SOME REMARKS ABOUT CHOW, HILBERT AND K-STABILITY OF RULED THREEFOLDS

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Abstract. Given a rank 2 holomorphic vector bundle E over a projective surface, we explain some relationships between the Gieseker stability of E and the Chow, Hilbert and K-stability of the polarized ruled manifold $\mathbb{P}E$ with respect to polarizations that make fibres sufficiently small.

In this paper, we pursue our study of the G.I.T stability of ruled manifolds given as projectivisation of rank 2 vector bundles over projective surfaces. The purpose of this note is to observe that the notion of Gieseker stability for the underlying vector bundle plays a key role in the Chow, Hilbert and K-stability of the associated ruled threefold when the first Chern class of the base is proportional to the considered polarization. This has to be compared with the simpler case of ruled manifolds over a curve where the notion of Mumford stability is central, and we refer to [1, 3] on this topic. We want to point that checking stability algebraically is a difficult problem. Our proofs rely mainly on two ingredients. One is coming from geometric analysis with the connection between existence of canonical Kähler metrics (namely Kähler metrics with constant scalar curvature) and stability notions. The other one is a brute force computation of the G.I.T weights for certain test configurations associated to the deformation to the normal cone of the projectivisation of a subbundle. These ingredients have appeared in [11] but in this paper we are carrying the computations to a greater extent and draw some simple and natural consequences from them. In particular we provide some new examples of asymptotically Hilbert or Chow semistable polarizations that are not asymptotically Hilbert or Chow stable.

1. About Chow stability, Hilbert stability and K-stability

In this section, we recall briefly some well known facts about Chow stability and K-stability of a polarized scheme. We refer to [16, 21, 7, 6] for details and examples.
Consider (X, L) a polarized subscheme of complex dimension n and $X \subset \mathbb{P}H^0(X, L^k)^* = \mathbb{P}V$ the closed immersion associated to the complete linear system $|L^k|$. Let $Z_X = \{P \in Gr(V, n-1) : P \cap X \neq \emptyset\}$ which is a divisor of degree $d = \deg L$ in the Grassmannian $\mathcal{G} = Gr(V, n-1)$. Thus there exists $s_{X,V} \in H^0(\mathcal{G}, \mathcal{O}_{G'}(d))$, such that one has $Z_X = \{s_{X,V} = 0\}$ and this induces a Chow point

$$\mathrm{Chow}(X) = [s_{X,V}] \in \mathbb{P}H^0(\mathcal{G}, \mathcal{O}_{G'}(d))$$

on which one can consider the action of $SL(V)$. The polarized scheme (X, L^k) is said to be Chow stable (resp. Chow semistable) if the Chow point $\mathrm{Chow}(X)$ is G.I.T stable (resp. G.I.T semistable).

We say that it is asymptotically Chow stable (resp. asymptotically Chow semistable) if (X, L^k) is Chow stable (resp. Chow semistable) for $k \gg 1$.

Let us discuss now Hilbert stability. For $X \subset \mathbb{P}V$ a closed subscheme such that the restriction map

$$\rho : H^0(\mathbb{P}V, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$$

is surjective, one sets

$$W_m = \bigwedge^{h^0(X,\mathcal{O}(m)} H^0(\mathbb{P}V, \mathcal{O}(m))^\vee.$$

Thus, from the map ρ and taking the wedge product, one can consider the m-Hilbert point

$$[X]_m = [\bigwedge^{h^0(X,\mathcal{O}(m)} H^0(\mathbb{P}V, \mathcal{O}(m)) \to \bigwedge^{h^0(X,\mathcal{O}(m)} H^0(X, \mathcal{O}(m))] \in \mathbb{P}(W_m).$$

The polarized scheme (X, L^k) is said to be Hilbert stable (resp. Hilbert semistable) if the induced m-Hilbert points $X \in \mathbb{P}H^0(X, L^k)$ defined by the closed immersion associated to the complete linear system $|L^k|$ are all G.I.T semistable (resp. G.I.T stable) for $m \gg 1$.
The polarized scheme (X, L) is said to be asymptotically Hilbert stable (resp. asymptotically Hilbert semistable) if (X, L^k) is Hilbert stable (resp. Hilbert semistable) for $k \gg 1$.

We recall now the notion of test configuration [5, 6].

Definition 1. A test configuration for a polarized scheme (X, L) is a polarized scheme $(\mathcal{X}, \mathcal{L})$ with:

- a $\mathbb{C}^\times$ action and a proper flat morphism $\pi : \mathcal{X} \to \mathbb{C}$ which is $\mathbb{C}^\times$ equivariant for the usual action on $\mathbb{C}$,
- a $\mathbb{C}^\times$ equivariant line bundle $\mathcal{L} \to \mathcal{X}$ which is ample over all fibers of π such that for $z \neq 0$, (X, L^s) is isomorphic to $(\mathcal{X}_z, \mathcal{L}_{\mathcal{X}_z})$ for some positive integer s, called the exponent.

A product test configuration is a test configuration with $\mathcal{X} \simeq X \times \mathbb{C}$. A test configuration is trivial in codimension 2 if it is $\mathbb{C}^\times$-equivariantly isomorphic to a product test configuration $X \times \mathbb{C}$, with trivial $\mathbb{C}^\times$-action, away from a closed subscheme of codimension at least 2.

From [18], we know that there is a correspondence between the data of a test configuration $(\mathcal{X}, \mathcal{L})$ of exponent s and the data of a 1-parameter subgroup of $GL(H^0(X, L^s))$. Thus using the Hilbert-Mumford criterion, it is sufficient to consider the weights of the $\mathbb{C}^\times$ action to check the stability of (X, L). More precisely, let us call $w(Ks)$ the total weight of the induced action on $\pi_* \mathcal{L}^K_{|0} = H^0_{\mathcal{X}_0}(\mathcal{L}^K)$ for $K \gg 0$, for a test configuration associated to (X, L^{Ks}). Remark that $w(Ks)$ is a polynomial of degree $n+1$ in the $k = Ks$ variable. Let us denote $P(k) = \dim H^0(X, L^k)$ which is equal to the Hilbert polynomial $\chi(X, L^k)$ for k large. The normalized weight after taking the $sP(s)$-th power of the $\mathbb{C}^\times$ action on $\pi_* \mathcal{L}^K_{|0}$ is

$$\tilde{w}(s, k) = w(k)sP(s) - w(s)kP(k) \tag{1}$$

which is a polynomial of degree $n+1$ in the k variable. It is the Hilbert weight of (X, L^s) and thus (X, L) is asymptotically Hilbert stable (resp.

asymptotically Hilbert semistable) if and only if $\tilde{w}(s,k) > 0$ (resp. $\tilde{w}(s,k) \geq 0$) for all $k \gg 1$ ($k > k_0(s)$ large enough), $s \gg 1$.

One can decompose $\tilde{w}(s,k)$ as

$$\tilde{w}(s,k) = \sum_{i=0}^{n+1} e_i k^i \tag{2}$$

where $e_i = \sum_{j=0}^{n+1} e_{i,j} s^j$ are polynomials of degree $n+1$ in the s variable with $e_{n+1,n+1} = 0$ due to the normalisation.

We refer to [16, Lemma 2.11] and [18, Theorem 3.9] for a proof of the next result.

Lemma 1. The coefficient $e_{n+1}(s)s^{n+1}(n+1)!$ is the Chow weight of $X \subset \mathbb{P}H^0(X, L^s)$. In particular, (X, L) is asymptotically Chow stable (resp. asymptotically Chow semistable) if and only if $e_{n+1}(s) > 0$ (resp. ≥ 0) for all $s \gg 1$. It is said to be asymptotically Chow polystable if it is asymptotically Chow semistable and any not strictly stable test configuration is a product test configuration.

The following definition is a refinement of Donaldson's definition of K-stability [6] and is due to Stoppa [20].

Definition 2. The polarized variety (X, L) is K-stable (resp. K-semistable) if for any test configuration which is non trivial in codimension 2, the leading coefficient $e_{n+1,n}$ of $e_{n+1}(s)$ is positive (resp. ≥ 0). It is said to be K-polystable if it is K-semistable and any not strictly stable test configuration is a product test configuration.

Let us finish this section by recalling certain well-known relationships between the various notions of stability that we shall use later (see [21, 13]):

Asymptotic Chow stability $\Leftrightarrow$ Asymptotic Hilbert stability $\Rightarrow$ Asymptotic Hilbert semistability $\Rightarrow$ Asymptotic Chow semistability $\Rightarrow$ K-semistability.

2. Rank 2 vector bundles over surfaces and the stability of their projectivisation

Let us fix B a projective surface polarized by L and $\pi : E \to B$ an indecomposable holomorphic vector bundle on B. We shall compute in our setting the Donaldson-Futaki invariant $F_1(\mathcal{T})$ induced by the degeneration $\mathcal{T}$ to the normal cone of $\mathbb{P}(F)$ where F is a subbundle of E and with respect to the polarization $\mathcal{L}_{r,m}$. Let us give now some explanations on this computation (we refer to [17, 11] for details of the test configuration we construct). We consider the family of bundles $\mathcal{E} \to B \times \mathbb{C} \to \mathbb{C}$ with general fibre E and central fibre $F \oplus G$ over $0 \in \mathbb{C}$ where G is the quotient bundle. Then $\mathcal{E}$ admits a $\mathbb{C}^*$ action that covers the usual action on the base $\mathbb{C}$, and whose restriction to $F \oplus G$ scales the fibres of F with weight 1 and acts trivially on G. Setting $\mathcal{X} = \mathbb{P}(\mathcal{E}) \to \mathbb{C}$ and

$$\mathcal{L}_{r,m} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(r) \otimes \pi^* L^m$$

with (r, m) such that $\mathcal{L}_{r,m}$ is ample, we obtain a flat family of polarized varieties with $\mathbb{C}^*$ action whose general fibre is the polarized ruled manifold $(\mathbb{P}E, \mathcal{L}_{r,m})$. It is a non trivial test configuration that we shall denote by $\mathcal{T}$.

Conventions: If $\pi\colon E \to B$ is a vector bundle then $\pi\colon \mathbb{P}(E) \to B$ shall denote the space of complex *hyperplanes* in the fibres of E. Thus $\pi_* \mathcal{O}_{\mathbb{P}(E)}(r) = S^r E$ for $r \geq 0$.

Notation 1. For L a line bundle (not necessarily ample) and $\mathcal{F}$ a coherent subsheaf over B, one can define the slope of $\mathcal{F}$ by the normalised degree of $\mathcal{F}$, i.e

$$\mu_L(\mathcal{F}) = \frac{\deg_L(\mathcal{F})}{rk(\mathcal{F})} = \frac{c_1(L)c_1(\mathcal{F})}{rk(\mathcal{F})},$$

and the normalised Hilbert polynomial by

$$\mathcal{P}_{\mathcal{F}}(k) = \frac{\chi(\mathcal{F}\otimes L^k)}{\mathrm{rk}(\mathcal{F})}.$$

We recall some well known definitions about stability of bundles.

Definition 3. Let L be an ample line bundle on the projective manifold B. A vector bundle E is said to be L-Mumford-Takemoto stable if for any proper coherent subsheaf $\mathcal{F}$ of E one has the slope inequality $\mu_L(\mathcal{F}) < \mu_L(E)$.
We say that E is Gieseker stable (resp Gieseker semistable) with respect to L if for all proper coherent subsheaves $F \subset E$ one has the following inequality for the normalized Hilbert polynomials

$$\mathcal{P}_F(k) < \mathcal{P}_E(k) \quad \text{for } k \gg 0 \qquad (\text{resp. } \leq),$$

and strictly Gieseker semistability E is Gieseker semistable but not Gieseker stable. A Gieseker semistable bundle is said to be Gieseker polystable if it is a direct sum of Gieseker stable bundles with respect to the same polarization.

These stability notions are related; using that $\mu_L(F)$ is the leading order term in k of $\mathcal{P}_F(k)$ one sees immediately that

$$\begin{matrix}\text{Mumford}\\ \text{stable}\end{matrix} \Rightarrow \begin{matrix}\text{Gieseker}\\ \text{stable}\end{matrix} \Rightarrow \begin{matrix}\text{Gieseker}\\ \text{semistable}\end{matrix} \Rightarrow \begin{matrix}\text{Mumford}\\ \text{semistable}\end{matrix}.$$

For simplicity we will work in the sequel of the paper with rank 2 vector bundles over surfaces.

Notation 2. Let us assume that the vector bundle E has rank $rk(E) = 2$ and B is a surface. We set

$$\begin{aligned}\delta_L &= \mu_L(E) - \mu_L(F)\\ \Delta &= \frac{\mathrm{ch}_2(E)}{2} - \mathrm{ch}_2(F) + \frac{1}{2}\delta_{K_B^*}\end{aligned}$$

so that one can write $\mathcal{P}_E(k) - \mathcal{P}_F(k) = k\delta_L + \Delta$.

In the following proposition, we express the Donaldson-Futaki invariant for the polarization $\mathcal{L}_{r,m}$ associated to the test configuration we have just described.

Proposition 1. The Donaldson-Futaki invariant of the test configuration $\mathcal{T}$ for a rank 2 vector bundle E over a polarized surface (B, L) induced by the deformation to the normal cone of $\mathbb{P}F$ where F is a subbundle of E is given by

$$F_1(\mathcal{T}) = \frac{r^6}{36}(\delta_{K_B^*})^2 - \frac{r^4}{72}\Gamma_1\delta_{K_B^*} + \frac{r^3}{24}\Gamma_2\left(m\delta_L + r\mathit{\Delta}\right),$$

with

$$\begin{aligned}\Gamma_1 &= r^2(c_1(E)^2 - 4c_1(F)^2) + 3c_1(F^r \otimes L^m)^2 + 4r^2\Delta + 12rm\delta_L \\ &\quad -3rc_1(B)c_1(F^r \otimes L^m), \\ \Gamma_2 &= (rc_1(E) + 2mc_1(L))^2 - 2rc_1(F^r \otimes L^m)c_1(B).\end{aligned}$$

Proof. The proposition is a consequence of [11, Proposition 19 - Corollary 21] where it is proved by a direct computation that

$$e_{4,3}(\mathcal{T}) = F_1(\mathcal{T}) = C_1 r^3 m^3 + C_2 r^4 m^2 + C_3 r^5 m + C_4 r^6 \tag{3}$$

where

$$\begin{aligned}
C_1(E,F) &= \frac{c_1(L)^2}{6}\left(\mu_L(E) - \mu_L(F)\right), \\
C_2(E,F) &= \frac{c_1(L)^2}{48}(c_1(E) - 2c_1(F))c_1(B) \\
&\quad + \frac{c_1(L)^2}{12}(\mathrm{ch}_2(E) - 2\mathrm{ch}_2(F)) \\
&\quad + \frac{1}{12}\left(2c_1(E)c_1(L) - c_1(B)c_1(L)\right)\left(\mu_L(E) - \mu_L(F)\right), \\
C_3(E,F) &= -\frac{1}{12}\deg_L(E)c_1(F)^2 + \frac{1}{12}\deg_L(E)\mathrm{ch}_2(E) \\
&\quad + \frac{1}{48}\deg_L(E)c_1(E)^2 - \frac{1}{24}\deg_L(F)c_1(E)^2 \\
&\quad + \frac{1}{24}c_1(L)c_1(B)\cdot c_1(F)^2 - \frac{1}{24}c_1(L)c_1(B)\cdot\mathrm{ch}_2(E) \\
&\quad + \frac{1}{24}\deg_L(F)c_1(E)c_1(B) - \frac{1}{24}\deg_L(E)c_1(B)c_1(F), \\
C_4(E,F) &= \frac{1}{288}c_1(E)^2\cdot c_1(B)c_1(E) - \frac{1}{144}c_1(E)^2\cdot c_1(B)c_1(F) \\
&\quad + \frac{1}{48}c_1(F)^2\cdot c_1(E)c_1(B) - \frac{1}{72}\left(c_1(B)c_1(F) + c_1(E)c_1(B)\right)\mathrm{ch}_2(E) \\
&\quad + \frac{1}{48}c_1(E)^2\left(\mathrm{ch}_2(E) - c_1(F)^2\right).
\end{aligned}$$

By a simple algebraic manipulation one obtains from (3) the expected result. □

Proposition 2. In the same setting as in Proposition 1 and with Notations 2, the Chow weight associated to the test configuration $\mathcal{T}$ is given by

$$\begin{aligned}\mathrm{Chow}_s(\mathcal{T}) = e_4(s) = &\frac{sr^4(rs-1)(rs+1)}{36}\delta_{K_B^*}{}^2 \\ &- \frac{sr^2(rs+1)}{72}A_1\delta_{K_B^*} \\ &+ \frac{sr^2(rs+1)}{24}A_2\left(m\delta_L + r\Delta\right)\end{aligned}$$

with

$$\begin{aligned}A_1 &= sr\Gamma_1 - A_1' \\ A_2 &= s\Gamma_2 - 4r\mathrm{Todd}_2(B).\end{aligned}$$

where we set $A_1' = \Gamma_1 + 3c_1(F^r \otimes L^m)^2 + 3rc_1(B)c_1(F^r \otimes L^m) + 6\mathrm{Todd}_2(B)$. Moreover,

$$\mathrm{Chow}_s = s^3 F_1 + s^2 F_2 + sF_3$$

with higher Futaki invariants F_2, F_3 given by

$$\begin{aligned} F_2 &= \left(\frac{1}{r}F_1 + rF_3\right), \\ F_3 &= -\frac{1}{36}r^4 {\delta_{K_B^*}}^2 + \frac{1}{72}r^2 A_1' \delta_{K_B^*} - \frac{1}{6}r^3 \mathrm{Todd}_2(B)\left(m\delta_L + r\Delta\right), \end{aligned}$$

with $\mathrm{Todd}_2(B)$ the second Todd class of B.

Proof. Writing the weigth of the action as $w(s) = \sum_{l=0}^{n+1} b_l s^{n+1-l}$ and $P(s) = \dim H^0(\mathbb{P}E, \mathcal{L}^s_{r,m}) = \sum_{l=0}^{n} a_l s^{n-l}$ with $n = 3$ and s large enough (see Section 1), we get

$$e_4(s) = \sum_{l=1}^{3} (b_0 a_l - a_0 b_l) s^{4-l} - a_0 b_4.$$

In the case we are considering, we have

$$\begin{aligned} a_0 =& \frac{1}{2}rm^2 c_1(L)^2 + \frac{1}{2}mr^2 \deg_L(E) + \frac{1}{6}r^3 \mathrm{ch}_2(E) + \frac{1}{12}r^3 c_1(E)^2, \\ a_1 =& \frac{r^2}{4}c_1(E)c_1(B) + \frac{m^2}{2}c_1(L)^2 + \frac{rm}{2}(c_1(L)c_1(B) + \deg_L(E)) + \frac{r^2}{2}\mathrm{ch}_2(E), \\ a_2 =& -\frac{r}{12}c_1(E)^2 + r\mathrm{Todd}_2(B) + \frac{r}{4}c_1(E)c_1(B) + \frac{m}{2}c_1(L)c_1(B) + \frac{r}{3}\mathrm{ch}_2(E), \\ a_3 =& \mathrm{Todd}_2(B), \end{aligned}$$

and

$$\begin{aligned} b_0 =& \frac{r^4}{24}c_1(E)^2 + \frac{r^4}{12}c_1(F)^2 + \frac{m^2 r^2}{4}c_1(L)^2 + \frac{mr^3}{6}(\deg_L(E) + \deg_L(F)), \\ b_1 =& \frac{r^3}{4}c_1(F)^2 + \frac{r^3}{12}c_1(F)c_1(B) + \frac{r^3}{12}c_1(E)c_1(B) + \frac{rm^2}{4}c_1(L)^2 \\ &+ \frac{mr^2}{4}(c_1(L)c_1(B) + 2\deg_L(F)), \\ b_2 =& \frac{r^2}{2}\mathrm{Todd}_2(B) + \frac{r^2}{6}c_1(F)^2 - \frac{r^2}{24}c_1(E)^2 + \frac{r^2}{4}c_1(F)c_1(B) \\ &+ \frac{rm}{3}\deg_L(F) - \frac{rm}{6}\deg_L(E) + \frac{rm}{4}c_1(L)c_1(B), \\ b_3 =& \frac{r}{2}\mathrm{Todd}_2(B) + \frac{r}{6}c_1(F)c_1(B) - \frac{r}{12}c_1(E)c_1(B), \\ b_4 =& 0. \end{aligned}$$

We refer to [11, Proposition 20] and [3] for the details of computing the terms a_l, b_l where most of them have been explicitly identified using Hirzebruch-Riemann-Roch theorem.

□

We dress now some easy consequences of the two previous results. We get the following theorem which strengthens [11, Proposition 21].

Theorem 1. Consider E an irreducible rank 2 holomorphic vector bundle on a polarized surface (B, L) with $c_1(B)$ proportional to $c_1(L)$.

(1) Assume that E is strictly Gieseker semistable and F is a subbundle of E with $\mathcal{P}_F = \mathcal{P}_E$ with respect to L. Then all the tensor powers of the polarization $\mathcal{L}_{r,m}$ are not Chow polystable, $\mathcal{L}_{r,m}$ is not asymptotically Chow polystable and not K-polystable.
(2) Assume that E is not Gieseker semistable and F is a destabilizing subbundle. Then $\mathcal{L}_{r,m}$ is not K-semistable and thus not asymptotically Chow semistable for $m \gg 0$.
(3) If $\mathcal{L}_{r,m}$ is K-stable (resp. K-polystable, resp. K-semistable) for all $m \gg 0$ then E is Gieseker stable (resp. Gieseker polystable, resp. strictly Gieseker semistable) with respect to L.

Proof. For (1), we consider the test configuration $\mathcal{T}$ of the deformation to the normal cone of $\mathbb{P}F$ described as before. From our assumption of Gieseker semistability we have $\delta_L = \Delta = 0$ while the assumption on the first Chern class gives $\delta_{K_B^*} = 0$ since $c_1(B) = 0$ or $c_1(B) = \lambda c_1(L)$. Therefore from Propositions 1 and 2, one has $F_1(\mathcal{T}) = \mathrm{Chow}_s(\mathcal{T}) = 0$ while the test configuration $\mathcal{T}$ is not a product test configuration. The point (2) can be treated in a similar way using the proof of Proposition 1. Actually the destabilizing subbundle leads to $C_1 = 0$ and $C_2 < 0$ or $C_1 < 0$ and thus $F_1(\mathcal{T}) < 0$. Remark that (2) strengthens a result of [17, Theorem 5.12] where it is shown that if E is not Mumford stable then $\mathcal{L}_{r,m}$ is not K-semistable.
Note that under the assumptions of (1) or (2), there is no Kähler metric with constant scalar curvature in the class $[c_1(\mathcal{L}_{r,m})]$ as a consequence of [14, 19, 4].
Now let us assume that $\mathcal{L}_{r,m}$ is K-stable. Then $C_1 \geq 0$ in the proof of Proposition 1 for all subbundles F of E. If the inequality is strict for any subbundle then E is Mumford stable. Actually, for a rank 2 bundle over a surface, it is sufficient to test stability with respect to subbundles. For any rank 1 torsion free subsheaf $\mathcal{F}$ of E, $\mathcal{F}^{**}$ is a reflexive rank 1 sheaf on the surface B and thus a line bundle. Now, if $C_1 = 0$ for a subbundle F of E, one has necessarily $C_2 \geq 0$. If $C_2 > 0$ then $\mathcal{P}_E > \mathcal{P}_F$. Now given $\mathcal{F}$ rank 1 torsion free subsheaf of E, one has $\mathcal{F} = F \otimes \mathcal{I}$ where F is a line bundle and $\mathcal{I}$ is an ideal sheaf with 0-dimensional support, the inequality $\mathcal{P}_E > \mathcal{P}_F$ only improves if F is replaced by $\mathcal{F}$ since $c_2(\mathcal{F})$ is the length of the support of $\mathcal{I}$ and thus is non-negative. Eventually if the inequality $C_2 > 0$ holds for all subbundles of E, then we have obtained that E is Gieseker stable. Consider now that $C_2 = 0$. Then we have $\delta_L = \delta_{K_B^*} = \Delta = 0$ and by Proposition 1, $F_1(\mathcal{T})$ vanishes. But the test configuration is not trivial so this leads to a contradiction. Therefore one has necessarily $C_2 > 0$ and we obtain Gieseker stability. The case of K-semistability is obtained by contraposition of (2).
In the case of K-polystability, the only case for which $C_2 = 0$ is when the rank 2 bundle E splits as a direct sum of two line bundles of same slope so is necessarily Mumford polystable. Since $C_3 \geq 0$, one has moreover Gieseker semistability. □

Remark that the case of K-unstability in (3) cannot be included since the base manifold B may be K-unstable which would induce a destabilizing test configuration for the projectivisation $\mathbb{P}E$.

Non simple semi-homogeneous rank 2 vector bundles over an abelian surface are Gieseker semistable and thus provide concrete examples of applications of our theorem, see [15, Section 6].

Conjecture 1. Consider E an irreducible rank 2 holomorphic vector bundle on a K-stable polarized surface (B, L) with $c_1(B)$ proportional to $c_1(L)$. For $m \gg 0$, the polarization $\mathcal{L}_{r,m}$ is K-stable (resp. K-polystable, resp. K-semistable) if and only E is Gieseker stable (resp. Gieseker polystable, resp. Gieseker semistable).

The conjecture is wrong if one removes the assumption on the first Chern class of B: in [11] it is constructed an example of a Gieseker stable bundle with $\mathcal{L}_{1,m}$ not K-semistable for $m \gg 0$. The hard sense of the conjecture is true under stronger assumption: on a surface with a constant scalar curvature Kähler metric and no non trivial holomorphic vector field, a Mumford stable bundle gives rise to a polarization $\mathcal{L}_{r,m}$ that admits a constant scalar curvature Kähler metric and thus is K-stable, see [8, 9, 10].

One can now wonder when the Futaki invariant as computed in Proposition 1 may vanish. We cannot say much for a fixed couple (r, m) but at the fiber or base limit we obtain the following result.

Proposition 3. Let (B, L) be a polarized surface such that its first Chern class satisfies $c_1(B) = 0$ or $c_1(B)c_1(L) \neq 0$ and E a rank 2 holomorphic vector bundle on B. Then, for the test configuration as in Proposition 1,

- the Futaki invariant $F_1(\mathcal{T})$ vanishes for all $m \gg 0$ (or all $r \gg 0$) if and only if the Chow weight $\mathrm{Chow}_s(\mathcal{T})$ vanishes for all $m \gg 0$ and any fixed $s > 0$ (or all $r \gg 0$ and $s \gg 0$).
- the Futaki invariant $F_1(\mathcal{T})$ is positive for all $m \gg 0$ if and only if the Chow weight $\mathrm{Chow}_s(\mathcal{T})$ is positive for all $m \gg 0$ and $s \gg 0$.

Proof. This comes from the computations of the Futaki invariant and Chow weight. Imposing $C_1 = C_2 = C_3 = 0$ in Proposition 1 implies firstly that $\delta_L = 0$, then $\Delta = \frac{1}{4}\delta_{K_B^*}$ and finally $\delta_{K_B^*}c_1(L)c_1(B) = 0$. Under our assumptions one gets in all the cases

$$\delta_L = \delta_{K_B^*} = \Delta = 0. \tag{4}$$

This forces obviously the Chow weight to vanish, see Proposition 2.
Conversely, if the Chow weight vanishes seen as a polynomial in the variables m, one gets from Proposition 2 that $\Delta = \frac{kr-2}{4kr}\delta_{K_B^*}$ and $\delta_{K_B^*}c_1(L)c_1(B) = 0$ and thus (4) holds which implies the vanishing of the Futaki invariant. Computations in the variables r are similar but slightly more involved. The second part of the result is using the same reasoning. □

Next we compute the Hilbert weight for the test configuration $\mathcal{T}$ for the deformation to the normal cone of $\mathbb{P}F$ where F is a subbundle of E. We remark that the Hilbert weight has a similar expression to the Chow weight and the Futaki invariant.

Proposition 4. In the same setting as in Proposition 1 and with Notations 2, the Hilbert weight associated to the test configuration $\mathcal{T}$ is given by

$$\begin{aligned}\mathrm{Hilb}_{s,k}(\mathcal{T}) =& \frac{r(rs-1)(rk+1)}{36}\beta_1(s,r)\delta^2_{K_B^*}\\ &+\frac{1}{72}(\beta_1(s,r)B_1-\beta_2(s,r)A_1)\delta_{K_B^*}\\ &+\left(\frac{\beta_2(s,r)}{24}A_2-\frac{(rk+2)\beta_1(s,r)}{6}\mathrm{Todd}_2(B)\right)(m\delta_L+r\Delta)\end{aligned}$$

with $\beta_1(s,r)=rks(rs+1)(k-s)(rk+1)$, $\beta_2(s,r)=rs^3(rs+1)^2(k-s)$, and

$$\begin{aligned}B_1 =& kr^2(c_1(E)^2+2c_1(F)^2+4\Delta+6\mathrm{Todd}_2(B))\\ &+6krm\deg_L(E)+6km^2c_1(L)^2\\ &+r(-c_1(E)^2+6\mathrm{Todd}_2(B)+8\Delta+6c_1(F)c_1(B)+4c_1(F)^2)\\ &+6mc_1(L)c_1(B)\end{aligned}$$

Proof. The result is obtained by a computation of the weight $\mathrm{Hilb}_{s,k}(\mathcal{T})=\tilde{w}(s,k)$ using (1) and the computations of a_i, b_i in Proposition 2. □

Proposition 3 can also be extended to Hilbert weights. We have also another obvious consequence.

Proposition 5. In the same setting as in Proposition 1, let us assume that $c_1(B)=0$. Then the Chow weight $\mathrm{Chow}_s(\mathcal{T})$ and the Hilbert weight $\mathrm{Hilb}_{s,k}$ are proportional to the Futaki invariant $F_1(\mathcal{T})$, and have same sign when one takes $k,s>0$ large enough.

Proof. This comes from the fact that when $c_1(B)=0$ one has $\delta_{K_B^*}=0$ and both quantities Γ_2 and A_2 do not depend on the bundle F. □

3. Strictly semistable examples

Inspired from [2], we construct a new example of a threefold which is Asymptotically Chow semistable and not Asymptotically Chow stable.

Let (B,L) be a polarized surface such that $c_1(L)$ admits a Kähler metric with constant scalar curvature and $Aut(B,L)/\mathbb{C}^\times$ is trivial and assume that the torus $\mathrm{Pic}^0(B)=H^1(B,\mathcal{O})/H^1(B,\mathbb{Z})$ parametrizing line bundles with trivial first Chern class is not trivial. Consider $E_0=G_1\oplus G_2$ a direct sum of two line bundles with $c_1(G_1)=c_1(G_2)$ over B. Then E_0 is Mumford polystable. On the polarized ruled manifold

$$(X_0,\mathcal{L}^0_{r,m})=(\mathbb{P}E_0,\mathcal{O}_{\mathbb{P}E_0}(r)\otimes\pi_0^*L^m)$$

there exists under our assumptions a Kähler metric with constant scalar curvature for all $m\gg 0$. Actually, the Futaki character associated to the Lie algebra $Lie(Aut(E_0)/\mathbb{C}^\times)$ vanishes thanks to Proposition 1, and one can apply [9, Corollary B]. Therefore, $(X_0,\mathcal{L}^0_{r,m})$ is K-polystable for all $m\gg 0$ from the work of Donaldson, Stoppa and Mabuchi [14, 19, 4].

Next, we do a small deformation of the trivial line bundle $T_0=\mathbb{C}\times B$ in order to obtain a line bundle T over B such that T^2 is non trivial. We can consider the following induced extension

$$0\to G_1\otimes T\to E\to G_2\otimes T^*\to 0. \tag{5}$$

Using Riemann-Roch formula we have $h^0(B, G_1 \otimes G_2^* \otimes T^2) - h^1(B, G_1 \otimes G_2^* \otimes T^2) + h^2(B, G_1 \otimes G_2^* \otimes T^2) = \mathrm{Todd}_2(B)$ since $c_1(G_1) = c_1(G_2)$. Now, if we assume $\mathrm{Todd}_2(B) < 0$, the space $Ext^1(G_2 \otimes T^*, G_1 \otimes T) = H^1(B, G_1 \otimes G_2^* \otimes T^2)$ has positive dimension and our extension (5) does not split. The ruled manifold

$$(X, \mathcal{L}_{r,m}) = (\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(r) \otimes \pi^* L^m)$$

is not K-polystable for $m \gg 0$. Actually for the choice $F = G_1 \otimes T$ one checks that the Futaki invariant $F_1(\mathcal{T})$ associated to the test configuration to the normal cone of $\mathbb{P}F$ vanishes for $m \gg 0$. Furthemore one obtains $\delta_L = \delta_{K_B^*} = \Delta = 0$. These relationships impose that the Chow weight Chow_s vanishes by Proposition 2. Therefore, $(X, \mathcal{L}_{r,m})$ cannot be asymptotically Chow stable.
On another hand, from the fact that all the higher Futaki invariants F_2, F_3 vanish simultaneously we can apply Mabuchi's main result in [12] (see also [3, Proposition 3.2, Theorem 3.5]). One concludes that $(X_0, \mathcal{L}_{r,m}^0)$ is asymptotically Chow polystable. By openness of the semistability condition in GIT, its small deformations are asymptotically Chow semistable and consequently $(X, \mathcal{L}_{r,m})$ is asymptotically Chow semistable.
Finally, in order to construct base manifolds that satisfy the assumptions as above, it is sufficient to consider for B a ruled surface as the projectivisation of a rank 2 Mumford stable bundle over a curve of genus > 1, see [11]. We have proved the following result.

Corollary 1. There are some ruled threefolds (projectivisation of rank 2 bundles over a surface endowed with a constant scalar curvature Kähler metric) that are asymptotically Chow semistable, but not asymptotically Chow stable.

One can also compare Corollary 1 with [22, Section 5] where other examples of non asymptotically Chow stable threefolds are discussed.

Since $(X_0, \mathcal{L}_{r,m}^0)$ is asymptotically Chow polystable, for the test configurations that have positive Chow weight asymptotically, the main result of [13] shows that they have also positive Hilbert weight asymptotically. Thanks to our assumptions on B, the product test configurations that have vanishing Chow weight Chow_s for $s \gg 0$ are associated to the splitting of E_0 and the deformation to the normal cone of $\mathbb{P}G_1$ or $\mathbb{P}G_2$. Thus one gets in both case for $m \gg 0$ that $\delta_L = \Delta = \delta_{K_B^*} = 0$. Proposition 4 shows that the Hilbert weight also vanishes. Consequently, $(X_0, \mathcal{L}_{r,m}^0)$ is asymptotically Hilbert polystable and thus its small deformation $(X, \mathcal{L}_{r,m})$ is also asymptotically Hilbert semistable. On another hand, considering the subbundle $F = G_1 \otimes T$ of E, one has for the test configuration associated to the deformation to the normal cone of $\mathbb{P}F$ that $\delta_L = \delta_{K_B^*} = \Delta = 0$ and so $\mathrm{Hilb}_{s,k} = 0$ for all s, k. Finally, $(X, \mathcal{L}_{r,m})$ for $m \gg 0$ cannot be asymptotically Hilbert stable since $\mathcal{T}$ is not a product test configuration.

Corollary 2. There are some ruled threefolds (projectivisation of rank 2 bundles over a surface endowed with a constant scalar curvature Kähler metric) that are asymptotically Hilbert semistable, but not asymptotically Hilbert stable.

Note that using [3, Proposition 4.1 and Corollary 4.4] our reasoning could also be applied to the case of Mumford semistable vector bundle over a curve of genus ≥ 2 to produce other similar examples to Corollaries 1 and 2. This will be discussed in more details in a forthcoming paper since one can be a little bit more precise in dimension one. For instance the following conjecture is true if the base manifold is a curve of genus $g > 1$.

Conjecture 2. Consider E a holomorphic vector bundle on a base manifold B polarized by L with $c_1(B) = 0$ or $c_1(B)c_1(L) \neq 0$. Then for $m \gg 0$, the following assertions are equivalent:

- the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is asymptotically Hilbert semistable,
- the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is asymptotically Chow semistable,
- the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is K-semistable.

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