Geometric Quantization of Complex Monge-Ampère Operator for Certain Diffusion Flows

Julien Keller

Laboratoire d'Analyse Topologie et Probabilités, C.M.I, Aix-Marseille Université
julien.keller@univ-amu.fr

In the 40's, C.R. Rao considered probability distributions for a statistical model as the points of a Riemannian smooth manifold, where the considered Riemannian metric is the so-called Fisher metric. When extended to the complex projective space, this metric is actually the Fubini-Study metric. For certain models, it is quite remarkable that one actually needs to consider data with *complex* values. For instance, for radars-sensors, one measures simultaneously the amplitude and phase of the electromagnetic vector. Thus the observation value of each radar cell is a vector $Z \in \mathbb{C}^N$ where N is the number of emitted waves. In the complex autoregressive model (C.A.R) developed by F. Barbaresco [1, 2], one considers Z as a realization of stationary Gaussian process which provides the covariance matrix $E[ZZ^*]$ that is hermitian definite positive and Toeplitz if the signal is stationary. In this context, the hermitian and the Kähler geometry are natural settings (that actually enjoy certain similarities, see [3]) to provide adapted mathematical tools. For the C.A.R model, the entropy of the signal is given by $-\log\det(E[ZZ^*])$ which is actually the Kähler potential of the Fisher metric information. Then, it is natural to wonder if the techniques developed by complex geometers during the last decades can bring a new insight on the target detection problem or in image processing.

For example certain natural geometric evolution flows for Kähler metrics have been interpreted as anisotropic filtering operators. These flows are typically given by highly non linear PDE and involve usually transcendental and non-constructive techniques. The main objective of this note is to show that in certain cases, Geometric Quantization theory helps to overcome these difficulties and provides new algorithms based on S.K. Donaldson's ideas. This paper is a shortened version of a preprint in preparation on the same subject.

1 Quantum Formalism and Emergence of Classical Geometry

A classical physical system can be mathematically described as a symplectic manifold X equipped with a symplectic form ω. In this setup, the observables on the classical phase space (X, ω) is the Poisson algebra $C^\infty(X, \mathbb{R})$ of real-valued functions on X. From this point of view a *quantization* of the physical system consists in associating on one hand a Hilbert space (the space of *wave*

F. Nielsen and F. Barbaresco (Eds.): GSI 2013, LNCS 8085, pp. 612–620, 2013.

functions) $H(X,\omega)$ to (X,ω) and on another hand hermitian operators on (X,ω) to the set of real-valued functions on X. Moreover, together with quantization should appear a small parameter $\hbar$ (seen as the Planck's constant) and in the limit $\hbar \to 0$ the classical setting should emerge from the quantum one. We refer to [4] as a general survey on this idea.

Berezin-Toeplitz quantization on Kähler manifolds is better understood than quantization on general symplectic manifolds, see for instance [5]. Thus, in this note, we decided to consider a holomorphic setting and X will be always a Kähler manifold of complex dimension n. As shown by B. Kostant, J-M. Souriau and others, any positively curved hermitian metric h on a holomorphic line bundle $L \to X$ induces a quantization. Let us be more precise. We fix $\hbar = 1/k$, where $k \in \mathbb{N}^*$ is large and denote ω the curvature of h that is, by assumption, a Kähler metric living in the integral Kähler class $[c_1(L)]$. In this setting one is considering the space of holomorphic sections of the higher tensor powers of L, namely $H(X,\omega) := H^0(X, L^{\otimes k})$ equipped with an L^2 metric induced by h (there are several possible choices for the volume form on X as we shall see later).

To any complex-valued function f it is associated a Toeplitz operator $T_f^{(k)}$ on $H^0(X, L^{\otimes k})$ with symbol f. Using the natural L^2-orthogonal projection P^k : $\mathcal{C}^\infty(X, L^{\otimes k}) \to H^0(X, L^{\otimes k})$ induced by the Hilbert space structure, the Toeplitz operator writes as $T_f^{(k)} = P_k(f\cdot)$. It is a hermitian operator if f takes real values.

We want to emphasize that *many* objects in the Kähler/projective complex geometry can be understood via this approach of quantum formalism. Let us give briefly of list of examples and let us take this opportunity to fix some notations.

– **Pointwise Riemann-Roch Formula.** Coming back to the work of D. Catlin and S. Zelditch using the micro-local analysis of L. Boutet de Monvel-J. Sjöstrand, one can consider the density associated to the spectrum $\mathcal{S}(T_f^{(k)})$ of $T_f^{(k)}$ and check that it converges towards the Monge-Ampère mass, i.e

$$\frac{1}{k^n} \sum_{\lambda_i^{(k)} \in \mathcal{S}(T_f^{(k)})} \delta_{\lambda_i^{(k)}} \to f_* (\omega^n) \qquad (1)$$

In particular, setting $f = 1$ and integrating over $\mathbb{R}$ gives back the asymptotic Riemann-Roch formula $N_k := \dim H^0(X, L^{\otimes k}) = k^n \int_X \omega^n + O(k^{n-1})$ relating the dimension of the quantum state to the volume of the phase state.

– **Density of the Bergman Space.** Let $\mathcal{H}_k$ be the set of all hermitian metrics on the vector space $H^0(X, L^{\otimes k})$, the *Bergman space at level k*. The map $A \mapsto A^*A$ clearly yields an isomorphism

$$GL(N_k, \mathbb{C})/U(N_k) \simeq \mathcal{H}_k \qquad (2)$$

turning $\mathcal{H}_k$ into a finite dimensional symmetric space. On another hand, we call $\mathcal{H}_\infty$ the space of all smooth hermitian metrics on L with positive curvature form. Fixing a reference metric h_0 with curvature form ω_0, any other hermitian metric on L may be written as $h_\phi = e^{-\phi} h_0$ with curvature form $\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi$, using the convention which makes the curvature form a *real* 2-form. Hence, one can do the following identification

$$\mathcal{H}_\infty = \{\phi \in C^\infty(X) : \omega_\phi := \omega_0 + i\partial\bar{\partial}\phi > 0\}$$

It is an infinite dimensional Riemannian symmetric space with non-positive sectional curvature. It is natural to compare both spaces $\mathcal{H}_k$ and $\mathcal{H}_\infty$. In order to do so, let us introduce the dequantization process, with the injective map 'Fubini-Study' map , $FS_k : \mathcal{H}_k \to \mathcal{H}_\infty$ such that

$$FS_k(H) = \frac{1}{k}\log(\sum_{i=1}^{N_k} |s_i^H|^2_{h_0}), \tag{3}$$

where (s_i^H) is any H-orthonormal basis of holomorphic sections of $H^0(X, L^{\otimes k})$ and $h_0 \in Met(L^k)$ a reference metric with curvature ω_0. Another way of thinking about $FS(H)$ is to remark that it is the rescaled pulled-back of the Fubini-Study metric on $\mathcal{O}(1) \to \mathbb{P}H^0(X, L^{\otimes k})^\vee$ induced by the metric H and the Kodaira embedding given by (s_i^H). A theorem of G. Tian [6] proves that the union of the images $FS(\mathcal{H}_k)$ is actually dense in C^∞-topology in $\mathcal{H}_\infty$. Another way of stating his result is to say that any metric $h \in \mathcal{H}_\infty$ is the limit of the sequence of the algebraic metrics $FS_k(Hilb_k(h))$ where one defined the map

$$Hilb_{\omega^n,k}(h) = \int_X \langle .,. \rangle_{h^k}\omega^n. \tag{4}$$

- **Geodesics in the Space of Kähler Potentials.** Let us fix $H_0, H_1 \in \mathcal{H}_k$ two Bergman metrics. There exist $\lambda_i \in \mathbb{R}$ with $1 \leq i \leq N_k$ and bases $(s_i^{H_0})$ and $(s_i^{H_1})$, orthonormal with respect to H_0 and H_1, such that $s_i^{H_1} = s_i^{H_0}e^{\lambda_i}$. The geodesic H_t in $\mathcal{H}_k$ (with respect to the Riemannian structure induced by (2)) between H_0 and H_1 is given by $H_t \in \mathcal{H}_k$ such that $s_i^{H_t} = s_i^{H_0}e^{t\lambda_i}$ is H_t-orthonormal. Given two metrics $h_0, h_1 \in \mathcal{H}_\infty$, Tian's theorem furnishes two sequences of metrics $H_0^{(k)}$ and $H_1^{(k)}$ in $\mathcal{H}_k$. Results of D.H. Phong- J. Sturm [7, 8] show that the convergence of the Bergman geodesic H_t towards the geodesic in the space of Kähler potentials between h_0 and h_1 when $k \to +\infty$. In the same spirit, we would like to underline that there is convergence of the rescaled geodesic distance on $\mathcal{H}_k$ to the geodesic distance on $\mathcal{H}_\infty$ equipped of the Mabuchi-Semmes-Donaldson metric (other metrics can be defined on $\mathcal{H}_\infty$ and their quantization needs to be investigated).
- **Flows in Kähler Geometry.** Consider $\mu : \mathbb{CP}^{N-1} \hookrightarrow Herm(N)$ embedding of the complex projective space into the space of hermitian $N \times N$ matrices, seen as an Euclidean space. We define the map μ by associating to a point of the projective space the orthogonal projection onto the line corresponding to the point. It is a moment map for the action of $U(N)$ on $\mathbb{CP}^{N-1}$. Let us consider a complex projective manifold $X \subset \mathbb{CP}^{N-1}$ and Ω a volume form on X. There are different possibilities to define the *center of mass* of X in $Herm(N)$ by considering the maps

$$\mu_\Omega = \int_X \mu\, \Omega \qquad \text{or} \qquad \tilde{\mu} = \int_X \mu\, \omega_{FS}^n, \tag{5}$$

where ω_{FS}^n is the volume form induced by the Fubini-Study metric restricted to X. In [9] and [10] are studied the downward gradient flow associated to $\tilde{\mu}$ and μ_Ω, called the balancing flow (respectively the Ω-balancing flow). Coming back to our original quantization setup, we fix $N = N_k$ and consider the Kodaira embedding of X in $\mathbb{P}H^0(X, L^{\otimes k})^\vee$. Then, at $k \to +\infty$, the balancing flow converges to the famous Calabi flow, a 4^{th} order PDE

$$\frac{\partial \omega_{\phi_t}}{\partial t} = \sqrt{-1}\bar{\partial}\partial \text{scal}(\omega_{\phi_t}) \tag{6}$$

for as long as the Calabi flow exists and the convergence is C^1 in time. Here we have denoted by $\text{scal}(\omega)$ the scalar curvature of the Kähler form ω. For the Ω-balancing Kähler flow, its quantum limit is not the Kähler-Ricci flow but the following Ω-Kähler flow, a 2^{nd} order parabolic diffusion flow:

$$\frac{\partial \phi_t}{\partial t} = 1 - \frac{\Omega}{\omega_{\phi_t}^n} \tag{7}$$

where $\omega_{\phi_t} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_t$. It is proved that it has actually a very similar behavior to the Kähler-Ricci flow in [11], and in particular converges to the solution of the Calabi's conjecture (as defined in Section 2).

Two ingredients are crucial in the proofs proofs of these convergence results. Firstly, the asymptotic expansion when $k \to +\infty$ of the "distortion Bergman function" or "density of states function". This function $B_{k,h}$ over X is the restriction to the diagonal of the Bergman kernel of the L^2 projection on $H^0(X, L^{\otimes k})$. $B_{k,h}$ depends on the metric $h \in \text{Met}(L)$ and the L^2 metric on $H^0(X, L^{\otimes k})$: let us choose $Hilb_{\Omega,k}(h) = \int_X \langle ., . \rangle_{h^k} \Omega$, then we have an asymptotic (see [12])

$$B_{k,h}(p) = k^n \frac{\omega^n}{\Omega} + k^{n-1}\frac{1}{8\pi}\frac{\omega^n}{\Omega}\left(\text{scal}(\omega) - 2\Delta_\omega\left(\frac{\omega^n}{\Omega}\right)\right) + O(k^{n-2}). \tag{8}$$

The second key ingredient is the fact that one is working with moment maps that encode the symmetry of the considered geometric structures. Morally, these moment maps are detecting the "best" projective embedding which are projectively equivalent to a given one $X \hookrightarrow \mathbb{P}H^0(X, L^{\otimes k})^\vee$. Such convenient embeddings (as called in the seminal paper of J-P. Bourguignon, P. Li and S.T. Yau [13]) are called *balanced* and enjoy nice algebraic and geometric properties. Let us mention briefly that one can associate to an embedded manifold $X \subset \mathbb{CP}^{N_k}$ of dimension n a point in the Chow variety $\mathcal{C}(N_k, d, n)$. This variety parametrizes all the subvarieties of $\mathbb{CP}^{N_k}$ of dimension n and fixed degree d. It is not possible to construct a good quotient of $\mathcal{C}(N_k, d, n)$ for the action of $PGL(N_k + 1)$ but Geometric Invariant Theory tells us what are the good $PGL(N_k + 1)$-orbits. They are the *Chow stable* points. This G.I.T stability can be translated in terms of symplectic geometry through the formalism of moment maps. In that different language, stable points correspond to special metrics (or equivalently, special embeddings), the *balanced* metrics. In the next section, we explain how this frame of ideas can be applied to provide numerical applications.

2 Calabi's Conjecture and Donaldson's Algorithm

2.1 Balanced Metric Associated to a Fixed Volume Form

Let us consider a smooth volume form Ω on X, and denote $Vol_\Omega(X) = \int_X \Omega$. Fix a Kähler class $[\omega]$ with volume equal to $Vol_\Omega(X)$. Calabi conjectured that one can find a Kähler metric in $[\omega]$ with prescribed point-wise Monge-Ampère mass, i.e a Kähler metric $\omega_\infty \in [\omega]$ such that $\omega_\infty^n = \Omega$. This is equivalent to solve the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = \Omega \tag{9}$$

with $\omega_\infty = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. The real version of this equation has many applications, including in data processing or cosmology. Note that in the Kähler setting, the Ricci curvature $\mathrm{Ric}(\omega)$ of the Kähler metric ω has a particular simple expression in terms of its Monge-Ampère mass, $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\omega^n)$. Thus, to solve (9) allows us to define the operator Ric^{-1} [14]. Furthermore, since the Ricci curvature has a natural geometric interpretation, it provides a link between Monge-Ampère equations and other mathematical fields like optimal transport theory, probability theory and physics. This is well explained in [15]. In [16], S.T. Yau solved Calabi's conjecture by proving the existence of a smooth (unique) solution of this non linear PDE by a continuity method. The proof is not constructive. Thirty years later, Donaldson gave the following definition [17].

Definition 1. *A metric $h \in \mathrm{Met}(L)$ is Ω-balanced or order k if the function $B_{k,h}$ is constant over the manifold.*

Note that if h is Ω-balanced, then we will say that $H = Hilb_{\Omega,k}(h)$ is Ω-balanced and we get for such a metric $\int_X \langle S_i, S_j\rangle_{FS(H)^k}\Omega = \delta_{i,j}$ for (S_i) an H-orthonormal basis. In other words, a balanced metric is a fixed point of the map $Hilb_{\Omega,k} \circ FS_k$. As shown by Donaldson the natural dynamical system induced by this map has a trivial behaviour:

Theorem 1. *The dynamical system induced by the iterations of $Hilb_{\Omega,k} \circ FS_k : \mathcal{H}_k \to \mathcal{H}_k$ has fixed attractive points, unique up to action of $U(N_k)$.*

The proof is natural using Kempf-Ness theory. Consider on $\mathcal{H}_k$ the functional

$$I_\Omega(H) = \frac{1}{kN_k}\log\det H + \int_M (FS_k(H) + \log(h_0))\Omega$$

where $h_0 \in Met(L)$ is a reference metric. Then I_Ω is convex on geodesics and proper. It is the *integral of the moment map* μ_Ω (in the sense of [18, 19]) and its critical points are Ω-balanced metrics. Using the arithmetico-geometric inequality and the concavity of the log, one can check that an iteration of $Hilb_{\Omega,k} \circ FS$ decreases I_Ω. Let us mention that I_Ω is the finite dimensional analog of the Aubin-Yau-Mabuchi [20] functional $F_\Omega^0 : \mathcal{H}_\infty \to \mathbb{R}$ which is given by the formula

$$F_\Omega^0(\omega_{\phi_t}) = \int_0^t \int_M \dot{\phi}_s(\Omega - \omega_{\phi_s}^n)ds$$

where $\{\phi_s\}_{s=0..t} \in \mathcal{H}_\infty$ is a path of potentials. This functional is related to the entropy functional which writes in that case $H_\Omega(\omega) = \int_M \log\left(\frac{\omega^n}{\Omega}\right)\omega^n$, see [21].

So we get for all k large enough a Ω-balanced metric in a **canonical** way. This provides a family of algebraic metrics. From the leading term of (8), it is clear that if the sequence of balanced metrics is convergent, its limit is necessarily the solution of (9). For the proof of the convergence in C^∞-topology, see [14].

2.2 Numerical Approximations of Solutions to the Complex Monge-Ampère Equation (9)

Now, we provide an algorithm to compute an approximation of the solution to the complex Monge-Ampère equation (9) based on [17]. The algorithm is based on the simple fact that an iteration of $Hilb_{\Omega,k} \circ FS$ decreases I_Ω towards a critical point.

1. Fix k large enough. Find a large number of points p_s over the manifolds (using charts, Monte-Carlo method, etc.)
2. Give Ω volume form, compute once the weights $\Omega(p_s)$.
3. Fix the space of holomorphic sections $H^0(L^{\otimes k})$. Use the symmetries if possible to reduce the dimension. Determine a basis s_i of $H^0(L^{\otimes k})$.
4. Fix a random invertible hermitian matrix $H_{[0]} \in \mathcal{H}_k$. $r := 0$.
5. Iteration of $Hilb_{\Omega,k} \circ FS$:
   (a) Compute the inverse $H_{[r]}^{-1}$.
   (b) Compute
   $$(H_{[r+1]})_{\alpha,\beta} = \sum_s \frac{s_\alpha(p_s)\bar{s}_{\bar{\beta}}(p_s)}{\sum_{i,j}(H_{[r]}^{-1})_{i\bar{j}}s_i(p_s)\bar{s}_{\bar{j}}(p_s)}\Omega(p_s).$$
   If $H_{[r+1]} \simeq H_{[r]}$, stop iteration otherwise $r := r+1$ and iterate.
6. Return $H_{[r+1]}$.

The output of this algorithm is an Ω-balanced metric (more precisely, an approximation of it) at order k. Now, we discuss some technical issues. For a complex n-dimensional manifold, the complexity of this algorithm is $\sim k^{4n}$, where most of the computations are done to evaluate the Bergman function (essentially a homogeneous "polynomial" of degree k in n variables) over the whole set of points picked on the manifold (how to choose the points on the manifolds will be discussed later). In particular it is clear that using this method, one can compute with one desktop computer a solution of (9) on complex surfaces. This will be compared with other methods.

The speed of convergence of the algorithm is exponential (so in practice a dozen of iterations of Step 5 are sufficient) and the common ratio, up to normalization, is actually the smallest positive eigenvalue of the Laplacian of ω_∞.

Furthermore, one can consider the r-th iterate $H_{[r]} = H_{[r]}(k) \in \mathcal{H}_k$ (at Step 5) and the induced Fubini-Study metric $h_{[r]} = FS_k(H_{[r]})$. Let us assume that $r = r(k)$ depends on the parameter k and that we make $k \to +\infty$. If $r(k)/k \to t$, then a result of R. Berman [22] asserts that $h_{[r(k)]}$ converges to the metric induced by the (normalized) Kähler-Ricci flow induced at time t, namely

$$\frac{\partial \phi_t}{\partial t} = \log\left(\frac{\omega_{\phi_t}}{\Omega}\right) \tag{10}$$

starting with the same initial metric than the one chosen in the algorithm ($h_0 = \lim_{k\to+\infty} FS_k(H_{[0]}(k))$. This provides an algorithm of quantization (10) (while Section 2.1 gives a discretization of the unormalized Ricci flow by unit time). With the balancing flow (resp. the Ω-balancing flow) we obtain also algorithms of quantization of the Calabi flow (6) (resp. the Ω-Kähler flow (7)).

3 Extra Remarks

We decided to restrict our attention to the complex Monge-Ampère equation of the type (9) over complex projective manifolds due to its historical importance and for pedagogical purpose. There are generalizations in various directions:
- One can consider Ω to be a non smooth or degenerate volume forms, see [23] where it is introduced the notion of probability measure of finite energy.
- One can also consider others Monge-Ampère type equations where the RHS depends on the unknown. They can be treated in a similar way, but convergence results are more subtle. In particular, keeping in mind (8), one can treat the cases of Kähler-Einstein metrics, Kähler-Ricci solitons, Kähler metrics with constant scalar curvature and extremal Kähler metrics. Note that in [24, 25] it is precisely Kähler metrics ω with $\mathrm{scal}(\omega) = cst$ that appear in the C.A.R model.
- One can consider certain non compact manifolds. Among them are bounded homogeneous complex domains, bounded pseudo-convex domains of $\mathbb{C}^n$ and ALE (asymptotically locally euclidean) manifolds on which an adapted version of Calabi's conjecture holds. For bounded homogeneous domains, it is possible to do Berezin quantizations and to define Ω-balanced metrics using Hilbert spaces, see [26]. Even if the mathematical theory needs substantial development to obtain convergence results, it is quite clear how to adapt the presented algorithm and that the Monge-Ampère operator arises. This will be discussed later.

On another side, one can restrict to the case of projective manifolds with large symmetry group, like toric manifolds. In that case, there is a correspondence between the manifold and a (simple, rational, smooth) *convex polytope in* $\mathbb{R}^n$, the Delzant polytope. With this correspondence, certain analytic questions are simplified and *real Monge-Ampère equations* appear naturally.

Acknowledgments. This research was supported by the French Agence Nationale de la Recherche - ANR project MACK (ANR-10-BLAN-0104).

References

[1] Barbaresco, F.: Innovative tools for radar signal processing based on Cartans geometry of SPD matrices and Information Geometry. In: IEEE International Radar Conference (2008)
[2] Barbaresco, F.: Interactions between Symmetric Cone and Information Geometries: Bruhat-Tits and Siegel Spaces Models for High Resolution Autoregressive Doppler Imagery. In: Nielsen, F. (ed.) ETVC 2008. LNCS, vol. 5416, pp. 124–163. Springer, Heidelberg (2009)
[3] Shima, H.: The geometry of Hessian structures. World Scientific Publishing Co. Pte. Ltd., Hackensack (2007)
[4] Ali, S.T., Engliš, M.: Quantization methods: a guide for physicists and analysts. Rev. Math. Phys. 17(4), 391–490 (2005)
[5] Berezin, F.A.: Quantization in complex symmetric spaces. Izv. Akad. Nauk SSSR Ser. Mat. 39(2), 363–402, 472 (1975)
[6] Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds. J. Differential Geom. 32(1), 99–130 (1990)
[7] Phong, D.H., Sturm, J.: The Monge-Ampère operator and geodesics in the space of Kähler potentials. Invent. Math. 166(1), 125–149 (2006)
[8] Phong, D.H., Sturm, J.: Monge-Ampère equations, geodesics and geometric invariant theory. In: Journées, E.D.P. (ed.) École Polytech., Palaiseau, No X, p. 15 (2005)
[9] Fine, J.: Calabi flow and projective embeddings. J. Differential Geom. 84(3), 489–523 (2010); With an appendix by Kefeng Liu and Xiaonan Ma
[10] Cao, H.D., Keller, J.: Balancing kähler flow for the calabi problem. To appear in Journal of European Math. Soc. ArXiv:1102.1097 (2011)
[11] Cao, H.D.: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. Invent. Math. 81(2), 359–372 (1985)
[12] Ma, X., Marinescu, G.: Holomorphic Morse inequalities and Bergman kernels. Progress in Mathematics, vol. 254. Birkhäuser, Basel (2007)
[13] Bourguignon, J.P., Li, P., Yau, S.T.: Upper bound for the first eigenvalue of algebraic submanifolds. Comment. Math. Helv. 69(2), 199–207 (1994)
[14] Keller, J.: Ricci iterations on Kähler classes. J. Inst. Math. Jussieu 8(4), 743–768 (2009)
[15] Bourguignon, J.P.: Ricci curvature and measures. Jpn. J. Math. 4(1), 27–45 (2009)
[16] Yau, S.T.: Nonlinear analysis in geometry. Enseign. Math (2) 33(1-2), 109–158 (1987)
[17] Donaldson, S.K.: Some numerical results in complex differential geometry. Pure Appl. Math. Q. 5(2), 571–618 (2009)
[18] Kempf, G., Ness, L.: The length of vectors in representation spaces. In: Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978). Lecture Notes in Math., vol. 732, pp. 233–243. Springer, Berlin (1979)
[19] Mundet i Riera, I.: A Hitchin-Kobayashi correspondence for Kähler fibrations. J. Reine Angew. Math. 528, 41–80 (2000)
[20] Mabuchi, T.: Some symplectic geometry on compact Kähler manifolds. I. Osaka J. Math. 24(2), 227–252 (1987)
[21] Berman, R., Boucksom, S., Eyssidieux, P., Guedj, V., Zeriahi, A.: Kähler-Einstein metrics and the Kähler-Ricci flow on log-Fano varieties. ArXiv 1111.7158 (2011)

[22] Berman, R.: Relative Kähler-ricci flows and their quantization. Arxiv 1002.3717 (2010)

[23] Berman, R., Boucksom, S., Guedj, V., Zeriahi, A.: A variational approach to complex Monge-Ampère equations. ArXiv 0907.4490 (2009)

[24] Barbaresco, F.: Information intrinsic geometric flows. In: MAX-ENT 2006 Conf. Paris, vol. 872, pp. 211–218 (2006)

[25] Barbaresco, F.: Etude et extension des flots de Ricci, Kähler-Ricci et Calabi dans le cadre du traitement de l'image et de la géométrie de l'information. Gretsi, Groupe d'Etudes du Traitement du Signal et des Images (2005)

[26] Loi, A., Mossa, R.: Berezin quantization of homogeneous bounded domains. Geom. Dedicata 161, 119–128 (2012)