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DYNAMIC GEOMETRY ENVIRONMENTS AS A SOURCE OF RICH LEARNING CONTEXTS FOR THE COMPLEX ACTIVITY OF PROVING

ABSTRACT. Is proof activity in danger with the use of dynamic geometry systems (DGS)? The papers of this special issue report about various teaching sequences based on the use of such DGS and analyse the possible roles of DGS in both the teaching and learning of proof. This paper is a reaction to these four papers. Starting from them, it attempts to develop a global discussion about the roles of DGS, by addressing four points: the variety of possible contexts for proof in a DGS, the dual nature of proof (cognitive and social) as reflected in the 'milieu' constructed around the use of a DGS, from observing to proving, and the overcoming of the opposition between doing and proving.

KEY WORDS: Deductive geometry, Dynamic geometry, Geometrical Construction, Justification, Learning and teaching proof, Learning environments, Secondary school

Not surprisingly, proof has given rise to many debates among researchers in mathematics education since it is the essence of mathematics and the teaching of proof in mathematics is a key issue which has been investigated over more than thirty years. Theoretical frameworks have been developed and numerous empirical data have been gathered in experimental settings inside or outside the mathematics classroom.

Across the world, debates and discussions have risen again with the increasing use of dynamic geometry computer environments. As mentioned in all the four papers in this special issue, it has often been claimed that the opportunity offered by such environments to 'see' mathematical properties so easily might reduce or even kill any need for proof and thus any learning of how to develop a proof.

The four papers decided to investigate this issue and to bring answers. All of them decided to construct answers within a theoretical framework and to set up experiments on the basis of their theoretical perspective. Even if their theoretical frameworks have some elements in common, each paper focuses on a specific aspect of a proof and the set of the four papers provides a multifaceted view on the concept of proof in students learning. This issue illustrates perfectly how the process of proving is complex and involves many different dimensions and aspects. In the following lines, we try to describe to what extent the four papers complement each other.

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VARIETY OF CONTEXTS FOR PROOFS IN A DYNAMIC GEOMETRY ENVIRONMENT

The collection of the four papers illustrates well the variety of possible uses of proof in mathematical activities. While three papers address the learning and teaching of proof for 15–16 year old students at a level of school in which proof is part of teaching, the paper by Jones deals with preparing younger students (12 year old) to deductive proof by making them aware of dependency between properties and enabling them to formulate such properties in a mathematical language.

The paper by Jones prepares proof with activities in which students must explain what is observed on the computer screen. The papers by Mariotti and by Marrades and Gutierrez report on a long term teaching experiments in which a social contract is introduced in the class according to which conjectures or constructions have to be justified: why are conjectures true or why are constructions valid? Hadas, Hershkowitz and Schwarz introduce the need for proof as a way of overcoming contradiction or uncertainty.

The set of papers illustrate the variety of functions and roles of proof which have been made explicit in previous theoretical frameworks (Balacheff, 1987; de Villiers, 1998; Hanna and Jahnke, 1996). The papers show very well how proof appears in a diversity of contexts for various reasons and how the teacher can play on various contexts and situations to motivate proof activities. By means of empirical evidence, the papers refute the current idea of proof being in danger by dynamic geometry environments; the situations and contexts proposed in the papers are actually based on the features of the environment.

The paper of Jones reports on a teaching unit made of three phases about the classification of quadrilaterals in which students had reproduce a figure which could not be 'messed up' and in some cases satisfying additional conditions. For example, they had to construct a rectangle in such a way that by dragging one of its vertices, it could be modified into a square. After constructing the figure the students had to explain why the constructed figure was the expected one. Explanation in these tasks prefigures proof in the sense that explaining consists of giving the conditions implying that the constructed figure is the expected type of quadrilateral. This task deals with the idea of implication between properties (or relations) which is necessary to understand how proof works. The context giving meaning to proof (or rather explanation) is the robustness of a figure under the drag mode. The explanations provided by the students give mathematical reasons for the fact that a figure remains a specified quadrilateral in the drag mode.

The paper by Mariotti reports on a long-term teaching experiment in which the system of axioms and theorems is constructed by students themselves as a system of commands introduced in the software which is empty[1] at the beginning of the teaching sequence. Proof is the means for justifying that the new command will provide the expected outcome. This is achieved by using what is known about previously implemented commands. The construction of the commands of the system is similar to the construction of theorems, as written by Mariotti: "it is done in parallel with the construction of the theory". Proof is here a means of being sure that the constructed DG system works as intended. But according to the rules established in the classroom, every student, confident in the validity of his/her construction, should defend it in front of his/her classroom mates. Proof fulfils thus a twofold role: establishing the validity of a construction for each individual and convincing the other students to accept the construction process.

Although differing from Mariotti's project, the long-term teaching sequence of Marrades and Gutierrez presents some common points. It is also based on a social organisation in which a solution proposed by a student must be accepted by the others. In both cases (Mariotti and Marrades and Gutierrez), the discussion is guided by the teacher since proof is not the most immediate social way of conviction among students. The teacher is the warrant of the respect of the discussion and justification rules established in the class. The students have a notebook of 'accepted' results which is updated at any time a new theorem has been proven. This notebook plays the role of the extendable system of commands of the DGS in Mariotti's classroom organisation. In both papers there is a specific social organisation in the classroom for assigning a social role to proof and increasing the need of having recourse to it.

Whereas formulating proof partly emerges in those papers for social reasons, in the paper of Hadas, Hershkowitz and Schwarz the need for proof is mainly due to cognitive reasons and disequilibria. These authors very carefully designed two sequences of tasks in which the order of the tasks led students to develop expectations which turned out to be obviously wrong when they checked them in the dynamic geometry environment. It created a conflict and an intellectual curiosity to know why this unexpected property is true. In the first activity, the 'false' guess was favoured by the first task about the sum of the angles of a polygon depending on the number of its sides. In the second task of the same activity, students were asked to guess the sum of the exterior angles and were thus predisposed to think that this sum depends on the number of sides. The second activity is a subtle succession of questions about conditions for triangles to be congruent or not congruent leading students to believe that some constructions were possible that turned out to be impossible and conversely.

Hadas, Hershkowitz and Schwarz created a climate of uncertainty which again compelled in students the need to understand better and not to simply check that what they guessed was wrong. It seems at a fist glance that the same sequence could be written for environments without DGS. Actually it would be impossible since the false conjectures came after students were convinced of other properties thanks to the DG system. These conjectures were elaborated in the continuation of valid conjectures. In other cases the DG environment gave a counter example to an expected result. This interplay of conjectures and checks, of certainty and uncertainty was made possible by the exploration power and checking facilities offered by the DG environment. Several examples given by the authors show how using the drag mode allowed the students to investigate whether it is possible to get non-congruent triangles with a given number of congruent elements.

THE DUAL NATURE OF PROOF AND THE 'MILIEU'

Although the four papers do not refer to the theoretical notion of 'milieu' (Brousseau, 1997), it seems worth introducing it to interpret the careful organisation of the context of proof production in those papers. To explain the solving processes carried out by a student in a task, Brousseau proposes to model these processes as resulting from mutual interactions between two systems: the learner and a system offering possibilities of actions and reactions, a system on which the student can act and which reacts to the actions of the student. This system (called 'milieu' by Brousseau) is a theoretical construct, which allows for explaining the strategies of the students. As the student strategies are affected by the context, it is clear that context and milieu are related. However the notion of context refers to all external elements, whilst the notion of milieu accounts for the elements of material as well as intellectual nature which are not controlled by the student and intervene in his/her mathematical behaviours in the task.

Proof is a target knowledge in the papers and the 'milieu', offering both feedback and action possibilities, includes the dynamic geometry environment in all of them. But a DGS itself without an adequately organised milieu would not prompt the need for proof. It is a common feature of all papers to have constructed a rich milieu with which the student is interacting during the solving process and the elaboration of a proof. In all papers, the milieu is developing and at each step in the sequence of tasks is constituted of all the results found at the previous steps.

The teaching unit presented by Jones is constructed according to this principle: at first the students gained understanding of a robust figure in the drag mode as a figure constructed by means of geometrical relations. It is only because they grasped this idea of robustness that they could be faced with the task of explaining why a constructed quadrilateral is of a specified type. In the same way, the question "why all parallelograms are trapeziums ?" (task 7 of phase 3) made sense to them only because in preceding activities, they could draw trapeziums and transform them into parallelograms by dragging. Although he does not analyse the succession of tasks in terms of milieu, Jones stresses this idea of progression, by speaking of 'progressive mathematisation' in which "mathematical models are developed through the successive positioning of contexts that embody the underlying structure of the concepts."

The organisation of the milieu has been achieved in the papers according to two different ways:

– a cognitive way consisting of a progressive construction of mathematical statements by means of tasks and systematically reconsidered and questioned by the following tasks
– a social way consisting of a construction of social rules of acceptance of results in the classroom.

This is not surprising that the dual nature of the constructed milieu, cognitive and social, corresponds to the dual nature of proof, so often stressed by various theoretical perspectives. A proof in mathematics is a specific kind of discourse meant both for validating the truth of a statement and for convincing the others of the validity of this assertion. In both cases, the milieu is evolving during the sequence of activities and this is the evolution of the milieu which is a catalyst for proof.

In the paper by Hadas, Hershkowitz and Schwarz it is because the students knew more about interior angles of a polygon that they could forge ideas of what could be the behaviour of the sum of exterior angles. These ideas turned out to be false thanks to the feedback from the DGS. Ideas and speculations did not emerge in a vacuum, they originated from existing knowledge. It is at this point that DGS can play a role in giving evidence that a conjecture is not valid. Cognitive conflict and/or surprise (as stressed by Aristotle, wonder is both source and end of knowledge) make the students eager to understand why. Understanding means grasping the mathematical reasons of the observed contradiction. Understanding cannot be achieved just through visual evidence as understanding requires restructuring the system of conceptions and ideas. Proof based on theoretical arguments becomes a means to understand.

It is because of social rules of discussion for accepting or rejecting statements that proof becomes a means of convincing the others as far as everybody can check the validity of a proposed reasoning through the system of rules. Theory may be the warrant of a democratic debate while not being based on authority arguments.

Both kinds of milieu have their limitations:

- students must understand and agree to enter a collective discussion, they must follow the rules and the role of the teacher is of course critical with regard to this aspect
- a cognitive conflict may not arise or if it arises, it may not lead to overcoming it. In the experiment of Hadas, Hershkowitz and Schwarz, around 20% of the answers in each of the activities did not mention any explanation but just stated affirmations.

It is also not surprising that in both cases, the organisation of the milieu is based on memory of the individual student and of the classroom (Brousseau and Centeno, 1991). Memory of what is already known, memory of what is already accepted. Proof in mathematics is by essence based on memory. Axioms and theorems constitute a collective memory of the mathematician's community on which they rely to go further and to produce proof of new statements. The deductive reasoning is based on memory. Memory is even reified in two teaching sequences under the form of the state of the DGS commands (Mariotti) and of a note book (Marrades and Gutierrez).

As soon as memory is part of the functioning of the milieu, time becomes a critical variable. Memory is not instantly built, memory requires not only accumulation of data but also sorting, eliminating and structuring of data. Memory is a process over time. The papers do not develop extensive comments on this dimension but actually this latter plays a decisive role in the evolution of students. The two papers involving collective discussions are based on long-term teaching sequences. Installing social rules requires obviously time. The system of tasks in all papers progressed at a pace related to the cognitive evolution of students. "The complexity of the theoretical system increased at a rate which the pupils were able to manage" (Mariotti). Hadas, Hershkowitz and Schwarz managed different paces in the sequence of the tasks of the congruence activity depending on the answers of the students.

CONCEPTUAL BREAK VERSUS CONTINUUM IN THE MOVE FROM OBSERVING TO PROVING

It has been often said in the past that deductive geometry differs deeply from geometry of observations: in the former every assertion is either a given or must be deduced from the givens, in the latter, what you see in the figure can be taken as granted. We would certainly claim that there is a deep change of status of the objects in the move from the geometry of visual evidence to the geometry of objects and relations involved in a deductive system as expressed in numerous research papers and confirmed by empirical evidence. But the existence of this conceptual cut does not imply that the former knowledge of students is not useful when faced with new task of proving. Nor does this imply that the solving processes of a proof problem are purely deductive, as we will come back to this point below.

The paper by Jones is very appropriate to illustrate how proof can be prepared in teaching with activities aimed at developing students' awareness of dependency between properties. If properties of figures are not conceived as dependent, a deductive reasoning has no meaning. The question of the validity of a property conditional on the validity of other properties would not arise in a world of unrelated properties. The sense of necessity links between properties must be developed to give a meaning to that question. Constructing robust figures under drag mode may reveal these necessary links. As soon as one constructs a parallelogram with four equal sides, one can observe that its diagonals remain perpendicular in the drag mode. However it may happen that this observation does not lead to conjecture a dependency between rhombus and perpendicular diagonals. It is interesting to note that in phase 2 devoted to construction tasks of robust figures under drag, students did not really formulate the dependency between properties: "It is a square because the sides are equal and the diagonals intersect. The diagonals are at right angles (90°)" (pair A). But in phase 3 where they had to modify a rectangle into a square and to explain why all squares are rectangle, they extracted the relevant additional property, which transforms a rectangle into a square. "A rectangle becomes a square when the diagonals become right angles where they meet" (same pair A).

Jones stresses this change in the formulations of students from purely descriptive relying on perception to more precise explanations, at first situated in the dynamic geometry environment and then related to the mathematical context. We see phase 3 as critical in this move, even if the formulations of students in this phase were referring to movement or drag. In the movement (change of space over time), the conjunction of two arising phenomena at a given time revealed to the students the link between the two phenomena for a rectangle: to become a square and to have perpendicular diagonals. In a robust construction under drag, the fact that all valid properties remain satisfied, is certainly a good indication that there is a dependency relation for experts or students aware of necessary links. It is not the case for novice students oscillating between a world of unrelated properties and a more structured world in construction. Change is more evident to be perceived by novices than permanence. This is why the simultaneity in the change of appearance is critical for novices because it is a strong external sign of a link between the two objects changing exactly at the same instant.

The continuity in the overcome of the break between empirical and formal ways of justification is present in all four papers, be it in the organisation of the tasks as in Mariotti or Hadas, Hershkowitz and Schwarz or in the solving processes of the students as in Marrades and Guttierez. These latter advocate in favour of the idea of a long and slow transition from empirical to formal justifications as reflected in their fine analysis of students solving problems in a DGS environment. They show how the deductive phase does not appear at the beginning of the solving process but after several empirical approaches and when it appears, how it is related to these empirical approaches.

A question arises from this convergence among the four papers. Would the continuity from empirical to more deductive not particularly be supported by DGS in that DGS offer a break with paper and pencil geometry? DGS contain within them the seeds for a geometry of relations as opposed to the paper and pencil geometry of unrelated facts. The break would be already entailed in the use of the DGS. This could explain why students do not enter immediately the new contract of construction of figures in a DGS, they must learn it. Instead of being faced with this the break at the level of formulations, students are faced earlier at the level of actions. We hypothesise that overcoming the break in action is easier for them for at least two reasons: from a cognitive point of view action very often precedes formulation, and feedback to actions is more easily recognised.

The continuity from empirical to deductive is reflected in the four papers by the interrelations between doing and proving which are visible in the behaviours of the students.

OVERCOMING THE OPPOSITION BETWEEN DOING AND PROVING

This interaction between student and the DG environment which has been organised in all the papers of this issue illuminates how proof is not separated of action. Hadas, Hershkowitz and Schwarz show very well how "some explanations had to be expressed by doing" and how "action pushed students towards deductive explanations". The fundamental principle underlying the teaching sequence in the paper by Mariotti relies on this dialectical link between action and proof, or more precisely between construction and proof. Mariotti explains in the example of students G and C how the command 'carry an angle' is both a construction command and an internal tool related to a theoretical control. The action of constructing a congruent angle in all possible ways in the triangle activity (Hadas, Hershkowitz and Schwarz) led the students to understand that there are non congruent angles with one congruent side and two congruent angles. Hoyles (1998) in a similar activity of investigating the minimal information to construct a triangle that is congruent to the one given, concluded about the matching between proof and construction.

Marrades and Guttierez develop a fine analysis of the interaction between the ascending phase in the solving process "characterised by an empirical activity" and a descending phase "where the solver tries to build a deductive justification". In several examples they show the intertwining of construction and proof. Students H and C closed the shape built by two parallel lines and a segment (Figure 1) by constructing a parallel line to segment AC and obtained thus a rhombus (Figure 2). This probably was just meant for satisfying a visual demand (according to the Gestalt psychology). But the shape rhombus allowed the students to recognise a line of symmetry (AK). This was the starting point of a construction leading to take into account one of the hypotheses of the problem statement (AB=AC) and to use congruence of triangles to deduce the expected property (BC is angle bisector of angle ACD).

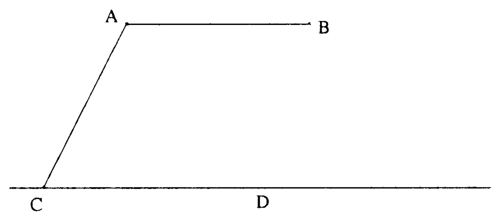

Figure 1.

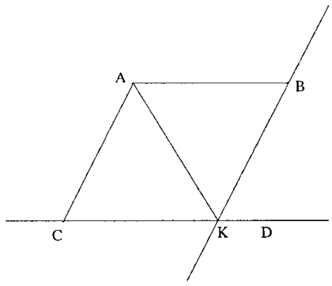

Figure 2.

Another example (the example of T and P), carefully reported in Marrades and Guttierez illuminates very well how a deductive justification takes its roots in a succession of trials just using drag mode and perceptive controls and of robust constructions meeting the conditions of the problem. The authors conclude to the dual nature of the justification: empirical and intellectual ("empirical justification by intellectual generic example"). They write: "The successive constructions added conceptual elements that helped the students to recognise and connect the different mathematical properties necessary to obtain the correct figure and, then, to justify its correctness". The new categorisation of proofs proposed by the authors reflects this complexity of the proving process made of various kinds of approaches.

THE ROLE OF A DYNAMIC GEOMETRY ENVIRONMENT

Without doubt the dynamic geometry environment fostered this interaction between construction and proof, between doing on the computer and justifying by means of theoretical arguments as claimed by Hoyles (*op.cit.*): "Some commentators may question whether the presence of the computer was necessary [...] it was to make construction methods explicit, to allow reflection on properties, to check things out and obtain immediate feedback but most crucially to foster [...] an experimental atmosphere that the teacher could exploit to introduce formal proofs in ways which matched rather than supplemented student constructions". We would further argue that dynamic geometry environment leads to analyse differently the processes involved in a proving activity as mentioned in Hadas, Hershkowitz and Schwarz and illustrated by the new categorisation of proofs proposed by Marrades and Guttierez. Following Jones and Mariotti, we could also claim in Vygotskian terms that DG environments afford possibilities of access to theoretical justifications through the semiotic mediation organised by the teacher around construction tools of dynamic geometry environments.

NOTES

1. Cabri allows the user to configure the available tools and menus by adding any new construction or suppressing any tool.

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