– Geometric Quantization of complex Monge-Ampère operator for certain diffusion flows –

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1 Kähler metrics

2 Geometric flows

3 Quantum formalism and intrinsic geometric operators

4 Other related geometries

- Consider M a Kähler manifold
- M complex manifold : differentiable manifold with atlas whose transition functions are holomorphic $\leftrightarrow$ integrable complex structure $J \in End(TM)$
- A Riemannian metric g on M is hermitian if the scalar product on TM is compatible with $J \Rightarrow$ a real $(1,1)$-form ω by $\omega(X,Y) = g(JX,Y)$, for all tangent vectors X, Y. Locally

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

and $\forall p \in M$, $h_{i\bar{j}}(p)$ is positive definite hermitian matrix
- If $d\omega = 0$, then ω is Kähler
- Example: the projective space endowed with Fubini-Study metric $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$, $U_i \simeq \mathbb{C}^n$, $\omega_{FS|U_i} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\left(\sum_{l\neq i}\frac{|z_l|^2}{|z_i|^2}\right)$

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Theorem (Yau -1978)

Let Ω a smooth volume form with $\int_M \Omega = Vol([\omega])$. Then there exists a smooth solution ϕ to the Monge-Ampère equation

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Ricci curvature of ω

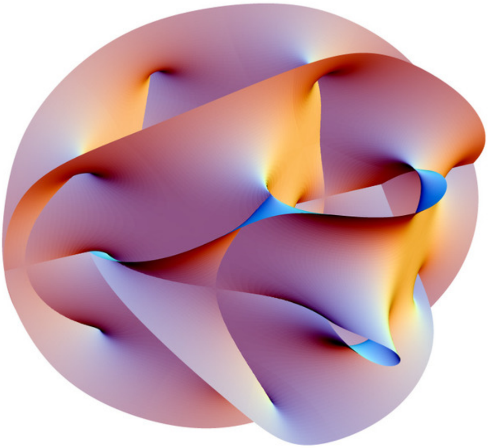
$$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\omega^n)$$

"The Ricci Curvature as organizing principle"

Scalar curvature $scal(\omega)$ = trace of the Ricci curvature

Some consequences of Yau's theorem

A new physics (Supersymmetric String Theory) $\leftrightarrow$ Ricci flat 3-folds



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Rao-Fisher metric	$\longleftrightarrow$	Calabi metric
$g_F(x,y)_{\vert\mu} = \int_M \frac{x}{\mu}\frac{y}{\mu}\mu$		$g_C(\alpha,\beta)_{\vert\omega_\phi} = \int_M \Delta_{\omega_\phi}\alpha\Delta_{\omega_\phi}\beta\frac{\omega_\phi^n}{n!}$

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$\hookrightarrow$ works in a more general setup (non compact, singular)

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Open questions for target detection:
– define a good distance between two Toeplitz covariance matrices
$\leftrightarrow$ geodesic distance on Kähler metrics
– give a reasonable definition of the average of covariance matrices
$\leftrightarrow$ balancing/barycenter condition

Quantum Field Theory

Classical system: Phase space (M,ω), observables $C^\infty(M,\mathbb{R})$
Quantized system: Hilbert space $\mathcal{H}(M,\omega)$, hermitian operators on $\mathcal{H}(M,\omega)$

Quantum phase space $\mathbb{P}(\mathcal{H})$: the Fubini-Study metric provides the means of measuring information in quantum mechanics

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Statistical manifold $\mathcal{P}_n^\star = \{p : \{x_1, \ldots, x_n\} \to \mathbb{R}, p(x_i) > 0, \sum_i p(x_i) = 1\}$ of non-vanishing probability distributions p on a discrete set $\{x_1, \ldots, x_n\}$.
Exponential representation for the tangent space at $p \in \mathcal{P}_n^\star$:

$$T_p \mathcal{P}_n^\star = \{u = (u_1, \ldots, u_n) \in \mathbb{R}^n | u_1 p(x_1) + \ldots + u_n p(x_n) = 0\}$$

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Define $\gamma : (T\mathcal{P}_n^\star, \omega_F) \to \{[z_1, .., z_n] | \forall i,\ z_i \neq 0\} \subset (\mathbb{CP}^n, \omega_{FS})$ universal covering map
$\Rightarrow$ local isomorphism of Kähler structures.

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Deformation of Kähler metrics

Intrinsic geometric flows in Riemannian geometry have been used for a few decades (Yamabe flow, Ricci flow, Calabi flow...) for theoretical purposes.

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$\hookrightarrow$ cf. works of X. D. Gu (Stony Brook), G. Zou (Wayne State University), E. Sharon & D. Mumford (Brown University), etc.

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(Normalized) Kähler-Ricci flow	$\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \lambda \omega_t, \quad \lambda \in \mathbb{R}$
Kähler Calabi flow	$\frac{\partial \phi_t}{\partial t} = scal(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t) - s$

These flows may develop singularities !

Perelman's functional

Boltzmann-Shannon entropy

$$\mathcal{E}_{BS} = -\int_M u \log u \, dV$$

with $u(t) = e^{-f(t)}$ probability density of a particle evolving under Brownian motion $\square^* u = 0$. Fisher information functional, the so-called *Perelman's functional*

$$\mathcal{F}(g,f) = \int_M (scal(g) + |\nabla f|^2) e^{-f} dV$$

since $\mathcal{F}$ is the rate of dissipation of entropy:

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Ricci flow is the gradient flow of $\mathcal{F}$.
$\hookrightarrow$ Important on Riemannian manifold (Poincaré's conjecture,...) but also for Kähler manifold (Hamilton-Tian's conjecture)

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Berezin Quantization and density of Bergman space

$Ka(\omega) = \{\phi \in C^\infty(M, \mathbb{R}) : \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$ ∞-dim Riemannian space.
Fix $k >> 0$, Planck constant $\hbar = 1/k$, $[\omega] = c_1(L)$ integral/rational class.
Space of Bergman metrics

$$\mathcal{B}_k = GL(N_k, \mathbb{C})/U(N_k)$$

set of all hermitian metrics on $\mathcal{H}_k = H^0(M, L^{\otimes k})$,
$N_k = \dim \mathcal{H}_k = k^n \int_M \omega^n + O(k^{n-1})$ relating dimension of quantum state to the volume of phase state (Riemann-Roch)

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Dequantization process: the injective 'Fubini-Study' map

$$FS_k : \mathcal{B}_k \to Ka(\omega)$$

given by $FS_k(H) = \frac{1}{k}\log(\sum_{i=1}^{N_k} |s_i^H|^2)$ where (s_i^H) is any H-orthonormal basis of holomorphic sections of $H^0(M, L^{\otimes k})$

Theorem (Tian - 1988, Bouche - 1990, ..)

The union of images $FS_k(\mathcal{B}_k)$ for $k >> 0$ is dense in C^∞-topology in $Ka(\omega)$.

A canonical approach to the Monge-Ampère equation

Building on ideas of Geometric Invariant Theory and the notion of moment map (J-M. Souriau), S.K. Donaldson introduced a dynamical system on $\mathcal{B}_k$ that depends only on Ω, volume form:

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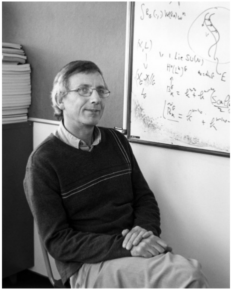
Theorem

Let ω_k^{bal} be the curvature of $FS_k(H_k^{bal})$. Then, for $k \to +\infty$,

$$\omega_k^{bal} \to \omega_\infty$$

such that

$$\omega_\infty^n = \Omega.$$



An algorithm

1. Fix $k >> 0$. Find points $p_s \in M$ over the manifold (using charts, Monte-Carlo method, etc.)
2. Give Ω volume form, compute the weights $\Omega(p_s)$.
3. Fix the space of holomorphic sections $H^0(L^{\otimes k})$ and a basis (s_i).
4. Fix a random invertible hermitian matrix $H_{[0]} \in \mathcal{B}_k$. $r := 0$.
5. Iteration of the T_k map:
   1. Compute the inverse $H_{[r]}^{-1}$.
   2. Compute

$$(H_{[r+1]})_{\alpha,\bar{\beta}} = \sum_s \frac{s_\alpha(p_s)\bar{s}_{\bar{\beta}}(p_s)}{\sum_{i,j}(H_{[r]}^{-1})_{i\bar{j}} s_i(p_s)\bar{s}_{\bar{j}}(p_s)} \Omega(p_s).$$

   If $H_{[r+1]} \simeq H_{[r]}$, stop iteration otherwise $r := r + 1$ and iterate.
6. Return $H_{[r+1]}$.

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- Balanced metric close to the solution of the Monge-Ampère equation: error $\sim O(1/k^3)$
- One can do a Newton method to get closer to the solution to the M-A equation:

$$\min_{\omega_k \in \mathcal{B}_k} \|\omega_k - \omega_\infty\|_{C^r(\omega_\infty)} = O(1/k^{\epsilon \log(k)^n})$$

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  Bergman spaces are getting exponentially close to $Ka(\omega)$.
- For the proofs, the key ingredient is the asymptotic behavior of the Bergman kernel (kernel of the L^2 projection onto $H^0(M, L^{\otimes k})$) and to consider coercive energy functionals

Extending the algorithm to other special metrics

- Kähler-Einstein metrics

$$Ric(\omega) = \lambda\omega$$

- Kähler metrics with constant scalar curvature

$$Scal(\omega) = cst$$

- Kähler-Ricci solitons

$$Ric(\omega) + L_X\omega = \lambda\omega$$

- Special metrics on bundles (solution to Vortex equation, Hermitian-Yang-Mills equation)

$$\sqrt{-1}\Lambda_\omega F_{h_E} = cst \times Id_E$$

$$\sqrt{-1}\Lambda_\omega F_{h_E} + \phi \otimes \phi^*_{h_E} = cst \times Id_E$$

- Extremal toric Kähler metrics (critical points of the Calabi functional)
- Weil-Petersson metrics (Lukic-Keller)

Extending the algorithm to geometric flows and operators

General principle:

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Moreover:

- Ricci operator on $Ka(\omega)$ can be quantized (Berman)
- Laplacian & Lichnerowicz operators +spectrum can be quantized (Fine)

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Toric geometry : from complex to real geometry

M^n Kähler manifold with effective action of the real n-dimensional torus $T_n = (S^1)^n$ preserving Kähler form and complex structure

Delzant $\leftrightarrow$ theorem

Integral polytope P in $\mathbb{R}^n$: the convex hull of a finite set of points in the lattice $\mathbb{Z}^n$, $P = \bigcap_{i=1}^{d} \{y \in \mathbb{R}^n, l_i(y) \geq 0\}$, l_i affine linear

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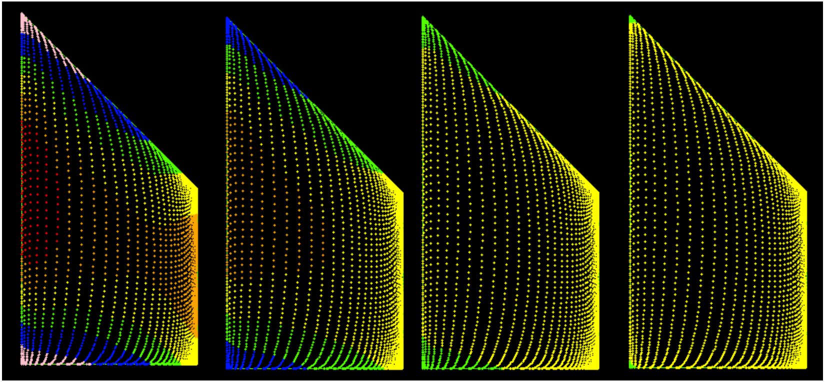
ϕ torus invariant Kähler potential in complex coordinates on M	Legendre $\leftrightarrow$ transform	$\sum_{i=1}^d l_i \log(l_i) + v$ strictly convex function in symplectic coordinates on P°, $v \in C^\infty(P, \mathbb{R})$

Toric geometry : from complex to real geometry

M^n Kähler manifold with effective action of the real n-dimensional torus $T_n = (S^1)^n$ preserving Kähler form and complex structure	Delzant $\leftrightarrow$ theorem	Integral polytope P in $\mathbb{R}^n$: the convex hull of a finite set of points in the lattice $\mathbb{Z}^n$, $P = \bigcap_{i=1}^d \{y \in \mathbb{R}^n, l_i(y) \geq 0\}$, l_i affine linear
ϕ torus invariant Kähler potential in complex coordinates on M	Legendre $\leftrightarrow$ transform	$\sum_{i=1}^d l_i \log(l_i) + v$ strictly convex function in symplectic coordinates on P°, $v \in C^\infty(P, \mathbb{R})$
$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = \Omega$ Complex Monge-Ampère equation	$\leftrightarrow$	$\det(\nabla^2\psi) = \Omega_{\mathbb{R}}$ with optimal transport map: $\nabla\psi : \mathbb{R}^n \tilde{\to} P^\circ$ wrt Lebesgue measure on P



$r = 1$ $r = 5$ $r = 10$ $r = 25$

Some iterations for constructing an extremal-balanced metric on an Hirzebruch surface
(we plot the scalar defect over the polytope (yellow means $< 1/100$)

Non compact Kähler manifolds

Let $D \subset \mathbb{C}^n$ (strictly) pseudoconvex bounded domain.

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with $f \in C^0(\bar{D} \times \mathbb{R})$, $f(z,\cdot)$ non-decreasing, $g \in C^0(\partial D)$. Then there exists a C^0 solution ϕ (viscosity solution)
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Theorem (Engliš - 2000)

Fix ϕ locally smooth bounded strictly psh on D. The Bergman kernel of $L^2_{hol}(D)$ with weight $e^{-k\phi}\mathrm{dvol}$ has the following expansion for $k \to +\infty$

$$K_k = k^n e^{k\phi} \frac{\det(\phi_{i\bar{j}})}{d\mathrm{vol}} + O(k^{n-1})$$

Non compact Kähler manifolds

Definition

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Conjecture

$D \subset \mathbb{C}^n$ pseudoconvex bounded domain.

- *Existence of $(k, d\mathrm{vol})$-balanced psh function $\phi_k^{bal} \in C^0(D)$ for $k >> 0$*
- *Convergence of ϕ_k^{bal} towards the solution of the Monge-Ampère equation*
- *Donaldson's algorithm can be extended*

The conjecture is partially checked for homogeneous bounded domains of $\mathbb{C}^n$.

Thank you for your attention

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