. In order to improve this, students have to develop a clear view of what they can expect from the CAS. Maybe this requires a thorough familiarity with the algebraic potential of the tool that the students in this study did not have yet. After one year, the graphing calculator had become an integrated part of (school)-life for many of the students. The symbolic calculator in this short period had not. Computer algebra is a more complex phenomenon than a graphing calculator. This suggests a fourth obstacle:

Obstacle 4 is the inability to decide when and how computer algebra can be useful.

Students do not appear to consider isolating one variable in an equation as solving it: solving means calculating a numerical result. This difficulty of using the symbolic calculator for algebraic manipulations appears to link with students' understanding of the concepts of variables and parameters. This is taken further in the next section.

#### *Parameters and Variables*

The restricted conception of solving an equation is also revealed in the example below. It indicates that some students had problems in understanding the role of variables and parameters.

Consider the well-known problem of a pipe that is to be carried horizontally around the corner of a corridor. The question is: How long can the pipe be? After concrete dimensions were given, the situation was generalized to corridors of dimensions  $p$  and  $q$  meters (see the left part of Figure 6).

The final question was to express the maximal length of the pipe that can pass in terms of  $p$  and  $q$ . Dennis and Niels started like this (see the right part of Figure 6) on paper:

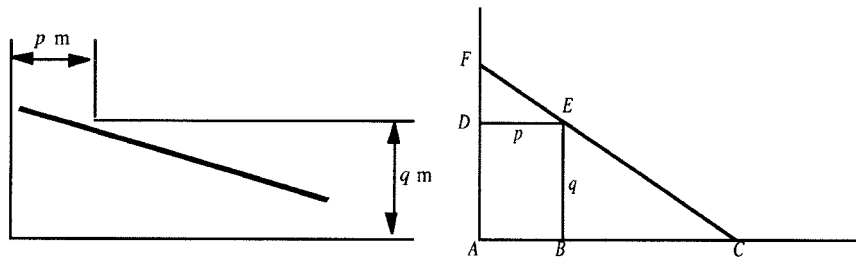


Figure 6. The pipe in the corridor.

$$BC = x$$

$$CE = \sqrt{x^2 + q^2}$$

Then they realized that triangle  $CBE$  is similar to  $CAF$ . The factor of multiplication is

$$\frac{x + p}{x}$$

so

$$CF = \frac{x + p}{x} \sqrt{x^2 + q^2}$$

Before this, they had entered these functions for specific values in the TI-92. While they were working, the teacher passed by and suggested that they made these functions more generic:

Teacher: You can use  $p$  and  $q$  in  $y1$  and  $y2$ , and then give  $p$  and  $q$  the values you want.

Dennis: (some time later) That goes very nice, with that  $p$  and  $q$  and then give values. We make the function super-general!

However, although they succeeded in solving the problem in terms of  $p$  and  $q$ , they did not include this in their report. During the classroom discussion afterwards, it became clear why:

[The teacher asks Dennis to explain his method. He wants Dennis to give the general solution, but Dennis says he can't:]

Dennis: You first have to fill in values for  $p$  and  $q$ , don't you?

Teacher: You can also solve immediately. Then you get the answer expressed in  $p$  and  $q$ .

As in the previous section, the interpretation of 'solve' is too narrow. Strictly speaking, this is not an obstacle encountered while performing an instrumentation scheme: the scheme was performed quite well. It is more the inability to deal with the results, and the difficulty of interpreting them correctly.



Of course, conceptual difficulties with variables and parameters are much older than CAS's; they exist independently from computer algebra. Much has been said about the different roles letters can play and the conceptual difficulties that students have with them (e.g., see Sfard and Linchevski, 1994; Warren, 1999).

But what is the specific role of a CAS in this? Is the computer algebra environment just making the difficulties with variables and parameters more explicit? I would say it is inherent in the use of symbolic manipulation software that variables and parameters appear in a more abstract context that enlarges possible misconceptions. As Usiskin (1988) pointed out, letters in a CAS are not placeholders for numbers, but just symbols. To the program, "all letters are equal" and one can operate with them very flexibly. For the user, this often is not the case. Adequately operating with symbols using computer algebra requires that students are aware of this, and that they really understand the concept of variables and parameters. Managing a CAS probably requires that the algebraic insight of the students is at the 'symbolic level' (see Harper, 1987). This suggests the last obstacle:

Obstacle 5 is the flexible conception of variables and parameters that using a CAS requires.

#### RELATING THE OBSTACLES TO RME

From the analysis of students' experience while working with computer algebra the following list of obstacles can be drawn up:

1. The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'.
2. The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.
3. The limitations of the CAS and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.
4. The inability to decide when and how computer algebra can be useful.
5. The flexible conception of variables and parameters that using a CAS requires.

These five obstacles are related to each other and have a common dual nature: there is a technological, machine-related component, but dealing appropriately with them also requires mathematical insight. Only the fifth obstacle can be considered as primarily mathematical; there, the existing lack of insight becomes more explicit while using a CAS.

Earlier, the possible pitfalls of using a CAS to realise the goals of RME were characterized by the words top-down tool, black box and idiosyncrasy. How do the obstacles above relate to these dangers?

The first obstacle, simplification, can indeed be considered as a result of the top-down behaviour of a CAS. It is difficult to influence the way a CAS simplifies expressions. Furthermore, the representations that the students would choose are related to the problem situation. The 'real' meaning of an algebraic expression is situated in the context. This CAS-representation, that is inappropriate (i.e. not natural) in relation to the context, may inhibit the two-directional process of horizontal mathematization. There is also an aspect of idiosyncrasy in this obstacle: the machine has its own rigid and sometimes seemingly illogical ways of rewriting expressions. Expertise is required to overcome this. The implicit way in which the TI-92 deals with numerical versus algebraic calculations is another example of the idiosyncrasy of the machine, although, in this case, not a hard one to overcome.

Generally speaking, students often have to 'come to the machine' instead of the other way around. That is also the case in the third obstacle: mathematical flexibility is needed in order to overcome CAS limitations; a sort of flexibility that requires much expertise in the field. It would be interesting to know whether students with a RME-background would do better at this point than students with a more traditional approach to mathematics education.

The fourth obstacle has much to do with the process of instrumentation. Considering the concept of schemes, the schemes that are involved here seem to be of a higher order than those mentioned by Lagrange: the capacity to find out how the tool can be useful is a 'meta-skill' compared to the ability to perform a specific utilisation scheme correctly.

The three 'RME-dangers' do not seem to account for the difficulties with variables and parameters. One might hope that previous experience of dealing with variables and parameters in a flexible and meaningful way would facilitate adequate handling in the CAS environment.

The presumed black-box character of computer algebra was not a direct obstacle while performing a utilisation scheme. At times, however, it seemed to be a somewhat emotional obstacle for some of the students who found themselves not very happy leaving differentiation to the machine. They did not know how the calculator 'did it' and were not able to check the results manually. An illustrative quotation:

[Esther solved an optimization problem graphically.]

Observer: It can also be done with differentiation.

Esther: But I cannot differentiate this function yet.

Observer: But the machine can.

Esther: Yeah, but then you don't know what you're doing!

Apparently some of the students wanted the 'boxes to be white'. This tendency has to be taken quite seriously, first because it indicates a critical mathematical attitude, and second because uncomfortable feelings can frustrate the learning process. With the theory of RME in mind, one can wonder whether the reluctance to adopt this black-box approach has to do with a lack of room for construction. No opportunities were created for the development of the conception of the rules for differentiation. The process of vertical mathematization on this point was, at least temporarily, 'overruled' by the machine. Referring to Buchberger, one can wonder whether the white-box/black-box approach was more natural than the inverse one as it was followed now. After the experiment, when the students studied the rules for differentiation without the machine, this omission was remedied. The teacher considered the chosen sequence useful because the students were really motivated to learn the rules of the differentiation technique after the experiment; his opinion on this is close to the resequencing ideas of Heid.

Although this paper focuses on the obstacles, it cannot end without saying a word on what went well. Classroom observations not cited in the examples indicated that students managed to solve optimisation problems in a meaningful way. They showed understanding of the concepts of mathematizing optimisation problems and of the strategy of solving them. The utilisation scheme of calculating the zeros of the derivative, close to the one described by Lagrange (1999b), was managed adequately.

## CONCLUSION

This research has extended to the algebraic representations of a CAS, as studied by Goldenberg for situations of interpreting graphical representations of a graphing software, and by Hillel et al. for graphical representations in a computer algebra environment. All these previous studies provide evidence that technological tools not only offer new possibilities but also may create obstacles. For example, in the case of the graphical tools the limited resolution of the screen and the dimensions of the viewing window are well-known obstacles. Research on obstacles created by the algebraic representations of a CAS environment was a natural next step. The research of Lagrange was helpful in analysing the students' instrumental behaviour, and Buchberger's idea of white and black boxes helped in the interpretation of students' reluctance to use the computer algebra device as a black box. In a similar manner, Heid's concept of resequencing was helpful in thinking of the structure of the learning trajectory.

The 'risks' of using computer algebra in mathematics education that were identified by reference to the theory of RME were shown to some

extent to have become reality. The top-down character of a CAS, its black-box style and its idiosyncracies produced obstacles during the performance of instrumentation schemes and during the interpretation of the results.

Five obstacles were identified, (although there is no reason to assume that this list is exhaustive). These obstacles encountered when using a CAS have consequences for teaching. Although these consequences were not within the scope of this study, I would like to conjecture on them. Generally speaking, as a pedagogical strategy, my feeling is that teachers should consider them seriously, pay attention to them and take advantage of them by making explicit the mathematics behind them. Trying to avoid these barriers might be counterproductive. Topics such as numerical and exact calculations, simplification of formulae and roles of variables and parameters deserve even more attention when a CAS is used than when it is not. As far as the black box use of a CAS is concerned the consequence of the experience in this study seems to be that one should be careful not to leave the student with a feeling of dependence on the technological tool. This may evoke the unpleasant feeling of working with an 'oracle' instead of an instrument. A careful designed 'instrumental genesis' as described by Lagrange and Guin and Trouche is worth considering.

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*Freudenthal Institute*  
*Utrecht University*  
*The Netherlands*

