Mémoire d'Habilitation à Diriger des Recherches

Mention Mathématiques et Applications

par

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Titre de la Thèse :

Canonical algebraic metrics and applications to various problems in Kähler geometry

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Part I

Overview - Présentation Générale

Chapter 1

Introduction

Dans cette thèse d'habilitation, nous présentons plusieurs résultats de géométrie complexe qui ont pour dénominateur commun l'utilisation de techniques de géométrie algébrique complexe dans le but d'appréhender l'existence de métriques Kählériennes à courbure spéciale sur des variétés algébriques projectives sur $\mathbb{C}$. En géométrie Kählérienne, certaines EDP non linéaires, qui proviennent le plus souvent de considérations des physiciens, sont particulièrement délicates à étudier. Il a été parfois possible d'en comprendre l'essence à l'aide de flots de la chaleur, des méthode de continuité ou des méthodes de viscosité mais ces techniques ont le désavantage d'être non constructives le plus souvent.

D'un autre côté, la quantification géométrique est l'art d'associer à un système classique physique décrit comme la donnée d'une variété symplectique et des observables (des fonctions à valeurs réelles sur cette variété), des espaces de Hilbert et des opérateurs hermitiens sur ces espaces. Ces espaces de Hilbert (les états quantiques) et les opérateurs hermitiens associés sont paramétrés par h, la constante de Planck, qui à la limite $h \to 0$ permettent de "reconstruire" le système classique originel et les observables selon le principe dit de correspondance. Comme cela apparut il y a un demi-siècle dans les travaux pionniers de J-M. Souriau [Sou67], F.A. Berezin, M. Cahen, S. Gutt, J. Rawnsley et beaucoup d'autres, la donnée d'une métrique à courbure positive sur un fibré en droites L^k au dessus d'une variété compacte X induit une quantification géométrique et dans ce cadre h s'identifie à $1/k$ où k est le paramètre de tensorisation du fibré. Dans le cadre compact et algébrique complexe, l'espace de Hilbert considéré est juste l'espace $H^0(X, L^k) = H^0(L^k)$ des sections holomorphes de $L^k \to X$. Le principe de correspondance s'exprime notamment par le fait que la dimension grandit asymptotiquement comme $k^n\mathrm{Vol}$ où Vol est le volume de l'espace classique des phases et n la dimension de X. Le fait crucial que l'espace de Hilbert soit de dimension finie va considérablement simplifier les choses. Il va nous permettre dans un premier temps de développer un formalisme d'application moment en dimension finie qui se réinterprète en termes de Théorie des Invariants Géométriques à la D. Mumford par les travaux classiques de G. Kempf- L. Ness. Cette nouvelle interprétation donnera des critères purement algébriques d'existence de solutions aux EDP considérées originellement. Il va aussi permettre dans un second temps d'obtenir des algorithmes implémentables et donc des approximations numériques des objets transcendants solutions de ces EDP, lorsque ces solutions existent a priori. Tout comme ce fut le leitmotiv de [Kel07], c'est l'idée générale de cette thèse que nous avons découpée en plusieurs parties qui sont en fait complémentaires.

Dans la Partie I (Chapitre 2), nous expliquons brièvement comment la quantification géométrique intervient dans le programme de S.K. Donaldson pour la conjecture dite de Yau-Tian-Donaldson au sujet de l'existence de métriques Kähler à courbure scalaire constante sur une variété projective complexe. C'est aussi l'occasion de fixer nos notations et de rappeler certaines définitions classiques qui seront utilisées dans le reste de la thèse.

Dans la partie II, nous expliquons comment il est possible d'obtenir à partir de la quantification géométrique et de la Théorie des Invariants Géométriques de nouveaux flots naturels sur l'espace de dimension infinie des potentiels Kähler. Dans le Chapitre 3, nous utilisons cette idée pour donner une nouvelle preuve du théorème de S.T. Yau (solution de la conjecture de Calabi). Ce chapitre se base sur la publication [CaoKel12] et quelques résultats de [Kel09]. Dans le Chapitre 4, en utilisant les techniques du Chapitre 3, nous étudions le J-flot de S.K. Donaldson et introduisons la notion de métrique équilibrée adaptée à ce contexte. L'objectif est ici d'obtenir une condition algébrique simple pour détecter des chambres dans le cône Kähler d'une variété projective qui contiennent des métriques à courbure scalaire constante ou K-polystables comme il est suggéré dans [Kel11].

En vue de la conjecture de Yau-Tian-Donaldson, il est naturel d'étudier des exemples concrets de variétés et de vérifier sous quelles conditions elles admettent des métriques à courbure scalaire constante. On pense au cas des surfaces, au cas des variétés toriques, ou encore au cas des variétés réglées. Dans la Partie III, nous regardons en détails le cas des variétés réglées données comme projectivisation de fibrés de rang 2 sur des surfaces. Les cas qui nous intéressent ici sont les cas "limite" des fibrés Mumford semistables qui sont Gieseker stables. Dans ce contexte, des phénomènes intéressants de stabilité apparaissent. Ils soulignent la subtilité de la conjecture de Yau-Tian-Donaldson. Par exemple, nous trouvons des exemples de variété Chow stables non asymptotiquement Chow stables. Ceci fait l'objet de l'étude du Chapitre 5 qui a été publié dans [KelRos12]. Dans le chapitre suivant, Chapitre 6, nous présentons certains raffinements, étudions des exemples concrets de manière algébrique et donnons quelques conjectures. En particulier nous trouvons des exemples de variétés strictement Hilbert ou Chow semistables asymptotiquement. Une grande partie de cette section a été publiée dans [Kel14b]. Puisque sur certaines variétés réglées, il n'existe pas de métriques à courbure scalaire constante, il est naturel de se demander quelles sont les métriques canoniques dont jouissent de telles variétés. On peut tout d'abord penser aux métriques extrémales (au sens de la fonctionnelle de Calabi) et elles furent étudiées dans les travaux de V. Apostolov, D. Calderbank, P. Gauduchon and C. Tønnesen-Friedman à l'aide du formalisme des 2-formes hamiltonniennes et des métriques admissibles dans le cas de sommes de fibrés stables au dessus des courbes. Toujours dans le cadre de métriques lisses, on peut imaginer des structures géométriques supplémentaires comme cela fut étudié dans [Gar09; KelTøn12]. Dans le Chapitre 7, qui se base sur [Kel14a], nous verrons que l'on peut alors introduire des métriques Kähler à courbure scalaire constante mais avec singularités (à singularités coniques) sur la projectivisation de fibrés semistables au dessus de courbes.

Dans la Partie IV, nous donnons des applications algorithmiques en calculant une méthode pour approcher la métrique de Weil-Petersson sur l'espace des modules de variétés Ricci plates. Nous montrons la pertinence de cette méthode en étudiant le cas des quintiques de $\mathbb{CP}^4$ et en comparant nos résultats avec les résultats bien connus des physiciens "cordistes" (Chapitre 8). Ce chapitre provient de [KelLuk12]. Par souci de cohérence et de concision, nous avons décidé de pas présenter les résultats de la publication [Kel09] où, entre autres choses, sont donnés des algorithmes sur les variétés toriques Fano. Leurs généralisations permettent en fait de résoudre numériquement certaines questions de transport optimal (deuxième problème avec donnée au bord pour l'équation de Monge-Ampère réelle).

Enfin, dans la Partie V, nous expliquons quels axes de recherche peuvent être développés à moyen et long terme à partir des résultats que nous avons prouvés dans les parties précédentes.

Chapter 2

A (very) brief survey about the constant scalar curvature problem in Kähler geometry

2.1 Historical background

Kähler geometry is at the intersection of various fields of research in pure mathematics and is a very active world for the last 40 years. Without being exhaustive, it is intrinsically related to symplectic geometry, complex analysis, algebraic geometry, Riemannian geometry, PDE analysis, deformation theory, quantization and has applications in all these fields and also in others like mathematical physics via String Theory. The origin of this extraordinary relationship lies in the very definition of a Kähler manifold that allows one to define the metric tensor using simply one potential function, implying a long list of "miracles". We refer to J-P. Bourguignon's enlightening paper "The unabated vitality of Kählerian geometry" (Mathematical works of E. Kähler, de Gruyter 2003) where the importance of the quest of Kähler metrics with special curvature properties and the impact of Kähler geometry on different fields are stressed.

Coming back to the definition of a Kähler manifold, one can ask if there are natural/canonical metric in a given Kähler class. In the early 80's, Calabi stated precisely this question and suggested Kähler extremal metrics as candidates. By extremal metric, we mean an extremal metric for the functional given by the L^2-norm of the scalar curvature, the trace of the Ricci curvature. These metrics turn out to be solutions of a 4-th order nonlinear PDE. Kähler Ricci-flat, Kähler-Einstein and constant scalar curvature Kähler metrics are the most commonly known examples of extremal Kähler metrics. Since the breakthrough of S. T. Yau in the 70's about Einstein's equations of general relativity, most efforts in this field are related to the so-called Yau-Tian-Donaldson conjecture.

The problem of finding a best metric on a given manifold goes back to the fundamental work of B. Riemann that led to the uniformization of Riemann surfaces. It is actually expected that special (Kähler) metrics give tools that allow geometers to classify (Kähler) manifolds, providing information on their topology and their underlying geometric complex structure. In a certain sense, Calabi's question is the natural extension to the complex world of R. Thom's, formulated in the fifties, "what are there best (or nicest or most distinguished) Riemannian structures on a smooth differentiable manifold?" which led generations of great geometers to the quest of Einstein metrics, see [Bes08].

A long time ago, S.T. Yau [Yau93] foresaw that, similar to the case of vector bundles, there should be an equivalence between an algebraic notion of stability for a polarized manifold and existence of special metrics in the Kähler class defined by the polarization. Actually, for holomorphic vector bundles, such a correspondence was proved (see [LT95]): an irreducible holomorphic hermitian vector bundle E is Mumford-Takemoto stable if and only if there exists an Hermitian-Einstein metric on E. When the considered Kähler class is integral, several algebraic notions of stability were introduced and tested (Chow stability, Hilbert stability, (uniform) K-stability, relative K-stability, b-stability etc., see Section 2.5) in view of Yau's idea. This led to the so-called Yau-Tian-Donaldson conjecture that predicts that K-polystability should be equivalent to the existence of a constant scalar curvature Kähler metric.

If the Yau-Tian-Donaldson conjecture for extremal metrics holds, it means that there is an algebraic method to test if a given integral Kähler class carries an extremal Kähler metric. This would be a very surprising result since the extremal Kähler condition is equivalent to solving a highly non-linear PDE. Moreover, it would set up a cartography of the Kähler cone - a fundamental object in complex geometry - of the underlying manifold in terms of stable chambers and walls. One needs to keep in mind that for the more linear case of vector bundles, the famous and fundamental Kobayashi-Hitchin-Donaldson-Uhlenbeck-Yau correspondence led to important and unexpected discoveries in topology (topological invariants,..), symplectic geometry, algebraic geometry (moduli space problems,..), complex geometry (classification of complex surfaces,..), mathematical physics (gauge theory,..), etc.

2.2 Notations and conventions

In the sequel, we will denote by M, X, $\mathcal{X}$, or B a complex manifold[1] (we will specify in the rare cases for which we need to allow singularities).

[1] We shall also use sometimes Σ for a surface and Q for a quintic threefold.

By a polarization, usually denoted L or $\mathcal{L}$, we mean an ample holomorphic line bundle on the considered manifold. Such bundle defines an (integral) Kähler class, denoted $c_1(L)$. On such bundles, it is well known that there exists a hermitian smooth metric h_L with positive curvature. This Chern curvature is denoted $c_1(h_L)$; it is a Kähler $(1,1)$-form that we usually denote also by $\omega = \frac{\sqrt{-1}}{2\pi}c_1(h_L)$ (we will omit the factor $\frac{\sqrt{-1}}{2\pi}$).

By $\mathrm{Vol}_L(M) = c_1(L)^n$, we denote the volume of the complex manifold M of complex dimension n with respect to the polarization L, omitting the subscript L when there is no possible confusion. Similarly we will write $H^i(L^k)$ for $H^i(M, L^k)$ when it is clear on which manifold we work with.

By $\mathrm{Met}(V)$ (resp $\Gamma(V)$) we denote the space of smooth hermitian metrics (resp. smooth sections) on the bundle V or the complex vector space V. We denote by $(.,.)$, $\langle .,. \rangle$ or $h(.,.)$ a hermitian inner-product (the fibrewise metrics are denote by h with subscripts).

By $\mathbb{P}^N = \mathbb{CP}^N$, we denote the complex projective space of complex dimension N.

When we write $k >> k_0$ we mean that we choose k large enough and larger than the integer k_0.

We use some abbreviations. For instance "cscK metric" stands for constant scalar curvature Kähler metric, and "G.I.T" for Geometric Invariant Theory.

2.3 Scalar curvature as moment map

Consider (X, ω) a finite dimensional symplectic manifold and for simplicity we shall assume that $H^1(X, \mathbb{R}) = 0$. We can consider the almost complex structures on M, i.e the set $\{J : TX \to TX : J^2 = -\mathrm{Id}\}$. An almost complex structure is compatible with ω if the tensor $\omega(X, JY)$ is symmetric and positive definite. Then, it provides a Riemannian metric $g_J = g_{J,\omega}$ and when J is integrable this metric is actually Kähler. Assuming now that X is also a Kähler manifold, we are interested in the infinite dimensional space $\mathcal{J}$ of almost complex structures on M compatible with ω. Its tangent space at $J \in \mathcal{J}$ consists in maps $v : TX \to TX$ such that $vJ + Jv = 0$ and $\omega(\overrightarrow{X}, v\overrightarrow{Y}) = \omega(\overrightarrow{Y}, v\overrightarrow{X})$ for all $\overrightarrow{X}, \overrightarrow{Y}$. It has a complex structure thanks to the fact that if $v \in T_J\mathcal{J}$, then $Jv \in T_J\mathcal{J}$, and can be equipped with an inner product depending on J,

$$\langle \overrightarrow{X}, \overrightarrow{Y} \rangle = \int_X \omega(\overrightarrow{X}, J\overrightarrow{Y})\frac{\omega^n}{n!}.$$

Thus, by integration, we get a Kähler structure on $\mathcal{J}$ simply by fixing at J,

$$\omega_J^{\mathcal{J}}(\overrightarrow{X}, \overrightarrow{Y}) = \langle J\overrightarrow{X}, \overrightarrow{Y} \rangle.$$

The group of ω-symplectomorphisms of X consists in diffeomorphisms of X preserving the form ω. It is an infinite dimensional Lie group that is a subgroup of the group of diffeomorphisms of X. As we shall see, it plays the role of a gauge group in the symplectic setting and we refer to the enlightening paper [Hit90] on this topic. Its Lie algebra is that of symplectic vector fields, the vector fields such that their Lie derivative annihilates the symplectic form. We denote by

$$\mathrm{Ham}(X, \omega)$$

the subgroup of Hamiltonian symplectomorphisms whose Lie algebra is given by the Hamiltonian vector fields (i.e vector fields such that its contraction with ω is exact). For X connected, there is a central extension of Lie algebras

$$0 \to \mathbb{R} \to C^\infty(X, \mathbb{R}) \to \mathit{Lie}(\mathrm{Ham}(X, \omega)) \to 0$$

which allows to identify $\mathit{Lie}(\mathrm{Ham}(X, \omega))$ with the smooth functions on X that have vanishing integral, the Hamiltonian functions on X. At the level of the structure, this identification is induced by the relation $[\overrightarrow{X}_{h_1}, \overrightarrow{X}_{h_2}] = -\overrightarrow{X}_{\{h_1,h_2\}}$ where $\{h_1, h_2\} = \omega(\overrightarrow{X}_{h_1}, \overrightarrow{X}_{h_2})$ is the Poisson bracket induced by ω for $\overrightarrow{X}_{h_i}$ the Hamiltonian vector field corresponding to h_i, i.e $\imath_{\overrightarrow{X}_{h_i}} \omega = dh_i$, $i = 1, 2$. The group $\mathrm{Ham}(X, \omega)$ acts on $\mathcal{J}$ by pullback of the complex structures i.e $\psi(J) = \psi_* J \psi_*^{-1}$, $\psi \in \mathrm{Ham}(X, \omega)$. This action preserves the Kähler form $\omega^{\mathcal{J}}$, hence one can ask for a moment map in the sense of J-M. Souriau.

A. Fujiki and S.K. Donaldson showed that the action of $\mathrm{Ham}(X, \omega)$ on $\mathcal{J}$ is hamiltonian and that the associated moment map is given by a nice geometric expression, which was a priori unclear:

$$\begin{aligned} \text{Moment map} : \mathcal{J} &\to \mathit{Lie}(\mathrm{Ham}(X, \omega))^* \\ J &\mapsto \mathrm{scal}(g_J) - \mathrm{scal}_0, \end{aligned}$$

with $\mathrm{scal}(g_J)$ the scalar curvature of the metric g_J and scal_0 its average over X (that does not depend on J). Here we see $\mathrm{scal}(g_J) - \mathrm{scal}_0$ as an element of the dual of $\mathit{Lie}(\mathrm{Ham}(X, \omega))^*$ thanks to the L^2 inner product on functions with respect to ω. Note that in the sequel, we restrict our attention to the case of J integrable, otherwise $\mathrm{scal}(g_J)$ is not the Riemannian scalar curvature. We denote by $\mathcal{J}_{int}$ the space of *integrable* almost complex structures compatible with ω. Once non empty, it is an infinite dimensional complex space of $\mathcal{J}$. In particular for X a Calabi-Yau manifold, the zeros of the moment map correspond to the Kähler Ricci-flat metrics. Of course, another way of stating the result of Fujiki and Donaldson is to say that for any $\overrightarrow{Y} \in T_J\mathcal{J}_{int}$ and h a smooth function on X with vanishing integral,

$$\langle D\mathrm{scal}(g_J)(\overrightarrow{Y}), h\rangle_{L^2} = \langle J\overrightarrow{Y}, L_{\overrightarrow{X}_h} J\rangle$$

where $L_{\overrightarrow{X}_h}$ is the Lie derivative with respect to the Hamiltonian vector field $\overrightarrow{X}_h$. We refer to the book (in preparation) of P. Gauduchon (*Calabi's extremal Kahler metrics an elementary introduction*) for details about this computation. Actually, this computation is difficult. In the next section we shall give of hint of deriving this result by a quantization method.

Eventually, we see that moduli of polarized varieties can be formally constructed as an infinite dimensional symplectic quotient, as it appeared in [Don97]. In fact, in the approach we overviewed, we are fixing the form ω and varying the complex structure. One can ask if it is similar to fix the complex structure and vary the form within the Kähler class. If J_1, J_2 are two complex structures with $\gamma^* J_1 = J_2$ then with above notations, $g_{J_2,\omega} = \gamma^* g_{J_1,(\gamma^{-1})^*\omega}$ which are isometric when γ belongs to $\mathrm{Ham}(X,\omega)$. This leads to consider the complexification of $\mathrm{Ham}(X,\omega)$ if we want a non trivial action. But the complexified group $\mathrm{Ham}_c(X,\omega)$ does not exist (at least as a group!) while it is possible to consider the complexified Lie algebra as the space of smooth functions on X with *complex* values and vanishing integral, usually denoted $C_0(X,\mathbb{C})$. It is also possible to think about its orbit as integral submanifolds of a certain distribution in $\mathcal{J}_{int}$ (in other words we have a foliation on $\mathcal{J}_{int}$ by the complexified infinitesimal action). The infinitesimal action by $\sqrt{-1}h$ is just, in the integrable case (op. cit.), the variation $JL_{\overrightarrow{X}_h}(J)$ i.e the action of $J\overrightarrow{X}_h$ on the complex structure[2]. But

$$L_{J\overrightarrow{X}_h}\omega = d\imath_{J\overrightarrow{X}_h}\omega = dJdh = -2\sqrt{-1}\partial\bar{\partial}h$$

and so we recover all the Kähler potential in the Kähler class. Hence we can formally identify the symmetric space $\mathrm{Ham}_c(X,\omega)/\mathrm{Ham}(X,\omega)$ with the space of Kähler potentials (up to normalization) for $[\omega]$, and the 'quotient' of $\mathcal{J}_{int}$ by $\mathrm{Ham}_c(X,\,\omega)$ as the set of isomorphism classes of integrable complex structures on (X,ω). Finally, this means that the problem of finding a cscK metric can be viewed as finding a zero of a certain moment map for the action of $\mathrm{Ham}(X,\omega)$ in the complexified J-orbit. Therefore, it is a particular case of the formalism of moment map type problems for Hamiltonian actions on Kähler manifolds but in *infinite dimension*.

[2]Note that the induced map $\upsilon_J : C_0(X,\mathbb{C}) \to T_J\mathcal{J}_{int}$ is not a Lie algebra homomorphism. Otherwise, $0 = \upsilon_J(\{h,\sqrt{-1}h\}) = [L_{\overrightarrow{X}_h}J, L_{J\overrightarrow{X}_h}J] = -L_{[\overrightarrow{X}_h,J\overrightarrow{X}_h]}J$ would imply that $\overrightarrow{X}_h$ is always a J-holomorphic vector field. This underlines the fact that there is no natural complexification of $Ham(X,\omega)$.

2.4 Quantization of the constant scalar curvature Kähler metrics

In [Don01b], Donaldson showed how this previous infinite dimensional quotient can be thought as the classical limit of a finite dimensional construction. Let us fix a polynomial $\chi(T) \in \mathbb{Q}[T]$ of degree n. One can consider the set $\mathcal{H}_\chi$ formed by couples (X, L) such that X is a projective variety of complex dimension n and L a polarization with Euler-Poincaré characteristic $\chi(X, L)$ satisfying

$$\chi(X, L^k) = \chi(k)$$

for k large enough. For each element (X, L) of $\mathcal{H}_\chi$, one obtains an embedding (which is not unique !) in a fixed projective space

$$\iota\colon X \hookrightarrow \mathbb{P}H^0(X, L^k)^* = \mathbb{P}^N$$

for k large enough, because $\mathcal{H}_\chi$ is a bounded family. From Grothendieck's results [Vie95], there is a quasi-projective scheme $Hilb(N, \chi)$ containing $\mathcal{H}_\chi$, the Hilbert scheme of subschemes of $\mathbb{P}^N$ with fixed Hilbert polynomial χ. Moreover, there is a universal family $Univ_{N,\chi} = \{((X, L), \iota(x)) : x \in X\}$ over $Hilb(N, \chi)$ such that $Univ_{N,\chi} \subset Hilb(N, \chi) \times \mathbb{P}^N$, i.e. one has the diagram

$$\begin{array}{lcl} Univ_{N,\chi} & \xrightarrow{\pi_2} & \mathbb{P}^N \\ \downarrow \pi_1 & & \\ \mathcal{H}_\chi & & \end{array}$$

In the sequel, we restrict our attention to $\mathcal{H}_\chi^\infty$ the subset of $\mathcal{H}_\chi$ formed by smooth projective manifolds. One can consider a natural $(1, 1)$ Kähler form on $\mathcal{H}_\chi^\infty$ by pulling-back the Fubini-Study form from $\mathbb{P}^N$:

$$\Omega_k = \pi_{1*}\left(\frac{(\pi_2^*\omega_{FS})^{n+1}}{(n+1)!} \cap [Univ_{N,\chi}]\right), \tag{2.1}$$

which simply corresponds to write at the point $(X, L) \in \mathcal{H}_\chi^\infty$,

$$\Omega_k(v_1, v_2) = \int_X \omega_{FS}(v_1, v_2)\frac{\iota^*\omega_{FS}^n}{n!}, \tag{2.2}$$

with v_1, v_2 vector fields along $T^{1,0}|_X\mathbb{P}^N$, normal to the subspace defined by the infinitesimal action of $SL(N + 1, \mathbb{C})$. More precisely, if $\Gamma\left(T^{1,0}|_X\mathbb{P}^N\right)$ denotes the space of smooth $(1, 0)$ vector fields on $\mathbb{P}^N$ restricted to $X \subset \mathbb{P}^N$, then $\Gamma\left(T^{1,0}|_X\mathbb{P}^N\right)$ decomposes, under the L^2 metric inherited from ω_{FS} on $\mathbb{P}^N$, as a direct sum

$$\Gamma\left(T^{1,0}|_X\mathbb{P}^N\right) = \Gamma\left(Lie(SL(N+1, \mathbb{C}))|_X\right) \oplus \Gamma\left(Lie(SL(N+1, \mathbb{C}))|_X\right)^\perp.$$

Here, $\Gamma\left(Lie(SL(N+1,\mathbb{C}))|_X\right)$ denotes the standard infinitesimal action of $SL(N+1,\mathbb{C})$ on $\mathbb{P}^N$ restricted to $X \subset \mathbb{P}^N$. Thus, the vector fields v_1, $v_2 \in \Gamma\left(Lie(SL(N+1,\mathbb{C}))|_X\right)^{\perp} \subset \Gamma\left(T^{1,0}|_X\mathbb{P}^N\right)$ in (2.2) are normal to $\Gamma\left(Lie(SL(N+1,\mathbb{C}))|_X\right)$ (this will be used in Section 8.4.3 where we shall reformulate the metric defined by (2.2)).

Of course, from the natural action of $SU(N+1,\mathbb{C})$ over $\mathbb{P}^N$, the group $SU(N+1,\mathbb{C})$ will act equivariantly on $Hilb(N,\chi)$ and $Univ_{N,\chi}$. With respect to Ω_k, this leads eventually to a natural moment map μ on the space of smooth maps from $X \in \mathcal{H}^\infty_\chi$ to $\mathbb{P}^N$. Given an orthonormal basis of sections $\{s_\alpha\}$ of $H^0(X,L^k)$ (which is equivalent to fix the embedding ι), one can write this associated moment map as

$$\mu(\iota) = \frac{N+1}{\mathrm{Vol}_L(X)} \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum_i |s_i|^2} \frac{\omega_{FS}^n}{n!} - \delta_{\alpha\bar{\beta}},$$

which is a trace free hermitian endomorphism. The zeros of μ correspond to “balanced” manifolds $(X,L^k) \in Hilb(N,\chi)^{ps}$ that are *polystable* in the sense of Geometric Invariant Theory (this will be described in details in Section 2.5.2 with the notion of Chow stability). For those manifolds, there does exist an embedding for which the center of mass with respect to the Fubini-Study form is zero. One can reformulate this by considering two natural maps on the space of metrics over L^k and the space of metrics over $H^0(X,L^k)$:

- The ‘Hilbertian’ map ,

  $$Hilb_k \colon \mathrm{Met}(L^k) \to \mathrm{Met}(H^0(X,L^k))$$

  such that

  $$Hilb = Hilb_k(h)(s,\bar{s}) = \int_X |s|_h^2 \frac{c_1(h)^n}{n!}$$

  with $c_1(h)$ the curvature of h.

- The injective ‘Fubini-Study’ map ,

  $$FS = FS_k \colon \mathrm{Met}(H^0(X,L^k)) \to \mathrm{Met}(L^k)$$

  such that for $H \in \mathrm{Met}(H^0(L^k))$, $\{s_i\}$ an H-orthonormal basis of $H^0(X,L^k)$ and for all $p \in X$,

  $$\sum_{i=1}^{\dim H^0(X,L^k)} |s_i(p)|^2_{FS_k(H)} = \frac{\dim H^0(X,L^k)}{\mathrm{Vol}_L(X)},$$

  which means that we fix pointwise the metric $FS_k(H) \in \mathrm{Met}(L^k)$.

> The curvature of $FS(H)$ is the pull-back of the Fubini-Study metric living in the projective space, using the embedding defined by the H-orthonormal basis $\{s_i\}$.

Equivalently, $\mu(\iota) = 0$ corresponds to the existence of a Hermitian metric H_k on the vector space $H^0(X, L^k)$ such that $Hilb_k(FS_k(H_k)) = H_k$, i.e to a fixed point of the map $T = Hilb_k \circ FS_k$ where

$$T\colon \mathrm{Met}(H^0(X, L^k)) \to \mathrm{Met}(H^0(X, L^k)).$$

Definition 2.4.1. We say that a fixed point $H_k \in \mathrm{Met}(H^0(X, L^k))$ of the T-map is a *balanced* metric at level k. We will also say that the induced metrics $FS_k(H_k) \in \mathrm{Met}(L^k)$ (resp. $\iota^*\omega_{FS} = \frac{1}{k}c_1(FS(H_k))$) are balanced on L (resp. in $c_1(L)$) at level k.

The next statement sums up the important results of [Don01b; Don05b; San06].

Theorem 2.4.2. *Let X be a smooth projective manifold with a constant scalar curvature Kähler metric in the class $c_1(L)$ and with $\mathrm{Aut}(X, \mathcal{L})$ group discrete*[3]*. Then, there exists a balanced metric $H_k \in \mathrm{Met}(H^0(X, L^k))$ for k sufficiently large.*
The sequence of Kähler forms $c_1(FS(H_k)^{1/k})$ converges in C^∞ topology when $k \to \infty$ to the unique cscK metric.
The T map admits a unique attractive fixed point, the balanced metric H_k, and iterates of the T map converges exponentially fast to the balanced metric.

Remark that the idea of approximating Kähler metrics by Fubini-Study type metrics via projective *balanced* embeddings, goes back to [BLY94]. The previous theorem also furnishes as a by-product a proof of the uniqueness of cscK metric in a given integral Kähler class.

One obtains a complete analog in a finite dimensional framework of the formal 'symplectic' quotient of $\mathcal{J}_{int}$ described in Section 2.3. The fact that at the quantum limit one recovers the setting of Section 2.3 is justified below. In terms of geometric analysis, a metric in $\mathrm{Met}(L^k)$ is balanced if and only if its associated *Bergman function* is constant over the manifold. Let us recall that given $h \in \mathrm{Met}(L)$ with positive curvature $c_1(h) = \omega$, the Bergman function associated to h^k is the restriction over the diagonal of the kernel

[3] This will not be essential in the rest of the thesis, but we explain shortly what it means. $\mathrm{Aut}(X, L)$ consists in couples (a, b) of bihlomorphisms of X and L such that $a \circ \pi = \pi \circ b$ where $\pi : L \to X$. There is the restriction map $\theta : Lie(\mathrm{Aut}(X, L)) \to Lie(\mathrm{Aut}(X))$ that has 1-dimensional kernel generated by let's say v. The required assumption that $\mathrm{Aut}(X, \mathcal{L})$ is discrete means that the image of θ is trivial, i.e $Lie(\mathrm{Aut}(X, L))/\mathbb{C} \cdot v$ is trivial. This implies by a result of Matsushima-Lichnerowicz that any holomorphic hamiltonian vector field for $\omega \in c_1(L)$ is trivial and thus that the Lichnerowicz operator associated to any Kähler metric in $c_1(L)$ has trivial kernel.

of the orthogonal L^2-projection (with respect to h^k and $Hilb(h^k)$) from the space of smooth sections of L^k to the subspace of holomorphic sections. This can be written as

$$\rho_k(h)(p) = \sum_{i=1}^{N+1} |s_i|^2_{h^k}(p)$$

where $p \in X$, $\{s_i\}_{i=1,..,N+1}$ is an $Hilb(h^k)$-orthonormal basis of $H^0(X, L^k)$. As we explain later in more details (Section 3.1) there is an asymptotic expansion for k large of ρ_k in C^∞ norm (Theorem 3.1.2) of the form

$$\rho_k = k^n + k^{n-1}\frac{\mathrm{scal}(\omega)}{2} + ...$$

so the scalar curvature of the curvature ω of h appears at the second term of the expansion. This is a crucial fact in the study of the Yau-Tian-Donaldson conjecture.

Now, we can come back to our previous discussion about the result of Fujiki and Donaldson. For k sufficiently large, there is an embedding of $\mathcal{J}_{int}$ into the Grassmannian $Gr(H^0(X, L^k), \Gamma(X, L^k))$ given by the map $J \mapsto \mathrm{Ker}(\bar\partial_J)$ (see also Section 8.4.1). Let us denote $\mathcal{Z}$ the symplectic quotient

$$\{(s_1, .., s_{N+1}, J) \in \Gamma(X, L^k)^{N+1} \times \mathcal{J}_{int}, \bar\partial_J s_i = 0, \{s_i\} \text{ basis}\}//SU(N+1),$$

i.e the set of couples (V, J) of $(N+1)$-dimensional subspace V of $\Gamma(X, L^k)$ generated by orthonormal sections that are holomorphic with respect to the structure J. There are obvious maps: the projection $\pi_1 : \mathcal{Z} \to \mathcal{J}_{int}$ and also $\pi_2 : \mathcal{Z} \to Gr(H^0(X, L^k), \Gamma(X, L^k))$ given by $\pi_2(s_1, .., s_{N+1}) = s_1 \wedge .. \wedge s_{N+1}$. On $\Gamma(X, L^k)$ there is a natural symplectic form associated to the hermitian metric $Hilb(h^k)$ which induces a symplectic form on the Grassmannian. One can pull-back this form by π_2 and push it forward by π_1 to obtain a Kähler form $\omega^{\mathcal{J},k}$ on $\mathcal{J}_{int}$. As sketched in [Don01a], one has the convergence

$$\lim_{k\to+\infty} \frac{1}{k^n}\omega^{\mathcal{J},k} = \omega^{\mathcal{J}}$$

when restricted to integrable almost complex structures compatible with ω. Let us give some details. Given g_0, g_1 two Kähler metrics on TX compatible with $J \in J_{int}$, we set $h_{K_X,0}$, $h_{K_X,1}$ the induced metrics on the canonical bundle K_X. A computation shows that one can write the local potential of $\omega^{\mathcal{J}}_J$ as

$$\int_X \log\left(\frac{h_{K_X,1}}{h_{K_X,0}}\right)\frac{\omega^n}{n!}.$$

Let's take k large enough so that $h^i(X, L^k) = 0$ for $i > 0$. Using the metric h on L, we get from g_0, g_1 the L^2 metrics $h_{L^2,0}$, $h_{L^2,1}$ on the determinant of the cohomology $\det H^0(X, L^k)$. One can invoke the work of J-M. Bismut, H. Gillet and C. Soulé to obtain that

$$\log\left(\frac{h_{L^2,1}}{h_{L^2,0}}\right) = k^n \int_X \log\left(\frac{h^2_{K_X,1}}{h^2_{K_X,0}}\right)\frac{\omega^n}{n!} + O(k^{n-1})$$

thanks to the asymptotic anomaly formula for the L^2 metrics, see [MM07, Theorem 5.5.12]. But the LHS of the previous equation is precisely the potential of the metric $\omega^{\mathcal{J},k}$.
A quite simple computation shows that the moment map associated to the action of $Ham(X,\omega)$ on $\mathcal{J}_{int}$ with respect to $\frac{1}{k^n}\omega^{\mathcal{J},k}$ is given by the function on X,

$$p \mapsto \frac{1}{k^n}\left(\frac{1}{2}\Delta_\omega + k\right)\sum_{i=1}^{N+1}|s_i|^2_{h^k}(p) - \frac{N+1}{k^{n-1}}, \tag{2.3}$$

for $\{s_i\}$ orthonormal basis for $H^0(X, L^k)$ with respect to the metric $\omega_J = c_1(h)$ induced by $J \in \mathcal{J}_{int}$. Actually, to do this computation it is sufficient to consider the diagonal action over $\{(s, J) \in \Gamma(L^k) \times \mathcal{J}_{int}, \bar\partial_J s = 0\} \subset \Gamma(L^k) \times \mathcal{J}_{int}$ of hermitian bundle maps of L^k that preserve the connection and cover the ω-symplectomorphisms on X. By [Don01b, Lemma 9], the associated moment map for this latter action is just $\left(\frac{1}{2}\Delta_\omega + k\right)|s|^2$ with s holomorphic section of L^k.
Eventually, from the asymptotic expansion of the Bergman function, one checks that the moment map (2.3) converges when $k \to +\infty$ towards the scalar curvature $\mathrm{scal}(\omega_J)(p) = \mathrm{scal}(g_J)(p)$, up to a normalizing factor, as expected.

2.5 Algebraic stability of varieties

In this section, we explain some notions of stability for manifolds in terms of Geometric Invariant Theory (G.I.T). These notions of stability will be related to the existence of canonical metrics as will appear shortly.

2.5.1 G.I.T stability

We need some basic material about G.I.T stability. We refer to [Tho06; Mum77; KN79] as general references, see also [Kel05b, Chapitre 1]. Suppose that $\mathcal{X} \in \mathbb{P}^N$ is a projective variety and G, a complex reductive[4] linear group, acts on $\mathcal{X}$ and the action is induced by a representation $G \to SL(N+1,\mathbb{C})$. G.I.T has been invented in order to provide a construction of "good" quotients of $\mathcal{X}$ by G.

[4] i.e it is the complexification of a maximal compact subgroup of G. Equivalently, one can ask that the uni-potent radical of G is trivial.

Through the action on $\mathbb{C}[x_0,..,x_N]$, G induces an action on the homogeneous coordinate ring of $\mathcal{X}$, $R(\mathcal{X})=\mathbb{C}[x_0,..,x_N]/\mathcal{I}$, where $\mathcal{I}$ is the prime ideal generated by the homogeneous polynomials vanishing on $\mathcal{X}$. The fact that G is reductive implies actually that the ring of invariants $R(\mathcal{X})^G$ is finitely generated as a $\mathbb{Z}$-graduated algebra. We can associate a projective variety $\mathrm{Proj}\,R$ by performing the following operation. Since $R(\mathcal{X})$ is graded by the degree, there is a decomposition $R(\mathcal{X})^G = \bigoplus_{k\geq 0} R(\mathcal{X})^G_k$ where $R(\mathcal{X})^G_k$ corresponds to the degree k. From Hilbert's Nullstellensatz, one knows that there is a correspondence between finitely generated graded $\mathbb{C}$-algebras generated in degree 1 without zero divisors and on another side projective varieties, seen as the zero set of a finite collection of homogeneous irreducible polynomials. We replace $R(\mathcal{X})$ by a $\mathbb{C}$-algebra generated in degree 1 (this corresponds to consider a higher power of the linearisation[5] on $\mathcal{X}$). It is true that the algebra $R'=\oplus_{k\geq 0}R(\mathcal{X})^G_{kd}$ for a certain $d>0$ is generated by elements in $R(\mathcal{X})^G_d$, and thus $\mathrm{Proj}R'$ is a projective variety, usually denoted $\mathcal{X}//G$. One thinks about this quotient as the set parametrizing orbits on which there is a least one non-vanishing G-invariant function in $R(\mathcal{X})$. By doing this quotient, we identify two orbits that cannot be distinguished by G-invariant functions.

Definition 2.5.1. The set of semistable points $\mathcal{X}^{ss}$ is the subset of $\mathcal{X}$ given by the points $x\in\mathcal{X}$ such that there exists a non-constant homogeneous function $f\in R(\mathcal{X})^G$ with $f(x)\neq 0$.
The set of polystable points $\mathcal{X}^{ps}$ is the subset of $\mathcal{X}^{ss}$ given by the points $x\in\mathcal{X}^{ss}$ such that the orbit $G\cdot x$ is closed in $\mathcal{X}^{ss}$.
The set of stable points $\mathcal{X}^{s}$ is the subset of $\mathcal{X}^{ps}$ given by the points $x\in\mathcal{X}^{ps}$ such that the stabilizer of x in G is finite.

Let us comment briefly this definition. First of all, it is independent of the choice of x in a fixed orbit, so we can speak of the stability of an orbit. The semistable points are those that the G-invariant functions can distinguish. The evaluation map from $\mathcal{X}\to\mathcal{X}//G$ is well defined on the locus $\mathcal{X}^{ss}$, which is Zariski open (but possibly empty). The closure of every semistable orbit contains a unique polystable orbit. The stable points x form a Zariski open and provide a "geometric quotient" $\mathcal{X}^s/G$ by separating orbits near x. This quotient has a quasiprojective structure.

The Hilbert-Mumford criterion provides a way to check in practice the semistability (resp. stability, polystability) of a point by restricting our

[5]We recall that to fix a linearisation L of the action of a linear algebraic reductive group G on $\mathcal{X}$ means to choose a line bundle L and a linear action of G on L inducing the one on $\mathcal{X}$. When $\mathcal{X}$ is Kähler, one can consider an extension of this definition for compact Lie group G such that its complexification acts holomorphically and G acts by symplectic diffeomorphisms. In practice, L is very ample and the linearisation implies that the action of G is induced by the projective embedding induced by L and a linear representation on $H^0(\mathcal{X},L)$. Then $R(\mathcal{X})$ can be identified to $\bigoplus_{k\geq 0} H^0(\mathcal{X},L^k)^*$.

attention to 1-parameter subgroups. Given $\lambda : \mathbb{C}^* \to G$ a 1-parameter subgroup and $x \in \mathcal{X}$, we can define the Hilbert weight

$$w(x, \lambda)$$

of the action by looking at the point $x' = \lim_{t\to 0} \lambda(t) \cdot x$ and doing the following computation. Since $\mathcal{X} \subset \mathbb{P}^N$, each point $x' \in \mathcal{X}$ can be lifted to $\hat{x}' \in \mathbb{C}^{N+1} \setminus \{0\}$. Since x' is fixed by the 1-parameter group, there is an integer $w(x, \lambda)$ such that[6]

$$\lambda(t) \cdot \hat{x}' = t^{-w(x,\lambda)} \hat{x}'.$$

If this weight is negative, then 0 belongs to the closure of the orbit of any lift of x.

Theorem 2.5.2 (Hilbert-Mumford)**.** *The point $x \in \mathcal{X}$ is semistable if and only if*

$$w(x, \lambda) \geq 0$$

for any 1-parameter subgroup λ.
The point $x \in \mathcal{X}$ is polystable if and only if

$$w(x, \lambda) > 0$$

for any 1-parameter subgroup λ such that $x' \notin G \cdot x$.
The point $x \in \mathcal{X}$ is stable if and only if

$$w(x, \lambda) > 0$$

for any nontrivial 1-parameter subgroup λ.

We conclude this section by explaining shortly how the moment map framework fits in the G.I.T construction. Assume $\mathcal{X}$ is now a projective submanifold of $\mathbb{P}^N$ and $G \subset SL(N+1, \mathbb{C})$ is acting on $\mathcal{X}$. Given $K \subset SU(N)$ a maximal compact subgroup of G, there is a moment map

$$\underline{\mu} : \mathcal{X} \to Lie(K)^*$$

with respect to the symplectic form given by the restriction of the Fubini-Study metric to $\mathcal{X}$. Note that fixing a linearisation on $\mathcal{X}$ is actually equivalent to fixing a G-equivariant moment map, see [Kir84, Section 8].

Theorem 2.5.3 (Kempf-Ness)**.** *A point $x \in \mathcal{X}$ is polystable (resp. stable) for the action of G if and only if the orbit $G \cdot x$ contains a zero of the moment map $\underline{\mu}$ (resp. and furthermore the stabilizer is discrete). It is*

[6] of course there is here a convention, we define the weight in such a way that stable points have positive weight.

unique up to the action of K. As sets, one can identify $\mathcal{X}//G$ with the Marsden-Weinstein reduction $\underline{\mu}^{-1}(0)/K$ that admits a symplectic structure.

The proof relies on considering the application $\mathfrak{I}_{\underline{\mu}} : G/K \to \mathbb{R}$ given by

$$g \mapsto \log \|g \cdot \hat{x}\|_h^2$$

where h is a K-invariant metric. A computation shows that g is a critical point of $\mathfrak{I}_{\underline{\mu}}$ if and only if $\underline{\mu}(g \cdot x) = 0$. Also $\mathfrak{I}_{\underline{\mu}}$ is convex along geodesics in G/K. On another hand, the orbit of x is closed exactly when the norm $\|g \cdot \hat{x}\|_h$ goes to infinity as g goes to infinity and this corresponds to say that $\mathfrak{I}_{\underline{\mu}}$ is proper. But the convexity property forces $\mathfrak{I}_{\underline{\mu}}$ to have a critical point and conversely. This natural idea will appear also several times in our work.

2.5.2 Chow, Hilbert and K-stability

In this section we recall some well known facts about Chow, Hilbert and K-stability of a polarized variety. We refer to the surveys [Biq06; Tho06; PS10; Fut12] and also to [Mum77; Don05a; Don02] for definitions and examples.

Consider (X, L) a polarized variety of complex dimension n and $X \subset \mathbb{P}H^0(L^k)^* = \mathbb{P}V$ the closed immersion associated to the complete linear system $|L^k|$. Let $Z_X = \{P \in Gr(V, n-1) : P \cap X \neq \emptyset\}$ which is a divisor of degree $d = \deg L$ in the Grassmannian $\mathcal{G} = Gr(V, n-1)$. Thus there exists $s_{X,V} \in H^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(d))$, such that one has $Z_X = \{s_{X,V} = 0\}$ and this induces a Chow point

$$\mathrm{Chow}(X) = [s_{X,V}] \in \mathbb{P}H^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(d))$$

on which one can consider the action of $SL(V)$. The polarized manifold (X, L^k) is said to be Chow stable (resp. Chow semistable) if the Chow point $\mathrm{Chow}(X)$ is G.I.T stable (resp. G.I.T semistable).
We say that it is asymptotically Chow stable (resp. asymptotically Chow semistable) if (X, L^k) is Chow stable (resp. Chow semistable) for $k \gg 1$.

Let us discuss now Hilbert stability. For $X \subset \mathbb{P}V$ a variety such that the restriction map

$$\rho : H^0(\mathbb{P}V, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m))$$

is surjective, one sets

$$W_m = \bigwedge^{h^0(X,\mathcal{O}(m))} H^0(\mathbb{P}V, \mathcal{O}(m))^*.$$

Thus, from the map ρ and taking the wedge product, one can consider the

m-Hilbert point in the projective space $\mathbb{P}(W_m)$ given by

$$[X]_m = [\bigwedge^{h^0(X,\mathcal{O}(m))} H^0(\mathbb{P}V, \mathcal{O}(m)) \to \bigwedge^{h^0(X,\mathcal{O}(m))} H^0(X, \mathcal{O}(m))] \in \mathbb{P}(W_m).$$

The polarized variety (X, L) is said to be Hilbert stable (resp. Hilbert semistable) if the induced m-Hilbert points $[X]_m$ defined by the closed immersion associated to the complete linear system $|L^m|$ are G.I.T semistable (resp. G.I.T stable) for all $m \gg 1$.
The polarized variety (X, L) is said to be asymptotically Hilbert stable (resp. asymptotically Hilbert semistable) if (X, L^k) is Hilbert stable (resp. Hilbert semistable) for $k \gg 1$.

We recall now the notion of test configuration [Don02; Don05a; Tia97].

Definition 2.5.4. A test configuration for a polarized variety (X, L) is a polarized scheme $(\mathcal{X}, \mathcal{L})$ with:

- a $\mathbb{C}^\times$ action and a proper flat morphism $\pi : \mathcal{X} \to \mathbb{C}$ which is $\mathbb{C}^\times$ equivariant for the usual action on $\mathbb{C}$,
- a $\mathbb{C}^\times$ equivariant line bundle $\mathcal{L} \to \mathcal{X}$ which is ample over all fibers of π such that for $z \neq 0$, (X, L^s) is isomorphic to $(\mathcal{X}_z, \mathcal{L}_{\mathcal{X}_z})$ for some positive integer s, called the exponent.

A product test configuration is a test configuration with $\mathcal{X} \simeq X \times \mathbb{C}$. A test configuration is trivial in codimension 2 if it is $\mathbb{C}^\times$-equivariantly isomorphic to a product test configuration $X \times \mathbb{C}$, with trivial $\mathbb{C}^\times$-action, away from a closed subscheme of codimension at least 2.

From [RT07], we know that there is a correspondence between the data of a test configuration $(\mathcal{X}, \mathcal{L})$ of exponent s and the data of a 1-parameter subgroup of $SL(H^0(X, L^s))$. Thus using the Hilbert-Mumford criterion, it is sufficient to consider the weights of the $\mathbb{C}^\times$ action to check the stability of (X, L). More precisely, let us call $w(Ks)$ the total weight of the induced action on $\pi_* \mathcal{L}^K_{|0} = H^0(\mathcal{X}_0, \mathcal{L}^K)$ for $K \gg 0$, for a test configuration associated to (X, L^{Ks}). Remark that $w(Ks)$ is a polynomial of degree $n+1$ in the $k = Ks$ variable. Let us denote $P(k) = \dim H^0(L^k)$ which is equal to the Hilbert polynomial $\chi(X, L^k)$ for k large. The normalized weight after taking the $sP(s)$-th power of the $\mathbb{C}^\times$ action on $\pi_* \mathcal{L}^K_{|0}$ is

$$\tilde{w}(s, k) = w(k) s P(s) - w(s) k P(k) \tag{2.4}$$

which is a polynomial of degree $n+1$ in the k variable. It is the Hilbert weight of (X, L^s) and thus (X, L) is asymptotically Hilbert stable (resp. asymptotically Hilbert semistable) if and only if $\tilde{w}(s, k) > 0$ (resp. $\tilde{w}(s, k) \geq 0$) for all $k \gg 1$ ($k > k_0(s)$ large enough), $s \gg 1$.

One can decompose $\tilde{w}(s,k)$ as

$$\tilde{w}(s,k) = \sum_{i=0}^{n+1} e_i k^i$$

where $e_i = \sum_{j=0}^{n+1} e_{i,j} s^j$ are polynomials of degree $n+1$ in the s variable with $e_{n+1,n+1} = 0$ due to the normalization.

We refer to [Mum77, Lemma 2.11] and [RT07, Theorem 3.9] for a proof of the next result.

Proposition 2.5.1. *The coefficient $e_{n+1}(s)s^{n+1}(n+1)!$ is the Chow weight of $X \subset \mathbb{P}H^0(X, L^s)$. In particular, (X, L) is asymptotically Chow stable (resp. asymptotically Chow semistable) if and only if $e_{n+1}(s) > 0$ (resp. ≥ 0) for all $s \gg 1$.*

Note that (X, L) is asymptotically Chow (resp. Hilbert) polystable if it is asymptotically Chow (resp. Hilbert) semistable and any not strictly stable test configuration is a product test configuration. This matches with the notion of polystability in terms of G.I.T.

We are ready to define the notion of K-stability. The following definition is a refinement of Donaldson's definition [Don05a] and is due to Stoppa [Sto08].

Definition 2.5.5. The polarized variety (X, L) is K-stable (resp. is K-semistable) if for any non trivial (in codimension 2) test configuration $\mathcal{T}$, the leading coefficient $e_{n+1,n}$ of $e_{n+1}(s)$ is positive (resp. ≥ 0). This leading coefficient is called the Donaldson-Futaki invariant of the test configuration and denoted $DF_1(\mathcal{T})$. Moreover (X, L) is said to be K-polystable if it is K-semistable and any not strictly stable test configuration is a product test configuration.

In the case that the test-configuration only involves a smooth central fibre, one recovers from this definition the classical invariant introduced by A. Futaki in the eighties (the test configuration providing a holomorphic vector field and conversely). Remember that in that case, one can write

$$\begin{aligned} DF_1(\overrightarrow{X}_h) &= \frac{1}{2}\int_X h(\mathrm{scal}(g_J) - \mathrm{scal}_0)\frac{\omega^n}{n!} \\ &= \frac{1}{2}\int_X h\frac{c_1(X)\wedge c_1(L)^{n-1}}{(n-1)!} - \frac{1}{2}\mathrm{scal}_0\int_X h\frac{c_1(L)^n}{n!} \end{aligned}$$

where scal_0 is the average of the scalar curvature for the class $[\omega] = c_1(L)$ and $\overrightarrow{X}_h$ a holomorphic vector field. Futaki proved that DF_1 depends actually only on the considered Kähler class and is a character over the Lie algebra of holomorphic vector fields of X.

As one can imagine easily, it is particularly challenging to test K-stability in practice. Y. Odaka has proved that it suffices to check the positivity of the DF_1 invariant on all "semi-test configurations" arising from ideals. More precisely, a flag ideal on X is a coherent ideal sheaf $\mathcal{I}$ on $X \times \mathbb{A}^1$ of the form

$$\mathcal{I} = I_0 + (t)I_1 + ... + (t^P),$$

where $I_0 \subseteq I_1 \subseteq ... \subseteq I_{P-1} \subset \mathcal{O}_X$ is a sequence of coherent ideal sheaves. Such a flag ideal $\mathcal{I}$ induces a coherent ideal sheaf on $X \times \mathbb{P}^1$, denoted the same way, and one can blow up $\mathcal{I}$ on $X \times \mathbb{P}^1$. Let us call π this map and E the associated exceptional divisor. Thus $\pi : \mathcal{B}_{\mathcal{I}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1$. Let $\mathcal{L}$ stand for $\pi^*(p^*L)$ where $p : X \times \mathbb{P}^1 \to X$ is the projection on X. Similarly $\mathcal{K}_X$ is the pull-back of K_X to

$$\mathcal{B} := BL_{\mathcal{I}}(X \times \mathbb{P}^1).$$

The natural $\mathbb{C}^\times$ action on $X \times \mathbb{P}^1$ acting trivially on X lifts to an action on $\mathcal{B}$. This gives rise to a "semi-test configuration" $(\mathcal{B}, \mathcal{L} - E)$ when $\mathcal{L} - E$ is relatively semi-ample over $\mathbb{P}^1$ and $\mathcal{B}$ is normal. This is a generalization of the deformations to the normal cone as used in Part III for which the parameter P is equal to 1. Then Y. Odaka [Oda13a] and independently X. Wang [Wan12] computed the following expression for the Donaldson-Futaki invariant:

$$\begin{aligned} DF_1(\mathcal{B}, \mathcal{L} - E) = & - n(L^{n-1}K_X)(\mathcal{L} - E)^{n+1} \\ & + (n+1)(L^n)(\mathcal{L} - E)^n(\mathcal{K}_X + K_{\mathcal{B}/X\times\mathbb{P}^1}), \end{aligned} \tag{2.5}$$

where $K_{\mathcal{B}/X\times\mathbb{P}^1}$ is the relative canonical bundle. We will explain in Part V how this formula can be used. To sum up, with formula (2.5) in hand, we have the following theorem [Oda13a, Corollary 3.11], [Wan12].

Theorem 2.5.6. *Assume that (X, L) is a normal polarized variety. Then (X, L) is K-stable (resp. K-semistable) if and only if*

$$DF_1(\mathcal{B}, \mathcal{L}^r - E) > 0, \qquad (\textit{resp.} \ \geq 0)$$

for all $r > 0$ and all flag ideals $\mathcal{I}$ with B normal and Gorenstein in codimension 1 and $\mathcal{L}^r - E$ semi-ample over $\mathbb{P}^1$.

Let us finish this section by recalling certain well-known relationships between the various notions of stability that we shall use later (see [Tho06; Mab08a]):

Asymptotic Chow stability $\Leftrightarrow$ Asymptotic Hilbert stability $\Rightarrow$ Asymptotic Hilbert semistability $\Rightarrow$ Asymptotic Chow semistability $\Rightarrow$ K-semistability.

Now, we can explain the deep relationship between Section 2.5 and Section 2.4. In [Zha96] and [Luo98], S. Zhang and H. Luo have reformulated the G.I.T notion of Chow (poly)stability in terms of existence of special metrics, the balanced metrics defined as previously. They proved the next theorem that we shall use several times in this thesis.

Theorem 2.5.7. *The embedding of the algebraic manifold X into the projective space via the linear series determined by L^k is Chow polystable if and only if there exists a hermitian metric on L that is balanced at level k.*

Therefore, together with Theorem 2.4.2, there exists a bridge between algebraic notions of stability (i.e G.I.T), existence of canonical *algebraic* metrics (balanced metrics) and existence of Kähler metrics with special curvature properties (cscK metrics). By algebraic metric, we mean metrics obtained as pull-back of Fubini-Study metrics by the Kodaira's embeddings of the manifold. We stress the fact that cscK metrics or more generally extremal metrics are transcendental objects solutions of non linear PDEs.

Part II

About balancing flows

Chapter 3

Ω-balancing flow

In this chapter we provide a new proof of the Calabi-Yau theorem over a projective complex manifold by introducing a new flow, called the Ω-Kähler flow that appears as limit of an algebraic construction through the moment map formalism in finite dimension. Firstly we give some definitions and recall some natural moment map considerations about Ω-balanced metrics. Then we introduce the two main flows of this chapter, the Ω-balancing flow and its quantum limit, the Ω-Kähler flow, and state our main results, Theorems 3.0.11 and 3.0.12. Eventually, we derive some consequences in Sections 3.2.3 and 3.4.

Assume that M is a smooth polarized manifold of complex dimension n and L an ample line bundle. We consider a smooth volume form Ω on M such that $\int_M \Omega = \mathrm{Vol}_L(M)$, the volume of M with respect to L.

In [Don09], S.K. Donaldson introduced a notion of Ω-balanced metric, adapted to the Calabi problem of fixing the volume of a Kähler metric in a given Kähler class. More precisely, given a (smooth) hermitian metric $h \in \mathrm{Met}(L^k)$, one can consider the Hilbertian map

$$Hilb_\Omega = Hilb_{k,\Omega} : \mathrm{Met}(L^k) \to \mathrm{Met}(H^0(L^k))$$

such that

$$Hilb_\Omega(h) = \int_M h(.,.)\,\Omega = \int_M \langle .,. \rangle_h\,\Omega$$

is the L^2 metric induced by the fibrewise h and the volume form Ω. On another hand, one can consider the Fubini-Study applications $FS = FS_k : \mathrm{Met}(H^0(L^k)) \to \mathrm{Met}(L^k)$ defined as page 19. One of the main result of [Don09] asserts that the dynamical system on $\mathrm{Met}(L^k)$ given by

$$T_{\Omega,k} = FS \circ Hilb_\Omega$$

has a unique attractive fixed point.

Definition 3.0.8. Let (M, L) be a polarized manifold, Ω a smooth volume form. Then for any sufficiently large k, there exists a unique fixed point h_k of the map $T_{\Omega,k} : \mathrm{Met}(L^k) \to \mathrm{Met}(L^k)$ which is called Ω-balanced. The metric $Hilb_\Omega(h_k) \in \mathrm{Met}(H^0(L^k))$ and the Kähler form $c_1(h_k) \in c_1(L)$, given by the curvature of h_k, will also be called Ω-balanced.

When k tends to infinity, one obtains from [Don09] and [Kel09, Theorem 3], the following result.

Theorem 3.0.9 ([Don09; Kel09]). *When $k \to \infty$, the sequence of Ω-balanced metrics $(h_k)^{1/k} \in \mathrm{Met}(L)$ converges to a hermitian metric $h_\infty \in \mathrm{Met}(L)$ in smooth topology and its curvature is a solution to the Calabi problem of prescribing the volume form*[1] *in a given Kähler class,*

$$c_1(h_\infty)^n = \Omega.$$

Let us denote in the sequel $N + 1 = N_k + 1 = \dim H^0(L^k)$. Another way of presenting the notion of Ω-balanced metric is to introduce a moment map setting. Let us consider first $\mu_{FS} : \mathbb{P}^N \to \sqrt{-1}Lie(U(N + 1))$ which is a moment map for the $U(N + 1)$ action and the Fubini-Study metric ω_{FS} on $\mathbb{P}^N$. Note that here we identify implicitly the Lie algebra $Lie(U(N+1))$ with its dual using the bilinear form $(A, B) = -\mathrm{tr}(AB)$. Given homogeneous unitary coordinates, one sets explicitly $\mu_{FS} = (\mu_{FS})_{\alpha,\beta}$ as

$$(\mu_{FS}([z_0, ..., z_N]))_{\alpha,\beta} = \frac{z_\alpha \bar{z}_\beta}{\sum_i |z_i|^2}. \tag{3.1}$$

Then, given an holomorphic embedding $\iota : M \hookrightarrow \mathbb{P}H^0(L^k)^*$, we can consider the integral of μ_{FS} over M with respect to the volume form:

$$\mu_\Omega(\iota) = \int_M \mu_{FS}(\iota(p))\Omega(p)$$

which induces a moment map for the $U(N + 1)$ action over the space of all bases of $H^0(L^k)$. Let us give some details on that point. On the space $\mathfrak{M}$ of smooth maps from M to $\mathbb{P}H^0(L^k)^*$, we have a natural symplectic structure ϖ_Ω defined by

$$\varpi_\Omega(a, b) = \int_M (a, b)\Omega.$$

for $a, b \in T_\iota\mathfrak{M} \subset \Gamma(M, T(\mathbb{P}^N)^*_{|M})$ and $(.,.)$ the Fubini-Study inner product on the tangent vectors. Let $\zeta \in Lie(U(N+1))$ and $X_\zeta \in H^0((\mathbb{P}^N)^*, T(\mathbb{P}^N)^*)$ be the induced holomorphic vector field on the dual space $(\mathbb{P}^N)^* = \mathbb{P}H^0(L^k)^*$.

[1] Note that in the following we shall forget the normalization factor $\frac{1}{n!}$ in front of the Monge-Ampère mass $c_1(h_\infty)^n$.

For all $Y \in \Gamma(M, T(\mathbb{P}^N)^*_{|M})$ we have that

$$\begin{aligned}\varpi_\Omega(X_{\zeta_{|M}}, Y) &= \int_M i_Y(i_{X_\zeta}\omega_{FS})\Omega \\ &= -\int_M \mathrm{tr}(d\mu_{FS}(Y)\cdot\zeta)\Omega \\ &= -\mathrm{tr}(d\mu_\Omega(Y)\cdot\zeta) \\ &= (d\mu_\Omega(Y), \zeta),\end{aligned}$$

and μ_Ω is Ad-equivariant as the integral of the Ad-equivariant moment map μ_{FS}. Thus, $U(N+1)$ acts isometrically on $\mathfrak{M}$ with the moment map given by

$$\iota \mapsto -\sqrt{-1}\left(\mu_\Omega(\iota) - \frac{\mathrm{tr}(\mu_\Omega(\iota))}{N+1}\mathrm{Id}_{N+1}\right) \in \sqrt{-1}Lie(SU(N+1)).$$

Note that if one defines a hermitian metric H on $H^0(L^k)$, one can consider an orthonormal basis with respect to H and the associated embedding, and thus it also makes sense to speak of $\mu_\Omega(H)$. As we shall see, in the Bergman space $\mathcal{B} = \mathcal{B}_k = GL(N+1)/U(N+1)$, we have a preferred metric associated to the volume form Ω and the moment map we have just defined, and this is precisely an Ω-balanced metric.

Definition 3.0.10. The embedding ι is Ω-balanced if and only if

$$\mu^0_\Omega(\iota) := \mu_\Omega(\iota) - \frac{\mathrm{tr}(\mu_\Omega(\iota))}{N+1}\mathrm{Id}_{N+1} = 0.$$

An Ω-balanced embedding corresponds (up to $SU(N+1)$-isomorphisms) to an Ω-balanced metric $\iota^*\omega_{FS}$ by pull-back of the Fubini-Study metric from $\mathbb{P}H^0(L^k)^*$, so our two definitions actually coincide (see [Don09]). Note that for $H \in \mathrm{Met}(H^0(L^k))$, it also makes sense to consider $\mu_\Omega(h)$ where $h = FS(H) \in \mathrm{Met}(L^k)$, i.e when h belongs to the space of *Bergman* type fibrewise metric that we identify with $\mathcal{B}$.

On the other hand, seen as a hermitian matrix, $\mu^0_\Omega(\iota)$ induces a vector field on $\mathbb{P}^N$. Thus, inspired from [Fin10], we study the following flow

$$\frac{d\iota(t)}{dt} = -\mu^0_\Omega(\iota(t)),$$

and we call this flow the Ω-balancing flow. To fix the starting point of this flow, we choose a Kähler metric $\omega = \omega(0)$ and we construct a sequence of hermitian metrics $h_k(0)$ such that $\omega_k(0) := c_1(h_k(0))$ converges smoothly to $\omega(0)$ providing a sequence of embeddings $\iota_k(0)$ for k sufficiently large. Such a sequence of embeddings is known to exist thanks to Theorem 3.1.2 (cf. [Tia90; Bou90]). For technical reasons, we decide to rescale this flow by considering the following ODE.

$$\frac{d\iota_k(t)}{dt} = -k\mu^0_\Omega(\iota_k(t)) \tag{3.2}$$

that we call the *rescaled Ω-balancing flow*. Of course, we are interested in the behavior of the sequence of Kähler metrics

$$\omega_k(t) = \frac{1}{k}\iota_k(t)^*(\omega_{FS})$$

when t and k tends to infinity. Here is one of the main results of this chapter.

Theorem 3.0.11 ([CaoKel12])**.** *For any fixed t, the sequence $\omega_k(t)$ converges in C^∞ topology to the solution $\omega + \sqrt{-1}\partial\bar{\partial}\phi_t$ of the following Monge-Ampère equation*

$$\frac{\partial \phi_t}{\partial t} = 1 - \frac{\Omega}{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n} \tag{3.3}$$

with $\phi_0 = 0$ and $\omega = \lim_{k\to\infty}\omega_k(0)$. Furthermore, the convergence is C^1 in the variable t.

We call the flow given by Equation (3.3), the Ω-Kähler flow. The proof of this theorem will be done in several steps. First we study the limit of a convergent sequence of rescaled Ω-balancing flows to identify the limit (Section 3.1), that we shall call the Ω-Kähler flow. Then, in Section 3.2, we study in details the behavior of the Ω-Kähler flow in any Kähler class and prove our second main result.

Theorem 3.0.12 ([CaoKel12])**.** *Let ϕ_t be the solution to Eq. (3.3) on the maximal time interval $0 \le t < T_{max}$. Let*

$$v_t = \phi_t - \frac{1}{\mathrm{Vol}_L(M)}\int_M \phi_t \frac{\omega^n}{n!}.$$

Then the C^∞ norm of v_t are uniformly bounded for all $0 \le t < T_{max}$ and consequently $T_{max} = +\infty$. Moreover, v_t converges when $t \to \infty$ to v_∞ in smooth topology and $\frac{\partial \phi_t}{\partial t}$ converges to a constant in smooth topology.

Finally, inspired from the work of [Don01b] and especially [Fin10] for the Calabi flow, we will prove Theorem 3.0.11 in Section 3.3. In Section 3.4.1, we give a moment map interpretation of the Ω-Kähler flow.

3.1 The limit of the rescaled Ω-balancing flow

In this section, we assume that the sequence $\omega_k(t)$ is convergent and we want to relate its limit to Equation (3.3). The goal of this section is to prove the following result.

Theorem 3.1.1 ([CaoKel12])**.** *Suppose that for each $t \in \mathbb{R}_+$, the metric $\omega_k(t)$ induced by Equation (3.2) converges in smooth topology to a metric ω_t and that this convergence is C^1 in $t \in \mathbb{R}_+$. Then the limit ω_t is a solution to the flow (3.3) starting at $\omega_0 = \lim_{k\to\infty} \omega_k(0)$.*

Given a matrix H in $\mathrm{Met}(H^0(L^k))$, we obtain a vector field X_H which induces a perturbation of any embedding $\iota : M \hookrightarrow \mathbb{P}H^0(L^k)^*$. The induced infinitesimal change in $\iota^*\omega_{FS}$ is pointwise given by the potential $\mathrm{tr}(H\mu_{FS})$ where μ_{FS} is given by (3.1). Thus, the corresponding potential in the case of the rescaled Ω-balancing flow is

$$-k\mathrm{tr}(\mu_\Omega^0 \mu_{FS}).$$

Since we are rescaling the flow in (3.2) and considering forms in the class $c_1(L)$, we are lead to understand the asymptotic behavior when $k \to \infty$ of the potentials

$$\beta_k = -\mathrm{tr}(\mu_\Omega^0 \mu_{FS})$$

We will need the following key result. Let us fix a Kähler form $\omega \in c_1(L)$ and write $\omega = c_1(h)$. Let us consider the Hilbertian map defined page 19.

Theorem 3.1.2 (Asymptotic expansion of the Bergman kernel)**.**
The Bergman function (or distortion function) associated to h^k has the following pointwise asymptotic expansion:

$$\rho_k(h)(p) := \sum_{i=1}^{N+1} |s_i|^2_{h^k}(p) = k^n + \sum_{i\geq 1} k^{n-i} a_i(h),$$

where $\{s_i\}_{i=1,..,N+1}$ is an $Hilb(h^k)$-orthonormal basis of $H^0(L^k)$. By $a_i(h)$ we mean terms depending on the curvature and its covariant derivatives that are uniformly bounded on M. If h is varying in a compact set (in smooth topology) in the space of hermitian metrics with positive curvature, then

$$\left\| \rho_k(h) - \left(k^n + \sum_{i=1}^{m} k^{n-i} a_i(h) \right) \right\|_{\mathrm{C}^r} \leq \frac{C}{k^{m+1}},$$

where C is uniform and only depends on r.
A direct consequence is the convergence of the sequence of Bergman metrics $\frac{1}{k} c_1(FS(Hilb(h^k)))$ to ω in smooth topology, i.e for all $r \geq 0$, i.e.

$$\left\| \frac{1}{k} c_1(FS(Hilb(h^k))) - \omega \right\|_{\mathrm{C}^r} = O\left(\frac{1}{k}\right). \tag{3.4}$$

Theorem 3.1.2 is usually called nowadays the Tian-Yau-Zelditch expansion. S. T. Yau conjectured the convergence of the Bergman metrics in

[Yau86, Section 6.1], while G. Tian proved it in [Tia90] for C^2 topology (and Y-D. Ruan for C^∞, see [Rua98]) and identified $a_0 = 1$. The existence of the asymptotic expansion was proved S. Zelditch [Zel98] (and independently by D. Catlin [Cat99]) using Boutet de Monvel-Sjöstrand techniques. The uniformity of the coefficients a_i appeared in [Lu00] and will be crucial in the rest of the chapter. We refer to [MM07] as a general survey on this topic and provides a historical perspective.

Remark 3.1.1. Away from the diagonal, the kernel $\sum_{i=1}^{N+1} \langle s_i(p_1), .\rangle_{h^k} s_i(p_2)$ vanishes asymptotically, so the geometric information is essentially carried by $\rho_k(h)$.

Equation (3.4) means that the embeddings ι_k induced by $Hilb(h)$ provides a sequence of metrics $\iota_k^*(\omega_{FS})$ by pull-back of the Fubini-Study metric, and this sequence is convergent towards the initial metric ω when $k \to \infty$. We will also use in the sequel the following technical result that can be proved with similar arguments to Theorem 3.1.2. See [Zel98; Cat99] and [Bou90] where is identified the first term of the asymptotic expansion.

Proposition 3.1.1. *Let (M, L) be a projective polarized manifold. Let $h \in \mathrm{Met}(L)$ be a metric such that its curvature $c_1(h) = \omega > 0$ is a Kähler form. Assume $\Omega > 0$ to be a volume form with continuous density. Then we have the following asymptotic expansion for $k \to \infty$,*

$$\sum_{i=1}^{N+1} |s_i|^2_{h^k} = k^n \frac{\omega^n}{\Omega} + O(k^{n-1}),$$

where $\{s_i\}$ is an orthonormal basis of $H^0(L^k)$ with respect to the L^2 inner product $\int_M h^k(.,.)\Omega = Hilb_\Omega(h^k)$. Here by $O(k^{n-1})$, we mean that for $r \geq 0$

$$\Big\| \sum_{i=1}^{N+1} |s_i|^2_{h^k} - k^n \frac{\omega^n}{\Omega} \Big\|_{C^r} \leq c_r k^{n-1}$$

where c_r remains bounded if h varies in a compact set (in smooth topology) in the space of hermitian metrics with positive curvature.

We will also need the following important technical result, see [LM07, Theorem 1], [MM12, Section 6]. Note that the C^r estimate below holds for any $f \in C^\infty(M, \mathbb{R})$.

Theorem 3.1.3. *Let us consider $h \in \mathrm{Met}(L)$ with positive curvature, $\omega = c_1(h)$ the induced Kähler form, Ω a smooth positive volume form and $\{s_a\}$ orthonormal basis of $H^0(L^k)$ with respect to $Hilb_\Omega(h^k)$. Then the operator on $C^\infty(M, \mathbb{R})$ given by*

$$Q_k(f)(p) = \frac{1}{k^n} \int_M \sum_{a,b} \langle s_a, s_b\rangle_{h^k}(q) \langle s_a, s_b\rangle_{h^k}(p) f(q)\Omega(q),$$

approximates the operator $\frac{\omega^n}{\Omega}\exp(-\frac{\Delta}{4\pi k})$ *in the following sense. For any* $r \in \mathbb{N}^*$*, there exists* $C > 0$ *such that for all* k *sufficiently large and any function* $f \in C^\infty(M, \mathbb{R})$*, one has*

$$\left\| \left(\frac{\Delta}{k}\right)^r \left(Q_k(f) - \frac{\omega^n}{\Omega}\exp\left(-\frac{\Delta}{4\pi k}\right) f\right) \right\|_{L^2} \leq \frac{C}{k}\|f\|_{L^2} \tag{3.5}$$

$$\|Q_k(f) - \frac{\omega^n}{\Omega} f\|_{C^r} \leq \frac{C}{k}\|f\|_{C^{r+2}} \tag{3.6}$$

where the norms are taken with respect to the induced Kähler form obtained from the fibrewise metric on the polarization L *and* Δ *is the Laplace operator for the induced Kähler metric. The estimate is uniform when the metric varies in a compact set of smooth hermitian metrics with positive curvature.*

We have the following first consequence.

Proposition 3.1.2 ([CaoKel12])**.** *Let* $h_k \in \mathrm{Met}(L^k)$ *be a sequence of metrics such that* $\omega_k := \frac{1}{k}c_1(h_k)$ *is convergent in smooth topology to the Kähler form* ω*. Then the potentials* $\beta_k = -\mathrm{tr}(\mu^0_\Omega \mu_{FS})$ *(induced by the embeddings given by* $Hilb_\Omega(h_k)$*) converge in smooth topology to the potential*

$$1 - \frac{\Omega}{\omega^n}.$$

Note that given a form ω, a sequence of Bergman metrics h_k is known to exist by the previous theorem.

Proof. Let $\{s_i\}$ be an orthonormal basis of $H^0(L^k)$ with respect to the metric $H_k := Hilb_\Omega(h_k)$. By the discussion at the beginning of Section 3.1, the balancing potential at $p \in M$ for the rescaled balancing flow is

$$\beta_k(H_k) = -\int_M \sum_{a,b} \left(\frac{\langle s_a, s_b\rangle(q)}{\sum_{i=1}^{N+1} |s_i(q)|^2} - \frac{\delta_{ab}}{N+1}\right) \frac{\langle s_a, s_b\rangle(p)}{\sum_{i=1}^{N+1} |s_i(p)|^2}\Omega(q),$$

where $\langle ., .\rangle$ stands for the fibrewise metric h_k. By the Riemann-Roch theorem, $N + 1 = k^n \mathrm{Vol}_L(M) + O(k^{n-1})$. From Proposition 3.1.1, the fact that ω_k is convergent to ω and the uniformity of the estimates, we obtain

$$\begin{aligned}\beta_k(H_k) =& 1 - \frac{k^n}{\sum_{i=1}^{N+1}|s_i(p)|^2}\int_M \sum_{a,b} \frac{\langle s_a, s_b\rangle(q)\langle s_a, s_b\rangle(p)}{k^n}\left(\frac{1}{\frac{\omega^n}{\Omega}(q) + O(\frac{1}{k})}\right)\Omega(q)\\ =& 1 - \frac{\Omega}{\omega^n}\int_M \frac{\langle s_a, s_b\rangle(q)\langle s_a, s_b\rangle(p)}{k^n}\left(\left(1 + O\left(\frac{1}{k}\right)\right)\frac{\Omega}{\omega^n}(q)\right)\Omega(q) + O\left(\frac{1}{k}\right).\end{aligned}$$

But now, from Theorem 3.1.3, one knows the asymptotic behavior of the quantification operator $Q_k(f)(p) = \frac{1}{k^n}\int_M \sum_{a,b} \langle s_a, s_b\rangle(q)\langle s_b, s_a\rangle(p) f(q)\Omega(q)$.

Then, for $k \to \infty$, from Inequality (3.6) and the uniformity of the constants, one obtains

$$\beta_k(H_k)(p) = 1 - \frac{\Omega}{\omega^n} Q_k \left(\frac{\Omega}{\omega^n} + O\left(\frac{1}{k}\right)\right) + O\left(\frac{1}{k}\right).$$

The convergence of $Q_k\left(\frac{\Omega}{\omega^n} + O\left(\frac{1}{k}\right)\right)$ to $1 + O(1/k)$ follows from the same arguments as in [Fin10, Pages 10-11] and is a consequence of (3.5). This gives finally the expected result. □

Independent of the considered flows, we have also a general result that complements Theorem 3.1.2.

Proposition 3.1.3 ([CaoKel12])**.** *Let $h(t) \in \mathrm{Met}(L)$ be a path of hermitian metrics on L with $c_1(h(t)) > 0$. Let us consider $h_k(t) = FS(Hilb_\Omega(h(t)^k))^{1/k}$ the path of induced Bergman metrics. Then $\frac{\partial h_k(t)}{\partial t}$ converges to $\frac{\partial h(t)}{\partial t}$ as $k \to +\infty$ in C^∞ topology. This convergence is uniform if $h(t)$ belongs to a compact set in the space of positively curved hermitian metrics on L.*

Proof. Let us assume that $h(t) = h_0 e^{\phi_t}$ and that $\dot{\phi} e^{\phi_t} h_0$ is the infinitesimal change of the fibrewise metric, say at $t = 0$. An infinitesimal change of the L^2 inner product corresponds to the hermitian matrix in the tangent space of the Bergman metrics

$$A = \int_M k\dot{\phi}\langle s_a, s_b\rangle \Omega,$$

and thus the potential associated to that infinitesimal change is, after rescaling to $\mathrm{Met}(L)$,

$$\frac{1}{k}\mathrm{tr}(A\mu_{FS}) = \frac{1}{k}\int_M k\dot{\phi} \sum_{a,b} \langle s_a, s_b\rangle(p) \frac{\langle s_a, s_b\rangle(q)}{\sum_{i=1}^{N+1} |s_i(p)|^2}\Omega(q),$$

where the $\{s_i\}_{i=1,..,N+1}$ form an orthonormal basis of holomorphic sections with respect to $Hilb_\Omega(h_0^k)$ and $h_0^k = \langle .,.\rangle$. Thus, using Proposition 3.1.1, one obtains that

$$\begin{aligned}\frac{1}{k}\mathrm{tr}(A\mu_{FS})(p) &= \frac{\int_M \dot{\phi}(q) \sum_{a,b}\langle s_a, s_b\rangle(p)\langle s_a, s_b\rangle(q)\Omega(q)}{k^n\left(\frac{\omega_0^n}{\Omega}(p) + O\left(\frac{1}{k}\right)\right)} \\ &= \frac{1}{\frac{\omega_0^n}{\Omega}(p) + O(1/k)} Q_k\left(\dot{\phi}\right)(p),\end{aligned}$$

and, as $k \to +\infty$, this converges, thanks to Theorem 3.1.3, towards $\dot{\phi}(p)$ after simplification. □

Remark 3.1.2. Thus we have obtained the convergence of the family $h_k(t)$ in C^1 topology with respect to the variable t. Note that the result cannot be improved, in the sense that, thanks to a direct computation, we don't expect a convergence in C^2 topology. Let us be more precise. An infinitesimal change at order 2 of the induced L^2 inner product along a smooth path of the form $h_0 e^{\phi_t}$ corresponds to a hermitian matrix

$$B = \int_M \left((k\dot{\phi})^2 + k\ddot{\phi} \right) \langle s_a, s_b \rangle \Omega$$

On another hand, the potential associated to this infinitesimal change at $p \in M$ is given after rescaling by the formula

$$\frac{1}{k} \left(\mathrm{tr}(B\mu_{FS}) - \mathrm{tr}(A\mu_{FS})^2 \right)(p) \tag{3.7}$$

Actually, if we write in an orthonormal basis the potential of the metric $FS(Hilb(h(t)^k))$,

$$\varphi(t) = \log \sum_\alpha \lambda_\alpha(t) |s_\alpha|^2$$

with $\varphi(0) = \log \sum_\alpha |s_\alpha|^2$, then $\ddot{\varphi}(t)_{|t=0} = \frac{\sum_\alpha (\lambda_\alpha)''(0)|s_\alpha|^2}{\sum_\alpha |s_\alpha|^2} - \left(\frac{\sum_\alpha (\lambda_\alpha)'(0)|s_\alpha|^2}{\sum_\alpha |s_\alpha|^2} \right)^2$ which shows (3.7). In order to simplify the computations, let us assume that h_0 is solution of the Calabi problem, i.e. $c_1(h_0)^n = \omega_0^n = \Omega$. Now, using this assumption, Proposition 3.1.1, and [MM07, Theorem 4.1.2],

$$\frac{1}{k}\mathrm{tr}(B\mu_{FS}) = \frac{1}{\left(1 + \frac{1}{4\pi}\frac{\mathrm{scal}(\omega_0)}{2k} + O(\frac{1}{k^2})\right)} Q_k\left(k\dot{\phi}^2 + \ddot{\phi}\right).$$

Then we can define the operator on $C^\infty(M, \mathbb{R})$, $\tilde{Q}_k(f) = \frac{1}{1+\frac{1}{4\pi}\frac{\mathrm{scal}(\omega_0)}{2k}} Q_k(f)$. We write

$$\frac{1}{k}\left(\mathrm{tr}(B\mu_{FS}) - \mathrm{tr}(A\mu_{FS})^2 \right) = \tilde{Q}_k(\ddot{\phi}) + k\left(\tilde{Q}_k(\dot{\phi}^2) - \tilde{Q}_k(\dot{\phi})^2 \right) + O\left(\frac{1}{k}\right)$$

Then using Theorem 3.1.3 and [MM12, Theorem 6.1] which gives the asymptotic expansion of Q_k at second order, $\frac{1}{k}\left(\mathrm{tr}(B\mu_{FS}) - \mathrm{tr}(A\mu_{FS})^2\right)$ is equal to

$$\begin{aligned}
=&\ddot{\phi} + O\left(\frac{1}{k}\right) \\
&+ \frac{1}{1 + \frac{1}{4\pi}\frac{\mathrm{scal}(\omega_0)}{2k}} k \left(\dot{\phi}^2 + \frac{1}{k}\left(\frac{\mathrm{scal}(\omega_0)}{8\pi}\dot{\phi}^2 - \frac{1}{4\pi}\Delta_{\omega_0}\dot{\phi}^2 \right) + O\left(\frac{1}{k^2}\right) \right) \\
&- \left(\frac{1}{1 + \frac{1}{4\pi}\frac{\mathrm{scal}(\omega_0)}{2k}} \right)^2 k \left(\dot{\phi} + \frac{1}{k}\left(\frac{\mathrm{scal}(\omega_0)}{8\pi}\dot{\phi} - \frac{1}{4\pi}\Delta_{\omega_0}\dot{\phi} \right) \right)^2
\end{aligned}$$

$$
\begin{aligned}
&=\ddot{\phi}-\frac{1}{4\pi}\Delta_{\omega_0}\dot{\phi}^2+2\dot{\phi}\frac{1}{4\pi}\Delta_{\omega_0}\dot{\phi}+O\left(\frac{1}{k}\right)\\
&=\ddot{\phi}-\frac{1}{2\pi}\|\nabla\dot{\phi}\|^2,
\end{aligned}
$$

which is different from $\ddot{\phi}$.

We are now ready for the proof of Theorem 3.1.1 which identifies the limit of the sequence of rescaled Ω-balancing flows for $k \to +\infty$.

Proof of Theorem 3.1.1. We write $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$. Using the C^1 convergence in t, $\dot{\phi}_t$ is continuous and unique up to a constant that we shall fix by setting $\int_M \dot{\phi}_t\omega_t^n = 0$. Consider the potential $\beta_k(\iota_k(t))$ induced by the embedding $\iota_k(t)$ given by the rescaled Ω-balancing flow at time t. Thanks to Proposition 3.1.3 and the fact that $\int_M \beta_k(\iota_k(t))\omega_k^n(t) \to 0$ when $k \to +\infty$, this sequence of potentials converges to $\dot{\phi}_t$. Moreover, using the balancing condition, we can apply Proposition 3.1.2 to get

$$
\dot{\phi}_t = \lim_{k\to\infty}\beta_k(\iota_k(t)) = 1 - \frac{\Omega}{\omega_t^n}.
$$

□

3.2 The Ω-Kähler flow and the proof of its convergence

3.2.1 The long time existence

We are now interested in the flow

$$
\frac{\partial\phi_t}{\partial t} = 1 - \frac{\Omega}{(\omega+\sqrt{-1}\partial\bar{\partial}\phi_t)^n} \tag{3.8}
$$

over a compact Kähler manifold (not necessarily in an integral Kähler class), where $\phi_0 = 0$ and ω is a Kähler form in a fixed class $[\alpha]$. Of course, this can be rewritten as

$$
(\omega+\sqrt{-1}\partial\bar{\partial}\phi_t)^n = \frac{1}{1-\frac{\partial\phi_t}{\partial t}}e^f\omega^n \tag{3.9}
$$

where f is a smooth (bounded) function defined by $f = \log(\Omega/\omega^n)$. We are interested in the long time existence of this flow and its convergence. We study now long time existence and convergence of this flow, following the ideas of [Cao85]. Note that after we wrote this article we have been informed that similar results were proved recently in [FLM11] , and we would like to thank Prof. Z. Blocki for pointing out this reference to us. In this section we will prove the following result.

Theorem 3.2.1 ([CaoKel12]). *Let ϕ_t be the solution of*

$$\frac{\partial \phi_t}{\partial t} = 1 - \frac{\Omega}{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n}$$

on the maximal time interval $0 \leq t < T_{max}$. Let $v_t = \phi_t - \frac{1}{\mathrm{Vol}_L(M)} \int_M \phi_t \omega^n$. Then the C^∞ norm of v_t are uniformly bounded for all $0 \leq t < T_{max}$ and $T_{max} = +\infty$.

We remark that if we look at the formal level of this equation in terms of cohomology class, we obtain directly

$$\frac{\partial[(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)]}{\partial t} = 0,$$

which shows that the Kähler form

$$\omega_t := \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$$

remains in the same class as $\omega + \sqrt{-1}\partial\bar{\partial}\phi_0$, i.e. $[\alpha]$.

Proposition 3.2.1 ([CaoKel12]). *The function $\frac{\partial \phi_t}{\partial t}$ and $\frac{1}{1-\frac{\partial \phi_t}{\partial t}}$ remain (uniformly) bounded in C^0 norm along the flow given by Equation (3.9).*

Proof. Let us differentiate Equation (3.8), we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi_t}{\partial t}\right) = \frac{\Omega}{\omega_t^n}\Delta_t \left(\frac{\partial \phi_t}{\partial t}\right)$$

with Δ_t the normalized Laplacian with respect to the metric $\omega + \sqrt{-1}\partial\bar{\partial}\phi_t$. We apply now the maximum principle for parabolic equations at the point where $\frac{\partial \phi_t}{\partial t}$ attains its maximum (respectively its minimum). Plugging this information in (3.8), we obtain

$$\frac{\partial \phi_t}{\partial t} \leq \sup_M (1 - e^f)$$

and moreover

$$\frac{\partial \phi_t}{\partial t} \geq \inf_M (1 - e^f).$$

On another hand,

$$\frac{\partial}{\partial t}\left(\frac{1}{1 - \frac{\partial \phi_t}{\partial t}}\right) = \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n}{\Omega}\Delta_t \left(\frac{\partial \phi_t}{\partial t}\right),$$

and one applies again the maximum principle to obtain the proposition. □

We denote Δ the Laplacian with respect to the Kähler form ω given at $t = 0$.

Lemma 3.2.1. *One has*

$$0 < n + \Delta\phi_t.$$

Proof. The fact that $\omega + \sqrt{-1}\partial\bar{\partial}\phi_t$ is a Kähler form implies by taking the trace that $n + \Delta\phi_t > 0$. □

We show now the upper bound for the Laplacian of the potential.

Proposition 3.2.2 ([CaoKel12])**.** *There exist positive constants C_1 and C_2 such that*

$$0 < n + \Delta\phi_t \leq C_1 e^{C_2(\phi_t - \inf_{M\times[o,T)}\phi_t)}, \quad \textit{for all } t \in [0, T).$$

Proof. In the proof we denote ϕ_t by ϕ, omitting the subscript for the sake of clearness. Moreover, g (resp g_t) denote the Riemannian metric associated to the Kähler form ω (resp. $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$).

First of all, using holomorphic normal coordinates system at any point $p \in M$, we have

$$\Delta_t(n + \Delta\phi) = g_t^{k\bar{l}}(g^{i\bar{j}}\phi_{i\bar{j}})_{k\bar{l}} = g_t^{k\bar{l}}R_{i\bar{j}k\bar{l}}\phi_{j\bar{i}} + g_t^{k\bar{l}}g^{i\bar{j}}\phi_{i\bar{j}k\bar{l}}.$$

Set

$$\hbar = \log\frac{\omega_t^n}{\Omega} = \log\omega_t^n - \log\omega^n - f.$$

so that

$$e^{-\hbar} = \frac{\Omega}{\omega_t^n}.$$

The idea of the proof is essentially to apply maximum principle to the quantity $(n + \Delta\phi)$ with the operator $e^{-\hbar}\Delta_t - \frac{\partial}{\partial t}$.

Now, by using holomorphic normal coordinates and direct computations, we get

$$\Delta\hbar = -g_t^{i\bar{q}}g_t^{p\bar{j}}\phi_{i\bar{j}k}\phi_{p\bar{q}\bar{k}} + g_t^{i\bar{j}}(-R_{i\bar{j}} + \phi_{i\bar{j}k\bar{k}}) + R - \Delta f.$$

Here $R_{i\bar{j}k\bar{l}}$ and $R = \mathrm{scal}(\omega)$ denote the curvature tensor and the scalar curvature of the metric $g_{i\bar{j}}$ respectively. Then

$$\begin{aligned}\frac{\partial}{\partial t}(n + \Delta\phi) =&\Delta(\frac{\partial\phi}{\partial t}) = -\Delta(e^{-\hbar}) = e^{-\hbar}(\Delta\hbar - |\nabla\hbar|^2)\\ =&e^{-\hbar}(g_t^{i\bar{j}}g^{k\bar{l}}\phi_{i\bar{j}k\bar{l}} - g_t^{i\bar{j}}R_{i\bar{j}} + R - \Delta f - g_t^{i\bar{q}}g_t^{p\bar{j}}\phi_{i\bar{j}k}\phi_{p\bar{q}\bar{k}} - |\nabla\hbar|^2).\end{aligned}$$

Thus

$$(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(n+\Delta\phi) = e^{-\hbar}[g_t^{k\bar{l}} g^{i\bar{j}}(\phi_{i\bar{j}k\bar{l}} - \phi_{k\bar{l}i\bar{j}}) + g_t^{k\bar{l}} R_{i\bar{j}k\bar{l}}\phi_{j\bar{i}} + g_t^{i\bar{j}} R_{i\bar{j}} - R + \Delta f + g_t^{i\bar{q}} g_t^{p\bar{j}} \phi_{i\bar{j}k}\phi_{p\bar{q}\bar{k}} + |\nabla\hbar|^2].$$

On the other hand, by commuting the covariant derivatives, we have

$$\phi_{i\bar{j}k\bar{l}} - \phi_{k\bar{l}i\bar{j}} = R_{i\bar{q}k\bar{l}}\phi_{q\bar{j}} - R_{i\bar{j}k\bar{q}}\phi_{q\bar{l}}.$$

Hence

$$(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(n+\Delta\phi) = e^{-\hbar}[2g_t^{k\bar{l}} R_{i\bar{j}k\bar{l}}\phi_{j\bar{i}} - g_t^{k\bar{l}} R_{k\bar{q}}\phi_{q\bar{l}} + g_t^{i\bar{j}} R_{i\bar{j}} - R + \Delta f + g_t^{i\bar{q}} g_t^{p\bar{j}} \phi_{i\bar{j}k}\phi_{p\bar{q}\bar{k}} + |\nabla\hbar|^2].$$

Moreover, if we choose another coordinates system so that $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\phi_{i\bar{j}} = \phi_{i\bar{i}}\delta_{i\bar{j}}$,

$$\begin{aligned} g_t^{k\bar{l}} R_{i\bar{j}k\bar{l}}\phi_{j\bar{i}} - g_t^{k\bar{l}} R_{k\bar{q}}\phi_{q\bar{l}} &= \sum_{i,k} R_{i\bar{i}k\bar{k}}(\frac{\phi_{i\bar{i}}}{1+\phi_{k\bar{k}}} - \frac{\phi_{k\bar{k}}}{1+\phi_{k\bar{k}}}) \\ &= \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\phi_{i\bar{i}}^2 - \phi_{i\bar{i}}\phi_{k\bar{k}}}{(1+\phi_{i\bar{i}})(1+\phi_{k\bar{k}})} \\ &= \frac{1}{2}\sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{(\phi_{i\bar{i}} - \phi_{k\bar{k}})^2}{(1+\phi_{i\bar{i}})(1+\phi_{k\bar{k}})}, \end{aligned}$$

and

$$\begin{aligned} g_t^{k\bar{l}} R_{i\bar{j}k\bar{l}}\phi_{j\bar{i}} + g_t^{i\bar{j}} R_{i\bar{j}} - R &= \sum_{i,k} R_{i\bar{i}k\bar{k}}(\frac{\phi_{i\bar{i}}}{1+\phi_{k\bar{k}}} + \frac{1}{1+\phi_{k\bar{k}}} - 1) \\ &= \frac{1}{2}\sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{(\phi_{i\bar{i}} - \phi_{k\bar{k}})^2}{(1+\phi_{i\bar{i}})(1+\phi_{k\bar{k}})}. \end{aligned}$$

Therefore,

$$(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(\Delta\phi) = e^{-\hbar}[\sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}(\phi_{i\bar{i}} - \phi_{k\bar{k}})^2}{(1+\phi_{i\bar{i}})(1+\phi_{k\bar{k}})} + \Delta f + g_t^{i\bar{q}} g_t^{p\bar{j}} \phi_{i\bar{j}k}\phi_{p\bar{q}\bar{k}} + |\nabla\hbar|^2]. \tag{3.10}$$

Now, we assume the curvature tensor $R_{i\bar{j}k\bar{l}}$ is bounded below by $-C_0$, for some constant $C_0 > 0$, so that

$$R_{i\bar{j}k\bar{l}} \geq -C_0(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}).$$

Then, from (3.10) we obtain

$$(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(\Delta\phi) \geq e^{-\hbar}[-2C_0(\sum_{i,k} \frac{1+\phi_{i\bar{i}}}{1+\phi_{k\bar{k}}} - n^2) + \Delta f + g_t^{i\bar{q}} g_t^{p\bar{j}} \phi_{i\bar{j}k}\phi_{p\bar{q}\bar{k}}]. \tag{3.11}$$

Finally, we consider the function $e^{-C\phi}(n+\Delta\phi)$ and compute

$$\begin{aligned}
\Delta_t(e^{-C\phi}(n+\Delta\phi)) =& \, C^2 e^{-C\phi}(n+\Delta\phi) g_t^{i\bar{j}}\phi_i\phi_{\bar{j}} \\
& - Ce^{-C\phi} g_t^{i\bar{j}}[\phi_i(\Delta\phi)_{\bar{j}} + (\Delta\phi)_i\phi_{\bar{j}}] \\
& - Ce^{-C\phi}(n+\Delta\phi)\Delta_t\phi + e^{-C\phi}\Delta_t(n+\Delta\phi) \\
\geq & - (n+\Delta\phi)^{-1} e^{-C\phi} g_t^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}} \\
& - Ce^{-C\phi}(n+\Delta\phi)\Delta_t\phi + e^{-C\phi}\Delta_t(n+\Delta\phi),
\end{aligned}$$

$$\frac{\partial}{\partial t}(e^{-C\phi}(n+\Delta\phi)) = -Ce^{-C\phi}(n+\Delta\phi)\frac{\partial}{\partial t}\phi + e^{-C\phi}\frac{\partial}{\partial t}(n+\Delta\phi).$$

Thus,

$$\begin{aligned}
(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(e^{-C\phi}(n+\Delta\phi)) \geq & - (n+\Delta\phi)^{-1} e^{-(C\phi+\hbar)} g_t^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}} \\
& + e^{-C\phi}(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(n+\Delta\phi) \\
& - Ce^{-C\phi}(n+\Delta\phi)(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})\phi.
\end{aligned}$$

Now observe that, by using $g_{i\bar{j}} = \delta_{i\bar{j}}$, $\phi_{i\bar{j}} = \phi_{i\bar{i}}\delta_{i\bar{j}}$ and (3.11), we have

$$\begin{aligned}
-(n+\Delta\phi)^{-1} & g_t^{i\bar{j}}(\Delta\phi)_i(\Delta\phi)_{\bar{j}} + (\Delta_t - \frac{\partial}{\partial t})(n+\Delta\phi) \\
\geq & - (n+\Delta\phi)^{-1}\sum_i (1+\phi_{i\bar{i}})^{-1}|\sum_k \phi_{k\bar{k}i}|^2 \\
& + \sum_{i,j,k}(1+\phi_{i\bar{i}})^{-1}(1+\phi_{k\bar{k}})^{-1}|\phi_{i\bar{j}k}|^2 + \Delta f \\
& - 2C_0(\sum_{i,k}\frac{1+\phi_{i\bar{i}}}{1+\phi_{k\bar{k}}} - n^2) \\
\geq & - 2C_0(\sum_{i,k}\frac{1+\phi_{i\bar{i}}}{1+\phi_{k\bar{k}}} - n^2) + \Delta f.
\end{aligned}$$

Therefore, by taking $C = C_0 + 1$,

$$\begin{aligned}
(e^{-\hbar}\Delta_t - \frac{\partial}{\partial t})(e^{-C\phi}(n+\Delta\phi)) \geq\ & e^{-(C\phi+\hbar)}(\Delta f + n^2 C_0) \\
& - Ce^{-(C\phi+\hbar)}(n+\Delta\phi)(n - e^{\hbar}\frac{\partial\phi}{\partial t}) \\
& + (C - C_0)e^{-(C\phi+\hbar)}(n+\Delta\phi)\sum_i \frac{1}{1+\phi_{i\bar{i}}} \\
\geq\ & e^{-(C\phi+\hbar)}(\Delta f + n^2 C_0) \\
& - Ce^{-(C\phi+\hbar)}(n+\Delta\phi)(n - e^{\hbar}\frac{\partial\phi}{\partial t}) \\
& + e^{-(C\phi+\hbar+\frac{f}{n-1})}(1 - \frac{\partial\phi}{\partial t})^{\frac{-1}{n-1}}(n+\Delta\phi)^{\frac{n}{n-1}},
\end{aligned} \tag{3.12}$$

where in the last inequality we have used the arithmetic-geometric inequality

$$\begin{aligned}
\sum_i \frac{1}{1+\phi_{i\bar{i}}} &\geq (\frac{\sum_i(1+\phi_{i\bar{i}})}{(1+\phi_{1\bar{1}})\cdots(1+\phi_{n\bar{n}})})^{1/n-1} \\
&= [e^{-f}(1 - \frac{\partial\phi}{\partial t})]^{1/(n-1)}(n+\Delta\phi)^{\frac{1}{n-1}}.
\end{aligned}$$

Now the proposition follows from the maximum principle and Proposition 3.2.1. Actually, at the point (p, t_0) where $(e^{-C\phi}(n+\Delta\phi))$ achieves its maximum, the left hand side of (3.12) is non positive and hence

$$(n + \Delta\phi(p,t_0))^{\frac{n}{n-1}} \leq C'(1 + (n + \Delta\phi(p,t_0)))$$

with C' independent of t. Finally, $(n + \Delta\phi(p,t_0)) \leq C_1$ which gives the result. ☐

Using the fact that we are working with plurisubharmonic potentials, we get the obvious fact:

Lemma 3.2.2. *Let us denote*

$$v_t = \phi_t - \frac{1}{\mathrm{Vol}_L(M)}\int_M \phi_t\omega^n$$

where ϕ_t is solution to Equation (3.9). Then, there exist constants c_2, c_3 such that

$$\sup_{M\times[0,T]} v_t \leq c_2,$$

$$\sup_{M\times[0,T]} \int_M |v_t|\omega^n \leq c_3.$$

Proposition 3.2.3 ([CaoKel12])**.** *There exists a constant $c_4 > 0$ such that*

$$\sup_{M\times[0,T]} |v_t| \leq c_4.$$

Sketch of the proof. We apply the Nash-Moser iteration argument. The only major difference with [Cao85, Lemma 3] is that in [Cao85, Equation (1.14)], the right hand side is bounded by the term

$$n! \int_M \frac{(-v_t)^{p-1}}{p-1} \left(\frac{e^f}{1 - \frac{\partial \phi_t}{\partial t}} - 1 \right) \omega^n.$$

But now, from Proposition 3.2.1, one can give the following upper bound for this term:

$$C \int_M \frac{(-v_t)^{p-1}}{p-1} \omega^n,$$

where C is a uniform positive constant. This ensures that one can applies the Nash-Moser argument. This implies in a similar way to the computations of [Cao85, page 364] the C^0 estimate. □

With Propositions 3.2.3 and 3.2.2 and Lemma 3.2.1, one obtains a uniform bound of the quantity $n + \Delta\phi_t = n + \Delta v_t$. This implies from the Schauder estimates a first order estimate

$$\sup_{M\times[0,T]} |\nabla v_t| \leq c_5 (\sup_{M\times[0,T]} |\Delta v_t| + \sup_{M\times[0,T]} |v_t|) \leq c_5'.$$

All the second order derivatives of the potential v_t are bounded. From the last inequality, one sees that in normal coordinates, the terms $1 + \phi_{i\bar{i}}$ is bounded from above, while from Proposition 3.2.3 and (3.3), the term $\prod_i (1 + \phi_{i\bar{i}})$ is bounded. So finally, $1 + \phi_{i\bar{i}}$ is uniformly bounded along the time.

From Calabi's work and similarly to [Yau78; Cao85], it is now standard that it implies also the third order estimate. Finally, using Schauder regularity theory [GT01] we have proved long time existence of the Ω-Kähler flow. This concludes the proof of Theorem 3.2.1.

3.2.2 The convergence

In this section, we are interested in the convergence of the Ω-Kähler flow.

Theorem 3.2.2 ([CaoKel12])**.** *Let us denote $v_t = \phi_t - \frac{1}{\mathrm{Vol}_L(M)} \int_M \phi_t \omega^n$ where ϕ_t is solution to Equation (3.9), the Ω-Kähler flow. Then, v_t converges when $t \to \infty$ to v_∞ in smooth topology and $\frac{\partial \phi_t}{\partial t}$ converges to a constant in smooth topology.*

Note that we also refer to [FLM11] for an independent proof of this result. To prove the convergence of the Ω-Kähler flow, we need some results of P. Li and S.T. Yau for the positive solution of the heat equation on Riemannian compact manifolds [LY86, Section 2]. This takes the following form.

Proposition 3.2.4. *Let M be a compact manifold of dimension n. Let $\gamma_{ij}(t)$ a family of Riemannian metrics on M such that*

1. $c_0\gamma_{ij}(0) \leq \gamma_{ij}(t) \leq c_0'\gamma_{ij}(0)$,
2. $|\frac{\partial \gamma_{ij}}{\partial t}|(t) \leq c_1\gamma_{ij}(0)$,
3. *for the Ricci curvature,* $R_{ij}(t) \geq -Kg_{ij}(0)$,

where c_0, c_0', c_1, K are positive constants independent of t. If we denote $\tilde{\Delta}_t$ the Laplace operator of the metric $\gamma_{ij}(t)$, and if $\phi(p,t)$ is a positive solution of the equation

$$\left(\tilde{\Delta}_t - \frac{\partial}{\partial t}\right)\phi(p,t) = 0$$

on $M \times [0,T)$, then one has the following Harnack type inequality for any $\alpha > 1$:

$$\sup_{p\in M}\phi(p,t_1) \leq \inf_{p\in M}\phi(p,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{n}{2}}\exp\left(\frac{c_3}{t_2-t_1} + c_4(t_2-t_1)\right)$$

where c_3 depends on c_0' and the diameter of M with respect to $\gamma_{ij}(0)$, c_4 depends on the quantities α, K, n, c_0', c_1, $\sup\|\nabla^2\log\phi\|$ and $0 < t_1 < t_2 < T$.

With Theorem 3.2.1 in our hands, we shall apply Proposition 3.2.4 with $\gamma_{ij}(t) = \frac{\omega_t^n}{\Omega}g_{i\bar{j}}(t)$ where $g_{i\bar{j}}(t)$ is the metric associated with the Kähler form $\omega + \sqrt{-1}\partial\bar{\partial}\phi_t$. Thus, $\tilde{\Delta}_t = \frac{\Omega}{(\omega+\sqrt{-1}\partial\bar{\partial}\phi_t)^n}\Delta_t$ and the potential ϕ_t solution of Equation (3.3) satisfies

$$\left(\tilde{\Delta}_t - \frac{\partial}{\partial t}\right)\frac{\partial\phi_t(p)}{\partial t} = 0.$$

We apply the same reasoning than in [Cao85, Section 2]. This turns out to show that the quantity

$$E(t) = \int_M\left(\frac{\partial\phi_t}{\partial t} - \frac{1}{\mathrm{Vol}_L(M)}\int_M\frac{\partial\phi_t}{\partial t}\omega_t^n\right)^2\omega_t^n$$

is (at least exponentially fast) decreasing to 0. The only difference with the computation in [Cao85] is that we need to show that the $\gamma_{ij}(t)$ are uniformly equivalent to $\gamma_{ij}(0)$. But this is clear because the metrics $g_{i\bar{j}}(t)$ and $g_{i\bar{j}}(0)$ are uniformly equivalent thanks to Theorem 3.2.1, and the same happens for their respective volume forms. So the first eigenvalue of the Laplacian $\tilde{\Delta}_t$ is under control.

Similarly to [Cao85, Proposition 2.2], we obtain now Theorem 3.2.2. Note that a consequence of Theorem 3.2.1 is the existence of a convergent sequence $v(p, t_n)$ in smooth topology (with $t_n \to \infty$ when $n \to \infty$) towards a smooth function v_∞.

3.2.3 Corollaries

A direct consequence of Theorem 3.2.2 is the convergence of the Ω-Kähler flow to the solution of the Calabi conjecture. Actually, the limit v_∞ satisfies

$$(\omega + \sqrt{-1}\partial\bar{\partial}v_\infty)^n = (\omega + \sqrt{-1}\partial\bar{\partial}\phi_\infty)^n = \Omega.$$

In other words, one can prescribe the volume form in a given Kähler class. This was first proved by S-T. Yau in [Yau78] and our proof uses essentially the same type of estimates. Of course, if the canonical bundle of M is trivial, then there is a global nowhere vanishing n-form and the limit metric is a Calabi-Yau metric (i.e Ricci-flat and Kähler).

We also remark that one can modify slightly Equation (3.8) if the manifold M has negative first Chern class. In that case, it is natural to introduce the following flow:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n = \frac{1}{1 - \frac{\partial\phi_t}{\partial t}} e^{f+\phi_t}\omega^n \tag{3.13}$$

where $\omega \in -c_1(M) > 0$, and f is the deviation of the Ricci curvature of ω, that is $Ric(\omega) + \omega = \sqrt{-1}\partial\bar{\partial}f$ and $\int_M \frac{1}{1-\frac{\partial\phi_t}{\partial t}} e^{f+\phi_t}\omega^n = \mathrm{Vol}_{K_M}(M)$. In that case similar computations to Section 3.2.1 will involve the operator $\Delta_t - \mathrm{Id}$ since by differentiating (3.13), one obtains

$$\frac{\partial}{\partial t}\left(\frac{\partial\phi_t}{\partial t}\right) = \frac{e^{f+\phi_t}\omega^n}{\omega_t^n}\left(\Delta_t\left(\frac{\partial\phi_t}{\partial t}\right) - \left(\frac{\partial\phi_t}{\partial t}\right)\right).$$

The uniform bound of the term $\frac{1}{1-\frac{\partial\phi_t}{\partial t}}$ can be proved in a similar way to Section 3.2.1 (Proposition 3.2.1) and by maximum principle, there is a uniform bound of the potentials ϕ_t. Thus, one obtains the convergence of ϕ_t when $t \to \infty$ and $\omega + \sqrt{-1}\partial\bar{\partial}\phi_\infty$ is a smooth Kähler-Einstein metric with negative curvature.

3.3 Proof of the convergence results

In this section (M, L) is a polarized manifold and we are only considering integral Kähler classes. The techniques we use in this section to prove

Theorem 3.0.11 are inspired from the techniques of [Fin10].

3.3.1 First order approximation

We know that from any starting point $\omega = \omega_0$, there exists a solution

$$\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$$

to the Ω-Kähler flow from the results of Section 3.2. We can write $\omega_t = c_1(h_t)$ where h_t is a sequence of hermitian metrics on the line bundle L. Furthermore, we can construct a natural sequence of Bergman metrics

$$\hat{h}_k(t) = FS(Hilb_\Omega(h_t^k))^{1/k}$$

by pulling back the Fubini-Study metric using sections which are orthonormal with respect to the inner product

$$\frac{1}{k^n}\int_M h_t(.,.)^k\Omega.$$

Using Proposition 3.1.1 we obtain the asymptotic behavior

$$\hat{h}_k(t) = \left(\frac{k^n c_1(h_t)^n}{\Omega} + O\left(\frac{1}{k}\right)\right)^{1/k} h_t$$

for k sufficiently large. Thus, the sequence $\hat{h}_k(t)$ converges to h_t as $k \to \infty$.

On the other hand, the rescaled Ω-balancing flow provides a sequence of metrics $\omega_k(t) = c_1(h_k(t))$ which are solutions to (3.2). Note that by construction, we fix $h_k(0) = \hat{h}_k(0)$ for the starting point of the rescaled Ω-balancing flow.

In this section, we wish to evaluate the distance between the two metrics $h_k(t)$ and $\hat{h}_k(t)$. Since we are dealing with algebraic metrics, we have the (rescaled) metric on Hermitian matrices given by

$$d_k(H_0, H_1) = \left(\frac{\mathrm{tr}\ (H_0 - H_1)^2}{k^2}\right)^{1/2}$$

on $\mathrm{Met}(H^0(L^k))$ which induces a metric on $\mathrm{Met}(L)$, that we denote by dist_k.

Proposition 3.3.1 ([CaoKel12]). *One has*

$$\mathrm{dist}_k(h_k(t), \hat{h}_k(t)) \leq \frac{C}{k},$$

for some constant $C > 0$ independent of k.

Proof. Let us consider $e^{\phi(t)}h_0$ a family of hermitian metrics with positive curvature, and denote

$$\omega_t = c_1(e^{\phi(t)} h_0).$$

The infinitesimal change at t in the L^2 inner product induced by this path and the volume form Ω is given by

$$\hat{U}_{\alpha,\beta}(t) = \frac{1}{k^n} \int_M \langle s_\alpha, s_\beta \rangle \, k\dot{\phi}(t) \, \Omega$$

for $\{s_\alpha\}$ an orthonormal basis of $H^0(L^k)$ with respect to the L^2-inner product

$$\frac{1}{k^n} \int_M e^{k\phi(t)} \Omega.$$

The formula is obtained by noticing that the variation occurs with respect to the fibrewise metric. Now, if furthermore $\phi(t)$ is a solution to the Ω-Kähler flow, this infinitesimal change is given at $\hat{h}_k(t)$ as

$$\hat{U}_{\alpha,\beta}(t) = \frac{1}{k^n} \int_M \langle s_\alpha, s_\beta \rangle \left(k \left(1 - \frac{\Omega}{\omega_t^n} \right) \right) \, \Omega, \tag{3.14}$$

with $\{s_\alpha\}$ satisfy the same assumption as above.

On another hand, the tangent (at the same point $\hat{h}_k(t)$) to the rescaled Ω-balancing flow (3.2) is given by directly by the moment map μ^0_Ω, and we write the infinitesimal change of the L^2 metric as

$$U_{\alpha,\beta}(t) = k \int_M \left(\frac{\delta_{\alpha\beta}}{N+1} - \frac{\langle s_\alpha, s_\beta \rangle}{\sum_{i=1}^{N+1} |s_i|^2} \right) \Omega, \tag{3.15}$$

where s_i are L^2 orthonormal with respect to the L^2 inner product induced by $h(t)^k$ and Ω. Again, from Proposition 3.1.1, one has asymptotically

$$U_{\alpha,\beta}(t) = \hat{U}_{\alpha,\beta}(t) + \frac{1}{k^n} \int_M \langle s_\alpha, s_\beta \rangle O(1) \, \Omega.$$

Here the term $O(1)$ stands implicitly for a (smooth) function which is bounded independently of the variables t and k. Thus, one has

$$\frac{\mathrm{tr}\, (\hat{U}_{\alpha,\beta}(t) - U_{\alpha,\beta}(t))^2}{k^2} = \langle \frac{1}{k} O(1), Q_k \left(\frac{1}{k} O(1) \right) \rangle_{L^2}.$$

We can use Theorem 3.1.3, Inequality (3.5) to obtain that

$$\frac{\mathrm{tr}\, (\hat{U}_{\alpha,\beta}(t) - U_{\alpha,\beta}(t))^2}{k^2} = O(k^{-2}).$$

This shows that $d_k(\hat{U}_{\alpha,\beta}(t), U_{\alpha,\beta}(t))) = O(1/k)$. If we denote by $\tilde{h}_k(t)$ the rescaled balancing flow passing through $\hat{h}_k(t_0)$ at $t = t_0$, we have just proved that $\tilde{h}_k(t)$ and $\hat{h}_k(t)$ are tangent up to an error term in $O(1/k)$ at $t = t_0$. On the other hand, it is clear that $\tilde{h}_k(t)$ and $h_k(t)$ are close when $t \to \infty$, because they are obtained through the gradient flow of the same moment map and this gradient flow converges to the unique Ω-balanced metric (this is a consequence of [Don09]). Thus $\mathrm{dist}(\tilde{h}_k(t), h_k(t)) = O(1/k)$. This finally proves the result. □

3.3.2 Higher order approximations

In this section, we improve the result of the last section by constructing a new time-dependent function

$$\psi(k,t) = \phi_t + \sum_{j=1}^{m} \frac{1}{k^j}\eta_j(t)$$

which is obtained by deforming the solution to the Ω-Kähler flow and which satisfies the property to be "as close" as we wish to the Ω-Balancing flow. We will need to compare this metric to the Bergman metric $h_k(t)$. Thus, we introduce the Bergman metric associated to $h_0 e^{\psi(k,t)}$, i.e

$$\overline{h}_k(t) = FS(Hilb_\Omega(h_0^k e^{k\psi(k,t)}))^{1/k}.$$

We wish to minimize the quantity

$$\mathrm{dist}_k(\overline{h}_k(t), h_k(t))$$

by showing an estimate of the form $\mathrm{dist}_k(\overline{h}_k(t), h_k(t)) < C/k^{m+1}$, with $C > 0$ a constant independent of $k >> 0$ and t. This is the parameter version of [Don01b, Theorem 26], and Proposition 3.3.1 shows that the result holds for $m = 0$. One needs to choose inductively the functions η_j and this is done by linearizing the Monge-Ampère operator.

Let us give some details for the first step of the induction, that is to find η_1. Consider the tangent to the path $\overline{h}_k(t)$, then similarly to (3.14), this tangent can be written as

$$\overline{T}_{\alpha\beta}(t) = \frac{1}{k^n}\int_M k\langle s_\alpha, s_\beta\rangle \left(1 - \frac{\Omega}{\omega_t^n} + \frac{\dot{\eta_1}}{k} + O(1/k)\right)\,\Omega,$$

where $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$ and $\{s_\alpha\}$ is L^2 orthonormal with respect to $e^{-\phi_t}h_0$ and the volume form Ω. On another hand, the tangent to the rescaled balancing flow at the point $\overline{h}_k(t)$ is given, similarly to (3.15) by

$$T_{\alpha\beta}(t) = \frac{1}{k^n}\int_M k\langle s_\alpha, s_\beta\rangle \left(1 - \frac{\Omega}{c_1(\overline{h}_k(t))^n} + O(1/k)\right)\,\Omega. \tag{3.16}$$

But now,

$$\frac{\Omega}{c_1(\overline{h}_k(t))^n} = \frac{\Omega}{\omega_t^n} - \frac{\Omega}{\omega_t^n}\Delta_t\left(\frac{1}{k}\eta_1\right) + O(1/k^2)$$

and we can write the error term $kO(1/k)$ from (3.16) as

$$kO(1/k) = \sum_{i\geq 0} \gamma_{1,i}k^{-i} = \gamma_{1,0} + O(1/k)$$

for $\gamma_{1,i}$ smooth functions with real values depending on the metric and obtained from the Bergman function asymptotics, so

$$T_{\alpha\beta}(t) = \frac{1}{k^n}\int_M \langle s_\alpha, s_\beta\rangle \left(k\left(1 - \frac{\Omega}{\omega_t^n} + \frac{\Omega}{\omega_t^n}\Delta_t\left(\frac{1}{k}\eta_1\right) + \frac{\gamma_{1,0}}{k}\right) + O(1/k)\right)\Omega.$$

If we wish to force $d_k(\overline{T}_{\alpha\beta}(t), T_{\alpha\beta}(t))$ to be of size $O(1/k^2)$, we need to find η_1 such that

$$\frac{\partial\eta_1(t)}{\partial t} - \frac{\Omega}{\omega_t^n}\Delta_t\eta_1(t) = \gamma_{1,0} \tag{3.17}$$

for all $t \geq 0$ and $\eta_1(0) = 0$. But, by the standard parabolic theory (see, e.g., [Bak11, Section 3.1] for a detailed exposition), a smooth solution η_1 to the above initial-value problem exists and is unique. Then, one obtains

$$\frac{\mathrm{tr}\,(\overline{T}_{\alpha\beta}(t) - T_{\alpha\beta}(t))^2}{k^2} = \langle O(1/k^2), Q_k(O(1/k^2))\rangle_{L^2}$$

and we can conclude with similar arguments to Section 3.3.1: there exists a constant $C > 0$ independent of t such that

$$\frac{\mathrm{tr}\,(\overline{T}_{\alpha\beta}(t) - T_{\alpha\beta}(t))^2}{k^2} \leq \frac{C}{k^4}. \tag{3.18}$$

This implies, by the same arguments as in the end of the proof of Proposition 3.3.1, that

$$\mathrm{dist}_k(h_k(t), \overline{h}_k(t)) \leq \frac{C}{k^2}.$$

Now, for higher order expansions, one considers higher order asymptotics in the expressions above. The same reasoning can be applied for the construction of higher order approximation. It will involve, knowing the terms $\eta_1, .., \eta_m$ to find η_{m+1}, solution of a similar equation to (3.17), where the (non constant) R.H.S will depend on the functions η_j $(1 \leq j \leq m)$ computed at previous step:

$$\frac{\partial\eta_{m+1}(t)}{\partial t} - \frac{\Omega}{\omega_t^n}\Delta_t\eta_{m+1}(t) = \gamma_{m+1,0}(\eta_1, ..., \eta_m). \tag{3.19}$$

Again, it is possible to solve (3.19) by inverting the operator $\frac{\Omega}{\omega_t^n}\Delta_t - \frac{\partial}{\partial t}$.

Finally, we have obtained

Theorem 3.3.1 ([CaoKel12])**.** *Given solution ϕ_t to the Ω-Kähler flow (3.3) and $k >> 0$, there exist functions $\eta_1, ..., \eta_m$, $m \geq 1$, such that the deformation of ϕ_t given by the potential*

$$\psi(k,t) = \phi_t + \sum_{j=1}^{m} \frac{1}{k^j}\eta_j(t)$$

satisfies

$$\mathrm{dist}_k(h_k(t), \overline{h}_k(t)) \leq \frac{C}{k^{m+1}}.$$

Here $\overline{h}_k(t) = FS(Hilb_\Omega(h_0^k e^{k\psi(k,t)}))^{1/k} \in \mathrm{Met}(L)$ is the induced Bergman metric from the potential ψ, $h_k(t) \in \mathrm{Met}(L)$ is the sequence of metric obtained by the rescaled balancing flow (3.2), and C is a positive constant independent of k and t.

Proof. The only point that we did not explain earlier is that C is independent of the variable $t \in \mathbb{R}_+$. This comes from the following facts. On one hand, the expansion of the Bergman function of a family of metrics h_t is uniform if the metrics h_t belong to a compact subset of hermitian positive metrics in $\mathrm{Met}(L)$, see Theorem 3.1.2. On the other hand, we have seen that the metrics involved in the Ω-Kähler flow are in a bounded set, since ω_t is convergent in smooth topology when $t \to +\infty$ thanks to Theorem 3.2.2. This completes the proof of Theorem 3.3.1. □

Furthermore, on can improve slightly this result by showing that one has C^1 convergence in t.

Proposition 3.3.2 ([CaoKel12])**.** *Under the same assumptions and notations of the previous theorem, one has*

$$\mathrm{dist}_k\left(\frac{\partial h_k(t)}{\partial t}, \frac{\partial \overline{h}_k(t)}{\partial t}\right) \leq \frac{C}{k^m},$$

where C is a uniform constant in k and t.

Proof. One needs essentially to give an estimate of the quantity

$$\mathrm{tr}\left(\frac{\partial \overline{T}_{\alpha\beta}(t)}{\partial t} - \frac{\partial T_{\alpha\beta}(t)}{\partial t}\right)^2.$$

Let us assume that we have fixed η_1 as in the proof of the theorem, that is

$m = 1$. Then, as the first step, we are lead to estimate

$$\frac{1}{k^n}\int_M \langle s_\alpha, s_\beta\rangle\, k \left(\frac{\partial}{\partial t}\left(\frac{\Omega}{\omega_t^n}\Delta_t(\frac{\eta_1}{k}) - \frac{\dot{\eta}_1}{k} + \frac{\gamma_{1,0}}{k} + O(\frac{1}{k^2})\right)\right)\Omega \tag{3.20}$$

$$+\frac{1}{k^n}\int_M k(k\frac{\partial \phi_t}{\partial t})\langle s_\alpha, s_\beta\rangle \left(\frac{\Omega}{\omega_t^n}\Delta_t(\frac{\eta_1}{k}) - \frac{\dot{\eta}_1}{k} + \frac{\gamma_{1,0}}{k} + O(\frac{1}{k^2})\right)\Omega \tag{3.21}$$

In (3.20), the term $O(1/k^2)$ stands for a smooth function $r(p,k,t)$, where $p \in M$, and is uniformly bounded over M and in the variable t. But we know that the asymptotics of the Bergman kernel is given by polynomial expressions of the curvature and its covariant derivative. We can write $r(p,k,t) = \sum_{i\geq 2}\frac{1}{k^i}r_i(p,t)$ where $r_i(p,t)$ are smooth in t and p variables. Thus $\|r(p,k,t)\|_{C^\infty} < \frac{C_1}{k^2}$ and $\left\|\frac{\partial r(p,k,t)}{\partial t}\right\|_{C^\infty} < \frac{C_2}{k^2}$, where C_1, C_2 do not depend on k, t and p. The independence in the variable t is again obtained from the fact that the metric ω_t along the Ω-Kähler flow is convergent (Theorem 3.2.2) and the uniformity of the expansion. Moreover, since η_1 is a smooth solution of (3.17) in t, one gets that the term (3.20) is uniformly bounded by C_3/k^2 using the same argument as in the proof of Theorem 3.3.1, Inequality (3.18).

On another hand, by the same reasoning, (3.21) is uniformly bounded by C_4/k, where C_4 is independent of t and k. This provides the result for $m = 1$.

The computations for $m > 1$ are completely similar. Also, higher order derivatives in t could be treated in a similar way. □

3.3.3 L^2 estimates in finite dimensional set-up

We start this section by fixing some notations and giving some definitions. Let us fix a reference metric $\omega_0 \in c_1(L)$. We denote $\tilde{\omega}_0 = k\omega_0$ the induced metric in $kc_1(L)$. We need the notion of R-bounded geometry in C^r [Don01b, Secion 3.2]. We say that another metric $\tilde{\omega} \in kc_1(L)$ has R-bounded geometry in C^r if $\tilde{\omega} > \frac{1}{R}\tilde{\omega}_0$ and $\|\tilde{\omega}-\tilde{\omega}_0\|_{C^r(\tilde{\omega}_0)} < R$. We say that a basis $\{s_i\}$ of $H^0(L^k)$ is R-bounded if the Fubini-Study metric induced by the embedding of M in $\mathbb{P}H^0(L^k)^*$ associated to the $\{s_i\}$ has R-bounded geometry.

The purpose to work with R-bounded metric is to avoid constants depending on k in the forthcoming estimates. Let us fix

$$H_A = \sum_{i,j} A_{ij}(s_i, s_j) = \mathrm{tr}(A\mu_{FS}) \in C^\infty(M,\mathbb{R}),$$

where $A = (A_{ij})$ is a Hermitian matrix, $\{s_i\}$ is a basis of $H^0(L^k)$, and $(.,.)$ denotes the fibrewise Fubini-Study inner-product induced by the basis $\{s_i\}$. This function corresponds to the potential obtained by an A-deformation of the Fubini-Study metric, i.e when one is moving the Fubini-Study metric in an $Lie(SU(N+1))$ orbit. Moreover, we denote $\|A\|_{op} = \max \frac{|A\zeta|}{|\zeta|}$ the operator norm, given by the maximum moduli of the eigenvalues of the hermitian matrix A, and the Hilbert-Schmidt norm $\|A\|^2 = \mathrm{tr}(A^2) = \mathrm{tr}(AA^*) \geq 0$. We will need the following result which is very general,

Proposition 3.3.3 ([Don01b, Lemma 24],[Fin10, Proposition 12])**.** *There exists $C > 0$ independent of k, such that for any basis $\{s_i\}$ of $H^0(L^k)$ with R-bounded geometry in C^r and any hermitian matrix A,*

$$\|H_A\|_{\mathrm{C}^r} \leq C\|\mu_\Omega(\iota)\|_{op}\|A\|$$

where ι is the embedding induced by the basis $\{s_i\}$.

Proof. By definition, $\mu_\Omega(\iota) = \int_M \mu_{FS}(\iota)\Omega$. Given a holomorphic section s of $L \to M$, one defines a holomorphic section $\tilde{s}$ of $\bar{L}^* \to \bar{M}$ (here $\bar{M}$ is just M with the opposite complex structure) thanks to the bundle isomorphism given by the fiber metric. Then, for the hermitian matrix A, one can define the section $\sigma_A = \sum A_{ij} s_i \otimes \tilde{s_j}$ and compute its L^2 norm over $M \times \bar{M}$. This L^2 norm is given by $\mathrm{tr}(A\mu_\Omega\mu_\Omega^* A^*)^{1/2}$. But one has an obvious upper bound for that term, by a standard inequality: for hermitian matrices G,F, $\mathrm{tr}(FGF) \leq \|F\|^2\|G\|_{op}$. Thus,

$$\|\sigma_A\| = \mathrm{tr}(A\mu_\Omega\mu_\Omega^* A^*)^{1/2} \leq \|\mu_\Omega(\iota)\|_{op}\|A\|. \tag{3.22}$$

On another hand, for any holomorphic section σ of a hermitian vector bundle $\tilde{L} \to Y$, one has the L^2 estimate, $\|\sigma\|_{\mathrm{C}^r(Y')} \leq C\|\sigma\|_{L^2(Y)}$ for a submanifold $Y' \subset Y$ and some constant C that depends on Y. This is described in [Don01b, Lemma 24]. Hence, applying this result with $Y = M \times \bar{M}$ and $Y' = M$, together with (3.22), one obtains the expected inequality. □

We will need the following lemma.

Lemma 3.3.1. *Let us fix $r \geq 2$. Assume that for all $t \in [0,T]$, the family of basis $\{s_i\}(t)$ of $H^0(L^k)$ have R-bounded geometry. Let us define by $h(t)$ the family of Bergman metrics induced by $\{s_i\}(t)$. Then the induced family of Fubini-Study metrics $\tilde{\omega}(t)$ satisfy*

$$\|\tilde{\omega}(0) - \tilde{\omega}(T)\|_{\mathrm{C}^{r-2}} < C \sup_t \|\mu_\Omega(\iota(t))\|_{op} \int_0^T \mathrm{dist}(h(s), h(0))ds,$$

and also

$$\begin{aligned}\left\|\frac{\partial\tilde{\omega}}{\partial t}(0) - \frac{\partial\tilde{\omega}}{\partial t}(T)\right\|_{\mathrm{C}^{r-2}} <& C^* \sup_t \|\mu_\Omega(\iota(t))\|_{op} \int_0^T \mathrm{dist}(\frac{\partial h}{\partial s}(s), \frac{\partial h}{\partial s}(0))ds \\ &+ C^* \sup_t \|d\mu_\Omega(\iota(t))\|_{op} \int_0^T \mathrm{dist}(h(s), h(0))ds,\end{aligned}$$

where C, C^* are uniform constants in k.

Proof. Thanks to [Fin10, Lemma 13], we just need to check the second inequality. For the deformation $A(t)$ of the L^2 metric induced along the path from 0 to T, one has

$$\begin{aligned}\Big\|\frac{\partial^2\tilde{\omega}(t)}{\partial t^2}\Big\|_{C^{r-2}} &= \Big\|\sqrt{-1}\partial\bar{\partial}\frac{\partial}{\partial t}H_{A(t)}\Big\|_{C^{r-2}} \\ &\leq \|\partial\bar{\partial}\mathrm{tr}(\dot{A}(t)\mu_{FS})\|_{C^{r-2}} + \|\partial\bar{\partial}\mathrm{tr}(A(t)\dot{\mu_{FS}}(\iota_t))\|_{C^{r-2}}. \quad (3.23)\end{aligned}$$

The first term of (3.23) can be bounded from above by $C\|\mu_\Omega(\iota(t))\|_{op}\|\dot{A}(t)\|$ using directly Proposition 3.3.3. For the second term, one needs to adapt the proof of Proposition 3.3.3, but this can be done with no major difficulty. Hence, the second term of (3.23) can be bounded from above by

$$\begin{aligned}\|\partial\bar{\partial}\mathrm{tr}(A(t)\dot{\mu_{FS}}(\iota_t))\|_{C^{r-2}} &\leq C\Big\|\int_M \dot{\mu_{FS}}(\iota(t))\Omega\Big\|_{op}\|A(t)\| \\ &\leq C'\|d\mu_\Omega(\iota(t))\|_{op}\|A(t)\|.\end{aligned}$$

Then by integration, one obtains the expected estimate. □

Corollary 3.3.1. *Let $\tilde{\omega}_k$ be a sequence of metrics with $R/2$-bounded geometry in C^{r+2} such that the norms $\|\mu_\Omega(\tilde{\omega}_k)\|_{op}$ are uniformly bounded. Then, there is a constant $C > 0$ independent of k such that if $\tilde{\omega}$ has $\mathrm{dist}_k(\tilde{\omega}, \tilde{\omega}_k) < C$, then $\tilde{\omega}$ has R-bounded geometry in C^r.*

Proof. The proof is completely similar to [Fin10, Lemma 14]. □

3.3.4 Projective estimates

In this subsection, we aim to control the operator norm of the moment map in terms of the Riemannian distance in the Bergman space

$$\mathcal{B} = GL(N+1)/U(N+1).$$

With this result in hand, we can launch the gradient flow of the moment map and show its convergence.

We start our investigation by the following result, which is a direct consequence of Theorem 3.1.2.

Proposition 3.3.4 ([CaoKel12]). *Let h be a hermitian metric on L with curvature $\omega = c_1(h) > 0$. Consider the sequence $h_k = FS(Hilb(h)) \in \mathrm{Met}(L^k)$ of Bergman metrics, approximating after renormalization h, thanks to Theorem 3.1.2. Let us call*

$$I_{\Omega,k} = \int_M \langle s_i, s_j\rangle_{h^k}\Omega$$

for $\{s_i\}_{i=1,..,N+1}$ *a basis of holomorphic sections of* $H^0(L^k)$ *with respect to* $Hilb(h)$. *Then, when* $k \to +\infty$,

$$\|\mu_\Omega(h_k) - I_{\Omega,k}\|_{op} \to 0$$

and the convergence is uniform for ω *lying in a compact subset of Kähler metrics in* $c_1(L)$.

Proof. Firstly, the matrix $I_{\Omega,k}$ does not depend on the choice of the orthonormal basis $\{s_i\}_{i=1,..,N+1}$. Thanks to the asymptotic expansion given by Theorem 3.1.2,

$$\mu_\Omega(h_k) = \int_M \langle s_i, s_j\rangle_{FS(Hilb(h))}\Omega = \int_M \langle s_i, s_j\rangle_{h^k}(1 + O(1/k))\Omega.$$

Finally, we can conclude the convergence by using [Don01b, Lemma 28] which ensures that for the operator norm,

$$\Big\|\int_M \langle s_i, s_j\rangle_{FS(Hilb(h))} \times O\left(\frac{1}{k}\right)\Omega\Big\|_{op} \leq \Big|\frac{\Omega}{\omega^n} O\left(\frac{1}{k}\right)\Big|_{L^\infty}.$$

The uniformity of the convergence is given by the uniformity of the expansion in the asymptotics, see Theorem 3.1.2. □

Given a tangent vector $A \in T_b\mathcal{B}$ with $b \in \mathcal{B}$, we have a vector field ζ_A on $(\mathbb{P}^N)^*$ and thus on M, corresponding to A. Of course, the fact that μ_Ω is a moment map gives straightforward the following fact.

Lemma 3.3.2. *For any pair of Hermitian matrices* $A, B \in T_b\mathcal{B}$, *one has*

$$\mathrm{tr}(Bd\mu_\Omega(A)) = \int_M (\zeta_A, \zeta_B)\Omega,$$

where $(.,.)$ *denotes the Fubini-Study inner product induced on the tangent vectors.*

By the fact that μ_{FS} is a moment map, we have the following lemma.

Lemma 3.3.3. *Let* $A, B \in T_b\mathcal{B}$. *Pointwisely over* $(\mathbb{P}^N)^*$, *one has*

$$H_A H_B + (\zeta_A, \zeta_B) = \mathrm{tr}(AB\mu_{FS}).$$

Lemma 3.3.4. *For any hermitian matrices* $A, B \in T_b\mathcal{B}$,

$$\mathrm{tr}(Bd\mu_\Omega(A)) + \langle H_A, H_B\rangle_{L^2(M,\Omega)} = \mathrm{tr}(AB\mu_\Omega).$$

Proof. We start from the previous lemma which says that at each point of M,

$$H_A H_B + (\zeta_A, \zeta_B) = \mathrm{tr}(AB\mu_{FS}).$$

Now, we integrate with respect to the volume form Ω and apply Lemma 3.3.2. □

Lemma 3.3.5. *For any hermitian matrix $A \in T_b\mathcal{B}$,*

$$\|H_A\|^2_{L^2(\Omega)} \leq \|A\|^2 \|\mu_\Omega\|_{op}.$$

Proof. From the last lemma,

$$\|H_A\|^2_{L^2(\Omega)} = \operatorname{tr}(A^2\mu_\Omega) - \operatorname{tr}(Ad\mu_\Omega(A)).$$

Now, by Lemma 3.3.2,

$$\operatorname{tr}(Ad\mu_\Omega(A)) = \int_M (\zeta_A, \zeta_A)\Omega \geq 0.$$

Hence,

$$\|H_A\|^2_{L^2(\Omega)} \leq \operatorname{tr}(A^2\mu_\Omega) \leq \|A\|^2 \|\mu_\Omega\|_{op}.$$

□

Lemma 3.3.6. *For any Hermitian matrix $A \in T_b\mathcal{B}$,*

$$\|d\mu_\Omega(A)\|_{op} \leq \|d\mu_\Omega(A)\| \leq 2\|A\|\|\mu_\Omega\|_{op}.$$

Proof. From Lemma 3.3.4, one has

$$\begin{aligned}
\|d\mu_\Omega(A)\|^2 &= \operatorname{tr}(d\mu_\Omega(A)^2) \\
&= \operatorname{tr}(Ad\mu_\Omega(A)\mu_\Omega) - \langle H_A, H_{d\mu_\Omega(A)}\rangle_{L^2(\Omega)} \\
&\leq \|A\|\|d\mu_\Omega(A)\|\mu_\Omega\|_{op} - \langle H_A, H_{d\mu_\Omega(A)}\rangle_{L^2(\Omega)}.
\end{aligned}$$

Then we can conclude by using the fact that

$$|\langle H_A, H_{d\mu_\Omega(A)}\rangle_{L^2(\Omega)}| \leq \|H_A\|_{L^2(\Omega)} \|H_{d\mu_\Omega(A)}\|_{L^2(\Omega)},$$

and the previous lemma. □

Finally, we obtain

Proposition 3.3.5 ([CaoKel12])**.** *Let $b_0, b_1 \in \mathcal{B}$. Then,*

$$\|\mu_\Omega(b_1)\|_{op} \leq e^{2\mathrm{dist}_k(b_0,b_1)} \|\mu_\Omega(b_0)\|_{op}.$$

Proof. We know that a geodesic in the space of Bergman metrics $\mathcal{B}$ is given by a line, i.e., the Hermitian metric involved along the geodesic is modified by e^{tA} and that $\mathrm{dist}(b_0, b_1) = \|A\|$. This can be rephrased by saying that if $\{s_i^0\}_{i=1,..,N+1}$ (resp. $\{s_i^1\}_{i=1,..,N+1}$) is an orthonormal basis of $H^0(L^k)$ with respect to b_0 (resp. b_1), then there exists $\sigma \in GL(N+1, \mathbb{C})$ such that

$\sigma \cdot s^0 = s^1$ and without loss of generality we can assume σ diagonal with entries $e^{\lambda_0}, .., e^{\lambda_n}$. Then the geodesic is just induced by the family of basis $\sigma^t \cdot s^0$ for $t \in [0,1]$. Now, we can conclude our proof by using Lemma 3.3.6 and the fact that the norm $\|.\|_{op}$ on the space of matrices is controlled from above by the Hilbert-Schmidt norm $\|.\|$. $\square$

We are now ready to give the proof of Theorem 3.0.11, that is to show the smooth convergence of Kähler metrics $\omega_k(t)$ involved in the rescaled balancing flow (3.2) towards the solution ω_t to the Ω-Kähler flow.

Proof of Theorem 3.0.11. Using Theorem 3.3.1, for any $m > 0$, we have obtained a sequence of Kähler metrics

$$\omega(k;t) = c_1(h_0 e^{\psi(k,t)})$$

such that $\omega(k;t)$ converges, when $k \to +\infty$ and in smooth sense, towards the solution $\omega_t = c_1(h_0 e^{\phi_t})$ to the Ω-Kähler flow. Moreover, one has, for k large enough and with $\overline{h}_k(t) \in \mathcal{B}$ the Bergman metric associated to $h_0 e^{\psi(k,t)} \in \mathrm{Met}(L)$, the estimate

$$\mathrm{dist}_k(h_k(t), \overline{h}_k(t)) \leq \frac{C}{k^{m+1}}, \tag{3.24}$$

where $h_k(t)$ is the metric induced by the rescaled Ω-balancing flow. Consequently, in order to get the C^0 convergence in t, all what we need to show is that

$$\|\omega_k(t) - c_1(\overline{h}_k(t))\|_{\mathrm{C}^r(\omega_t)} \to 0. \tag{3.25}$$

The idea is to consider the geodesic in the Bergman space between these two points.

Firstly, we will get that along the geodesic from $\overline{h}_k(t)$ to $h_k(t)$ in $\mathcal{B}$, $\|\mu_\Omega\|_{op}$ is controlled uniformly if we can apply Proposition 3.3.5. This requires to prove that $\overline{h}_k(t)$ is at a uniformly bounded distance of $h_k(t)$ and that $\|\mu_\Omega(\overline{h}_k(t))\|_{op}$ is bounded in k. But, this comes from the fact that one can choose precisely $m \geq n+1$ in Inequality (3.24) and one can apply Proposition 3.3.4. For the latter, one needs to notice the estimate

$$\|I_{\Omega,k}\|_{op} \leq \sup_M \frac{\Omega}{\omega^n}$$

from [Don01b, Lemma 28].

Secondly, we show that the points along this geodesic have R-bounded geometry. This is a consequence of Corollary 3.3.1, applied with the reference metric ω_t to the sequence $c_1(\overline{h}_k(t))$. On one side, $\|\mu_\Omega(\overline{h}_k(t))\|_{op}$ is under control as we have just seen. On another side, $c_1(\overline{h}_k(t))$ are convergent to ω_t in C^∞ topology (hence in C^{r+4} topology), thus they have $R/2$-bounded geometry. Given $m \geq n+2$, one obtains, thanks to Corollary 3.3.1 and inequality (3.24), that all the metrics along the geodesic from $\overline{h}_k(t)$ to $h_k(t)$ have R-bounded geometry in C^{r+2}.

Thirdly, we are exactly under the conditions of Lemma 3.3.1. It gives, by renormalizing the metrics in the Kähler class $c_1(L)$ and by (3.24), that

$$\begin{aligned}\|k\omega_k(t) - kc_1(\overline{h}_k(t))\|_{\mathrm{C}^r(k\omega_t)} &\leq C\|\mu_\Omega(\overline{h}_k(t))\|_{op}k^{n+2}\mathrm{dist}_k(h_k(t),\overline{h}_k(t)),\\ \|\omega_k(t) - c_1(\overline{h}_k(t))\|_{\mathrm{C}^r(\omega_t)} &\leq C\|\mu_\Omega(\overline{h}_k(t))\|_{op}k^{n+2-m-1+r/2},\end{aligned}$$

where we have used that the geodesic path from 0 to 1 is just a line. Here $C>0$ is a constant that does not depend on k. If we choose $m > r/2+1+n$, we get the expected convergence in C^r topology, i.e Inequality (3.25). Of course, this reasoning works to get the uniform C^0 convergence in t for $t \in \mathbb{R}_+$, because all the Kähler metrics ω_t that we are using are uniformly equivalent (we have convergence of the Ω-Kähler flow, Theorem 3.2.2) and because we have uniformity of the expansion in Theorem 3.1.2 and Theorem 3.1.3.

We now prove that one has C^1 convergence in t of the flows $\omega_k(t)$. Again, we need to show the C^1 convergence of $\omega_k(t)$ to $c_1(\overline{h}_k(t))$, because we already know the convergence of $c_1(\overline{h}_k(t))$ to ω_t by Proposition 3.1.3. We are under the conditions of Lemma 3.3.1 by what we have just proved above. So we have, using again that our path is a geodesic,

$$\begin{aligned}\left\|k\frac{\partial\omega_k(t)}{\partial t} - k\frac{\partial c_1(\overline{h}_k(t))}{\partial t}\right\|_{\mathrm{C}^r} \leq & C^*\|\mu_{k,\chi}(\overline{h}_k(t))\|_{op}k^{n+2}\mathrm{dist}_k\left(\frac{\partial h_k(t))}{\partial t},\frac{\partial\overline{h}_k(t)}{\partial t}\right)\\ &+ C^*\|d\mu_{k,\chi}(\overline{h}_k(t))\|_{op}k^{n+2}\mathrm{dist}_k(h_k(t),\overline{h}_k(t)).\end{aligned}$$

Here the C^r norm is computed with respect to $k\omega_t$. If we apply Lemma 3.3.6, Theorem (3.3.1) and Proposition 3.3.2, we can bound from above the RHS of the last inequality, and get

$$\begin{aligned}\left\|\frac{\partial\omega_k(t)}{\partial t} - \frac{\partial c_1(\overline{h}_k(t))}{\partial t}\right\|_{\mathrm{C}^r(\omega_t)} \leq & C'\|\mu_{k,\chi}(\overline{h}_k(t))\|_{op}k^{n+2-m-r/2}\\ &+ C''\|\mu_{k,\chi}(\overline{h}_k(t))\|_{op}k^{n+2+r/2}k^{-m-1}k^{-m-1}\\ \leq & C'''k^{n+2-m-r/2}.\end{aligned}$$

Finally, we choose $m > r/2+n+2$ to obtain C^1 convergence. This completes the proof of Theorem 3.0.11. □

Finally, if we apply Definition-Proposition (3.0.8) that asserts that an Ω-balanced metric does always exist and is a zero of the moment map μ^0_Ω, Theorem 3.0.11, and the convergence of the Ω-Kähler flow towards a solution to the classical Calabi problem, we obtain directly the following result.

Corollary 3.3.2 ([CaoKel12]). *Under the same setting as above, the sequence of balanced metric $h_k(\infty)^{1/k} \in \mathrm{Met}(L)$, obtained as the limit of the balancing flow at $t = +\infty$, converges in smooth topology towards h_∞, a solution of the Calabi problem,*

$$(c_1(h_\infty))^n = \Omega.$$

Note that this is a new proof of Theorem 3.0.9, but which uses a priori the existence of a solution to the Calabi problem (compare with [Kel09]).

3.4 The infinite dimensional setup

3.4.1 A symplectic approach to the Calabi problem

In this section we develop the moment map set-up on the infinite dimensional space of Kähler potentials related to the Ω-Kähler flow. Let us assume that (M, L) is a polarized manifold. Let us fix $\omega \in c_1(L)$ and Ω a smooth volume form on M with $\int_M \Omega = \mathrm{Vol}_L(M)$. We introduce $\mathcal{C}$ the infinite dimensional space of integrable hermitian connections on L with Kähler form as curvature, with respect to a fixed complex structure. It means that we consider unitary connections ∇ on L such that if $F_\nabla \in \Omega^2(M, End(L))$ is the curvature connection, then $F_\nabla^{0,2} = F_\nabla^{2,0} = 0$ and $F_\nabla^{1,1}$ is a positive form with respect to the complex structure on M. Consider the abelian gauge group $\mathcal{G}$ of maps $L \to L$ that cover the identity on M. By duality, the Lie algebra $Lie(\mathcal{G})$ can be identified with the space of smooth functions from M to $\mathbb{R}$ with zero integral, since one can identify $\mathcal{G}$ with $C^\infty(M, S^1)$. The tangent space at $\mathcal{C}$ is given by the 1-forms with values in $End(L)$. For simplicity, we assume that M is simply connected and we fix the following symplectic form on $\mathcal{C}$ at the point $\nabla \in \mathcal{C}$,

$$\nu_\nabla(a, b) = \int_M a \wedge b \wedge F_\nabla^{n-1}$$

which is a symplectic form invariant under the action of $\mathcal{G}$. Note that ν is invariant under the action of the group $\mathcal{G}$.

We have a natural paring $Lie(\mathcal{G}) \times Lie(\mathcal{G})^* \to \mathbb{R}$ given by

$$(\zeta, \theta) \mapsto \int_M \zeta\theta = \int_M \langle \zeta, \theta \rangle. \tag{3.26}$$

We are in a moment map setting. Actually, we have the following simple proposition that shows that prescribing the volume form in a Kähler class is related to finding the zero of a certain moment map. Note that given $\nabla \in \mathcal{C}$, we set A_∇ is the real connection (S^1 invariant) 1-form associated to ∇ on the natural S^1-principal bundle $\pi : P \to M$ that we can associate with $L \to M$. It acts on an element of $\zeta \in Lie(\mathcal{G})$ by decomposing in a vertical and horizontal parts and fibrewise this vertical part corresponds to a rotation which is eventually parametrized by the real function $\langle A_\nabla, \zeta\rangle$ over M.

Proposition 3.4.1 ([CaoKel12])**.** *There is a moment map $\overline{\mu} : \mathcal{C} \to Lie(\mathcal{G})^*$ associated to the action of $\mathcal{G}$ on $(\mathcal{C}, \nu)$ given by*

$$\overline{\mu}(\nabla) = \langle A_\nabla, .\rangle((F_\nabla)^n - \Omega).$$

Proof. We need to check that for any $\zeta \in Lie(\mathcal{G})$ and any a vector field v, we have

$$\langle d\overline{\mu}(\nabla)(v), \zeta\rangle = \nu_\nabla(v, X_\zeta),$$

where X_ζ is the vector field on $\mathcal{C}$ defined by the infinitesimal action of $\zeta \in Lie(\mathcal{G})$. More explicitly, X_ζ is given by $X_\zeta = L_\zeta A_\nabla = d\langle A_\nabla, \zeta\rangle + \iota_\zeta dA = d\langle A_\nabla, \zeta\rangle + \iota_{\pi_* \zeta} F_\nabla = d\langle A_\nabla, \zeta\rangle$, since the elements of $\mathcal{G}$ cover the identity on M. Now, we have

$$\begin{aligned}\nu_\nabla(v, X_\zeta) &= \int_M v \wedge d\langle A_\nabla, \zeta\rangle \wedge F_\nabla^{n-1} \\ &= \int_M \langle A_\nabla, \zeta\rangle dv \wedge F_\nabla^{n-1}.\end{aligned}$$

But the change in F_∇ by the vector field v is precisely given by dv, so

$$\langle d\overline{\mu}(\nabla)(v), \zeta\rangle = \int_X \langle A_\nabla, \zeta\rangle dv \wedge F_\nabla^{n-1},$$

since the elements of $\mathcal{G}$ cover the identity on M. □

Note that the moment map that we have just defined is obviously not unique. Let us denote as in the proof above by X_ζ the vector field associated to $\zeta \in Lie(\mathcal{G})$. Now, using the pairing (3.26) and the natural $\mathcal{G}$-invariant norm on $Lie(\mathcal{G})$, one can consider $\overline{\mu}$ with values in $Lie(\mathcal{G})$, which means that we write $\overline{\mu}(\nabla) = \frac{(F_\nabla^n - \Omega)}{F_\nabla^n}$. Then, we consider the downward gradient flow

$$\frac{d}{dt}\|\overline{\mu}(\nabla_t)\|^2 = -\|X_{\overline{\mu}(\nabla_t)}\|^2$$

where the norm on the R.H.S is computed with respect to ν_{∇_t}. This is actually equivalent to

$$\frac{dA_{\nabla_t}}{dt} = IX_{\overline{\mu}(\nabla_t)},$$

with I the complex multiplication on the tangent vectors in $\mathcal{C}$. This equation can be rephrased in terms of flow over 1-forms by

$$\frac{dF_{\nabla_t}}{dt} = L_{I\overline{\mu}(f_t)} F_{\nabla_t}$$

If we use the notations of the previous sections, $\omega_t = F_{\nabla_t}$ is an evolving Kähler form, then this (negative) gradient flow reads as

$$\frac{d\omega_t}{dt} = \sqrt{-1}\partial\bar{\partial}\left(\frac{\omega_t^n - \Omega}{\omega_t^n}\right).$$

Then, using the fact that the kernel of the operator $\sqrt{-1}\partial\bar{\partial}$ is given by constants (since M is compact), one recovers precisely the equation of the Ω-Kähler flow (3.8). Finally, we would like to mention that J. Fine has developed in the preprint [Fin11] a more general theory that covers the results presented in this section (see [Fin11, Section 3.2]).

3.4.2 Integral of a moment map

In this section we deal with a very general setup. Consider the case of a Kähler manifold (Ξ, ω) polarized by the line bundle L and a moment map μ_0 associated to the action of a linear reductive group Γ such that its complexified acts holomorphically. To the moment map μ_0 corresponds canonically a functional

$$\Psi : \Xi \times \Gamma^{\mathbb{C}} \to \mathbb{R}$$

that we call "*integral of the moment map μ_0*" and that satisfies the following two properties:

- For all $p \in \Xi$, the critical points of the restriction Ψ_p of Ψ to $\{p\} \times \Gamma^{\mathbb{C}}$ coincide with the points of the orbit $Orb_{\Gamma^{\mathbb{C}}}(p)$ on which the moment map vanishes;
- the restriction Ψ_p on the "lines" $\{e^{\lambda u} : u \in \mathbb{R}\}$ where $\lambda \in Lie\left(\Gamma^{\mathbb{C}}\right)$ is convex.

This is well known in the projective case from the seminal work of G. Kempf and L. Ness [KN79]. We refer also to [Mun00] for a more general setting.

Theorem 3.4.1. *There exists a unique map $\Psi : \Xi \times \Gamma^{\mathbb{C}} \to \mathbb{R}$ that satisfies:*

1. $\Psi(p, e) = 0$ *for all* $p \in \Xi$;
2. $\frac{d}{du}\Psi\left(p, e^{i\lambda u}\right)_{|u=0} = \langle \mu_0(p), \lambda \rangle$ *for all* $\lambda \in Lie(\Gamma)$.

Let us sum up some of the main properties of the integral of the moment map.

Proposition 3.4.2. *The functional Ψ is $\Gamma-$invariant (for the left action) and satisfies the cocyclicity relation*

$$\Psi\left(p,\gamma\right)+\Psi\left(\gamma p,\gamma'\right)=\Psi\left(p,\gamma'\gamma\right)$$

for all $p\in\Xi$, $\gamma,\gamma'\in\Gamma^{\mathbb{C}}$, and the relation of equivariance

$$\Psi\left(\gamma p,\gamma'\right)=\Psi\left(p,\gamma^{-1}\gamma'\gamma\right)$$

for all $p\in\Xi$, $\gamma\in\Gamma,\gamma'\in\Gamma^{\mathbb{C}}$.

Moreover, $\frac{d^2}{du^2}\Psi\left(p,e^{i\lambda u}\right)\geq 0$ for all $\lambda\in Lie\left(\Gamma\right)$ with equality if and only if the vector field $X_\lambda\left(e^{i\lambda u}p\right)=0$.

Let us apply the previous results in our set-up. We introduce some classical functionals on the space of Kähler potentials. The energy functionals I, J, introduced by T. Aubin in [Aub84] (see also [Tia00]), are defined for each pair $(\omega,\omega_\phi:=\omega+\sqrt{-1}\partial\bar\partial\phi)$ by

$$\begin{aligned}I(\omega,\omega_\phi)&=\frac{1}{\mathrm{Vol}(M)}\int_M\sqrt{-1}\partial\phi\wedge\bar\partial\phi\wedge\sum_{i=0}^{n-1}\omega^i\wedge\omega_\phi^{n-1-i}\\&=\frac{1}{\mathrm{Vol}(M)}\int_M\phi(\omega^n-\omega_\phi^n),\end{aligned}$$

$$J(\omega,\omega_\phi)=\frac{1}{(n+1)\mathrm{Vol}(M)}\int_M\sqrt{-1}\partial\phi\wedge\bar\partial\phi\wedge\sum_{i=0}^{n-1}(n-i)\omega^i\wedge\omega_\phi^{n-1-i},$$

where we have skipped again the normalization of the volume form by the factor $n!$ for the sake of clearness. Note that one has the relationship

$$J(\omega,\omega_\phi)=\int_0^1\frac{I(\omega,\omega+s\sqrt{-1}\partial\bar\partial\phi)}{s}ds.$$

It is well known that I,J and $I-J$ are all non-negative and equivalent. One may also define these functionals via a variational formula and they are very natural from this point of view. We refer to the recent work of [Ber+13] where this idea is exploited in details. We shall see that they are also natural in the context of the Ω-Kähler flow. If ω_{ϕ_t} is a smooth path in the Kähler cone, a direct computation gives

$$\frac{d}{dt}J(\omega,\omega_{\phi_t})=\frac{1}{\mathrm{Vol}(M)}\int_M\dot\phi_t(\omega^n-\omega_{\phi_t}^n).$$

We obtain

Proposition 3.4.3 ([CaoKel12])**.** *The integral of the moment map associ-*

ated to $\overline{\mu}:\mathcal{C}\to Lie(\mathcal{G})^*$ is given by the functional

$$F^0_\Omega(\omega,\omega_\phi)=J(\omega,\omega_\phi)+\frac{1}{\mathrm{Vol}(M)}\int_M\phi(\Omega-\omega^n).$$

This functional is decreasing along the Ω-Kähler flow. Along the flow (3.8), one has

$$\frac{d}{dt}F^0_\Omega(\omega,\omega_{\phi_t})=\int_M\dot\phi_t(\Omega-\omega^n_{\phi_t})=-\int_M\dot\phi_t^2\omega^n_{\phi_t}\leq 0.$$

Furthermore, for the second derivative along the Ω-Kähler flow, we observe using the fact that $\int_M\dot\phi_t\omega^n_{\phi_t}=0$ and $\ddot\phi_t=\frac{\Omega}{\omega^n_{\phi_t}}\Delta_t\dot\phi_t=(1-\dot\phi_t)\Delta_t\dot\phi_t$ that

$$\begin{aligned}\frac{d^2}{dt^2}F^0_\Omega(\omega,\omega_{\phi_t})&=\int_M\ddot\phi_t\,\Omega\\&=2\int_M(1-\dot\phi_t)\partial\dot\phi_t\wedge\bar\partial\dot\phi_t\wedge\omega^{n-1}_{\phi_t}\end{aligned}$$

Since $1-\dot\phi_t$ is positive from Equation (3.8), we get that this functional $t\mapsto F^0_\Omega(\omega,\omega_{\phi_t})$ is convex along the Ω-Kähler flow.

Moreover, let us compute the second derivative of $F^0_\Omega(\omega,\omega_{\phi_t})$ along a geodesic in the space of Kähler potentials that satisfies the geodesic equation $\ddot\phi_t=\frac{1}{2}|\nabla\dot\phi_t|^2_{\phi_t}$; we find

$$\frac{d^2}{dt^2}F^0_\Omega(\omega,\omega_{\phi_t})=\int\ddot\phi_t\,\Omega-\int_M(\ddot\phi_t+\Delta_t\dot\phi_t)\omega^n_t,$$

but the first integral is obviously non-negative and the second integral vanishes since its integrand is a divergence when ϕ_t is a geodesic.

Finally, it is not difficult to check that this functional satisfies the cocyclicity property

$$F^0_\Omega(\omega,\omega_{\phi_1})=F^0_\Omega(\omega,\omega_{\phi_2})+F^0_\Omega(\omega_{\phi_2},\omega_{\phi_1}),$$

for $\omega_{\phi_1},\omega_{\phi_2}$ Kähler metrics in the class of $[\omega]$.

Chapter 4

The J-balancing flow

In this chapter we explain how the techniques used in the previous chapter can be modified to obtain a finite dimensional approach to Donaldson J-flow. The main results of this chapter are Theorems 4.3.3, 4.3.4 and the consequences drawn in Section 4.4 (Theorem 4.4.1, Corollary 4.4.5).

4.1 The setting

We consider M a smooth projective manifold equipped with two polarizations $L_1, L_2 > 0$ and of complex dimension $n \geq 2$. Let us fix $h_1 \in \mathrm{Met}(L_1)$, $h_2 \in \mathrm{Met}(L_2)$ such that the curvatures $c_1(h_1) = \omega$, and $c_1(h_2) = \chi$ are both two Kähler forms. The Donaldson J-flow is given by the following parabolic PDE in the (smooth) ω-potentials ϕ_t:

$$\frac{\partial \phi_t}{\partial t} = \gamma - \frac{\chi \wedge (\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^{n-1}}{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t)^n}, \tag{4.1}$$

where γ is the topological constant $\frac{c_1(L_2)c_1(L_1)^{n-1}}{c_1(L_1)^n}$ denoted as the *J-constant*. A *critical metric* for the Donaldson J-flow is a solution to

$$\chi \wedge (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^{n-1} = \gamma(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n. \tag{4.2}$$

This flow is very natural and we shall explain in the next section its interest. If one considers the manifold $\mathrm{Diff}(M)$ of diffeomorphisms $f : M \to M$ homotopic to the identity and equipped with a natural symplectic form $\Omega_{\chi,\omega}(a,b) = \int_M \chi(a,b)\frac{\omega^n}{n!}$, then there is a moment map setting [Don99]. Here $a, b \in \Gamma(f^*(TM))$ as we identify the tangent space of $\mathrm{Diff}(M)$ at f to the space of smooth sections of $f^*(TM)$. The group $\mathcal{G}$ of ω-symplectomorphisms of M acts on $\mathrm{Diff}(M)$ and preserves $\Omega_{\chi,\omega}$. Since we can identify the Lie algebra $Lie(\mathcal{G})$ with the set

$$\{f \in C^\infty(M, \mathbb{R}), \int_M f\, \omega^n = 0\}$$

we can express simply the associated moment map $\mu_J : \mathrm{Diff}(M) \to Lie(\mathcal{G})^*$ for the group action and this is

$$\mu_J(f) = \frac{f^*(\chi) \wedge \omega^{n-1}}{\omega^n} - \gamma. \tag{4.3}$$

This moment map induces also a gradient flow f_t of the function $\|\mu_J(f_t)\|^2$. The J-flow is just the gradient flow expressed using $(f_t^*)^{-1}(\omega)$ on M, using the same argument as at page 17.

4.2 Some results about Donaldson J-flow

From [Che04], it is known long time existence of Donaldson J-flow for all time, that is (4.1) admits a smooth solution for all $t \geq 0$. If it exists, the critical metric is actually unique. Donaldson observed that a necessary condition for the existence of a critical metric is the following inequality on the Chern classes

$$n\gamma[\omega] - [\chi] = n\gamma c_1(L_1) - c_1(L_2) > 0. \tag{4.4}$$

He conjectured that it is sufficient.

Applications of Donaldson's conjecture and the existence of critical metrics are provided in [Wei06; SW08; Che00]. The key point is that it is expected that the J-flow describes around a class with cscK metric, the classes in the Kähler cone that admit a cscK metric. In that direction, it has been proved by X.X. Chen that if $[\chi] \in -c_1(M) < 0$ and there exists a critical metric, then the Mabuchi energy for $[\omega]$ is bounded from below. In other words, if one obtains a Chern inequality similar to (4.4) that would imply the existence of critical metrics, it would be possible to describe more precisely the Kähler cone of a given manifold in terms of "chambers" for cscK metrics (at least one can expect such a result for general type manifolds, see for instance [Ros06, Section 4] or [SW08] where it is described a conical neighborhood of the anticanonical class for which the Mabuchi energy is proper). Keeping in mind the Yau-Tian-Donaldson conjecture, a similar guess is natural in terms of K-semistability or K-stability, see Part V.

In dimension 2, it is proved in [Che04; Wei04] that Donaldson's conjecture holds, the problem being proved equivalent to solve a Monge-Ampère equation in [Che04] (see also the study of the convergence of the J-flow in [Wei04]).

As far as we know, the problem is not well understood in higher dimension. The best result in that direction is that if there exists a Kähler metric $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\psi \in c_1(L_1)$ satisfying

$$(n\gamma\omega' - (n-1)\chi) \wedge (\omega')^{n-2} \wedge u \wedge \bar{u} > 0$$

for all $(1,0)$-form u, then there is convergence of the flow towards a critical metric [SW08; Wei06]. Unfortunately, it seems pretty hard to check on a given manifold this condition. For instance it is not even clear whether it depends on the class only (and not on the forms). Conversely if there is convergence such a metric ω' does exist. For instance, this condition holds if ω and χ belong to the same class.

We explain now by a simple construction why Donaldson's conjecture does not hold in higher dimension. This suggests that the problem is actually quite subtle.

A counterexample to Donaldson's conjecture in higher dimension. Let Σ be a complex surface and ω and χ Kähler metrics. Let γ the J-constant appearing in (4.1). Assume that $n\gamma[\omega] - [\chi]$ is not Kähler. For some integer $k > 0$, the class $n(\gamma + k)[\omega] - [\chi]$ is Kähler. Consider the manifold $X = \mathbb{CP}^k \times \Sigma$, π_1 the projection from X to $\mathbb{P}^k$ and π_2 from X to Σ. Consider on X the product Kähler form $\chi' = \pi_1^*\omega_{FS} + \pi_2^*\chi$. On X, the class $[\pi_1^*\omega_{FS}] + [\pi_2^*\omega]$ has J-constant $\gamma' = \gamma + k$ (with respect to the class $[\chi']$) by direct computation.
Let us assume now that Donaldson's conjecture is true. Then, there exists a critical metric in $[\pi_1^*\omega_{FS}] + [\pi_2^*\omega]$ and let's call it Θ. As we shall see soon, it must be of the form

$$\Theta = \pi_1^*\omega_{FS} + \pi_2^*\theta$$

where $\theta \in [\omega]$ is a metric on Σ and which is critical for χ. To see why, note that the form χ' and the class $[\pi_1^*\omega_{FS}] + [\pi_2^*\omega]$ are both invariant under the action of $U(k+1)$, so the critical metric must be too (by uniqueness of the critical metric). It follows that the restriction of Θ to each copy of $\mathbb{P}^k$ is the Fubini-Study metric. Now it is not difficult to check that because Θ is closed it is determined by its fibrewise restriction up to sums of forms pulled back from the base. This forces the decomposition $\Theta = \pi_1^*\omega_{FS} + \pi_2^*\theta$ where θ is a Kähler metric on the base. Now writing out the critical equation we see that θ must be critical for χ as claimed with J-constant γ. But θ critical (i.e we have a stationary solution of the J-flow in the class $[\omega]$) implies that $n\gamma[\theta] - [\chi] = n\gamma[\omega] - [\chi]$ is Kähler as pointed out above. Contradiction!

To finish this section, let us mention that a more technical counterexample appeared in the recent preprint [LS13].

4.3 Finite dimensional approach to the J-flow

Given the hermitian metric $h \in \mathrm{Met}(L_1^k)$ with positive curvature, one can consider the Hilbertian map

$$Hilb_\chi = Hilb_{k,\chi} : \mathrm{Met}(L_1^k) \to \mathrm{Met}(H^0(L_1^k))$$

such that

$$Hilb_\chi(h) = \int_M \langle .,. \rangle_h \ \frac{\chi \wedge c_1(h)^{n-1}}{\gamma}$$

is the L^2 metric induced by the fibrewise h and the volume form $\chi \wedge c_1(h)^{n-1}$. On another hand, one can consider the Fubini-Study applications $FS = FS_k : \mathrm{Met}(H^0(L_1^k)) \to \mathrm{Met}(L_1^k)$, cf. page 19. We also define the map

$$T_{k,\chi} = FS \circ Hilb_\chi.$$

Definition 4.3.1 ([Kel13])**.** A fixed point h_k of the map $T_{k,\chi} : \mathrm{Met}(L_1^k) \to \mathrm{Met}(L_1^k)$ is called a J-balanced metric at level k.

Let us denote in the sequel $N = N_k = \dim H^0(L_1^k) - 1$. We introduce a moment map setting in finite dimension. Let us consider first $\mu_{FS} : \mathbb{P}^N \to \sqrt{-1}Lie(U(N+1))$ which is a moment map for the $U(N+1)$ action and the Fubini-Study metric ω_{FS} on $\mathbb{P}^N$. Given homogeneous unitary coordinates, one sets explicitly $\mu_{FS} = (\mu_{FS})_{\alpha,\beta}$ as in (3.1). Then, given an holomorphic embedding $\iota : M \hookrightarrow \mathbb{P}H^0(L_1^k)^*$, and the Fubini Study form ω_{FS} on the projective space, we can consider

$$\mu_{k,\chi}(\iota) = \int_M \mu_{FS}(\iota(p)) \ \frac{\chi \wedge \iota^*(\omega_{FS}^{n-1})}{\gamma}(p). \tag{4.5}$$

Claim. The map $\mu_{k,\chi}$ is a moment map for the $U(N+1)$ action over the space of all bases of $H^0(L_1^k)$.

Let us give some details. On the space $\mathfrak{M}$ of smooth maps from M to $\mathbb{P}H^0(L_1^k)^*$, we have a natural symplectic structure ϖ defined by

$$\varpi(a,b) = \int_M (a,b) \ \frac{\chi \wedge \omega_{FS}^{n-1}}{\gamma}.$$

for $a, b \in T_\iota\mathfrak{M}$ and $(.,.)$ the Fubini-Study inner product induced on the tangent vectors. Let $\zeta \in Lie(U(N+1))$ and $X_\zeta \in H^0((\mathbb{P}^N)^*, T(\mathbb{P}^N)^*)$ be the induced holomorphic vector field on $(\mathbb{P}^N)^* = \mathbb{P}H^0(L_1^k)^*$. For all

$Y \in \Gamma(M, T(\mathbb{P}^N)^*_{|M})$ we have that

$$\begin{aligned}
\gamma\varpi(X_{\zeta|M}, Y) =& \int_M i_Y(i_{X_\zeta}(\omega_{FS}\wedge\chi\wedge\omega_{FS}^{n-1}))\\
=& \int_M \omega_{FS}(X_\zeta, Y)\chi\wedge\omega_{FS}^{n-1} - \int_M i_{X_\zeta}(\omega_{FS})\wedge i_Y(\omega_{FS})\wedge\chi\wedge\omega_{FS}^{n-2}\\
=& \int_M \omega_{FS}(X_\zeta, Y)\chi\wedge\omega_{FS}^{n-1} - \int_M \partial\mu_{FS}(\zeta)\wedge\bar\partial\mu_{FS}(Y)\wedge\chi\wedge\omega_{FS}^{n-2}\\
=& \int_M \omega_{FS}(X_\zeta, Y)\chi\wedge\omega_{FS}^{n-1} + \int_M \mathrm{tr}(\mu_{FS}\zeta)\bar\partial\partial\mu_{FS}(Y)\wedge\chi\wedge\omega_{FS}^{n-2},\\
=& \langle d\int_M \mu_{FS}\,\chi\wedge\omega_{FS}^{n-1}(Y), \zeta\rangle.
\end{aligned}$$

Also $\mu_{k,\chi}$ is Ad-equivariant as the integral of the Ad-equivariant moment map μ_{FS}. Thus, $U(N+1)$ acts isometrically on $\mathfrak{M}$ with the moment map given by

$$\iota \mapsto -\sqrt{-1}\left(\mu_{k,\chi}(\iota) - \frac{\mathrm{tr}(\mu_{k,\chi}(\iota))}{N+1}\mathrm{Id}_{N+1}\right) \in \sqrt{-1}Lie(SU(N+1)).$$

Note that if one defines a hermitian metric H on $H^0(L_1^k)$, one can consider an orthonormal basis with respect to H and the associated embedding, and thus it also makes sense to speak of $\mu_{k,\chi}(H)$. In the Bergman space of metrics $GL(N+1)/U(N+1)$, we have a preferred metric associated and this is precisely a J-balanced metric.

Definition 4.3.2 ([Kel13])**.** The embedding ι is J-balanced if and only if

$$\mu^0_{k,\chi}(\iota) := \mu_{k,\chi}(\iota) - \frac{\mathrm{tr}(\mu_{k,\chi}(\iota))}{N+1}\mathrm{Id}_{N+1} = 0.$$

A J-balanced embedding corresponds (up to $SU(N+1)$-isomorphisms) to a J-balanced metric $\iota^*\omega_{FS}$ by pull-back of the Fubini-Study metric from $\mathbb{P}H^0(L_1^k)^*$ so our both definitions of J-balanced metric and embedding actually agree. Note that for $H \in \mathrm{Met}(H^0(L_1^k))$, it also makes sense to consider $\mu_{k,\chi}(h)$ where $h = FS(H) \in \mathrm{Met}(L_1^k)$, i.e when h belongs to the space of *Bergman* type fibrewise metric that we identify with $\mathcal{B}$.

On the other hand, seen as a hermitian matrix, $\mu^0_{k,\chi}(\iota)$ induces a vector field on $\mathbb{P}^N$. Thus, like in the previous chapter, we are lead to study the following flow

$$\frac{d\iota(t)}{dt} = -\mu^0_{k,\chi}(\iota(t)),$$

and we call this flow the J-balancing flow. To fix the starting point of this flow, we choose a Kähler metric $\chi = \chi(0)$ and we construct a sequence of hermitian metrics $h_k(0)$ such that $\omega_k(0) := c_1(h_k(0))$ converges smoothly to

$\chi(0)$ providing a sequence of embeddings $\iota_k(0)$ for $k >> 0$. For technical reasons, we decide to rescale this flow by considering the following ODE.

$$\frac{d\iota_k(t)}{dt} = -k\mu^0_{k,\chi}(\iota_k(t)) \tag{4.6}$$

that we call the *rescaled J-balancing flow*. In the following sections, we are interested in the behavior of the sequence of Kähler metrics

$$\omega_k(t) = \frac{1}{k}\iota_k(t)^*(\omega_{FS}),$$

when t and k tends to infinity.

4.3.1 The limit of the rescaled J-balancing flow

In this section, we assume that the sequence $\omega_k(t)$ is convergent and we want to relate its limit to Equation (4.1).

Theorem 4.3.3 ([Kel13]). *Suppose that for each $t \in \mathbb{R}_+$, the metric $\omega_k(t)$ induced by Equation (4.6) converges in smooth topology to a metric ω_t and that this convergence is C^1 in $t \in \mathbb{R}_+$. Then the limit ω_t is a solution to the Donaldson J-flow (4.1) starting at $\omega_0 = \lim_{k\to\infty}\omega_k(0)$.*

The proof is essentially identical to Theorem 3.1.1 and use similar arguments to the one developed in Section 3.1. The only difference is that we are dealing with orthonormal basis of holomorphic sections $\{s_i\}$ of L_1^k with respect to $Hilb_{k,\chi}(h^k)$. But in that case, the asymptotic of the Bergman function stands as

$$\sum_{i=1}^{N_k+1} |s_i|^2_{h^k} = k^n \frac{\gamma\omega^n}{\chi\wedge\omega^{n-1}} + O(k^{n-1}) \tag{4.7}$$

where $\omega = c_1(h)$ thanks to Proposition 3.1.1. In particular, the potentials $-\mathrm{tr}(\mu^0_{k,\chi}\mu_{FS})$ converge in smooth topology to the potential $1 - \frac{\chi\wedge\omega^{n-1}}{\gamma\omega^n}$ when $k \to +\infty$. Proposition 3.1.3, with respect to the metric induced by the $Hilb_{k,\chi}$ map is also true in our context.

4.3.2 Convergence result for the J-balancing flow

Theorem 4.3.4 ([Kel13]). *Fix $T > 0$. For any $t \in [0,T]$, the sequence $\omega_k(t)$ converges in C^∞ topology to the solution of the Donaldson J-flow (4.1) with $\phi_0 = 0$ and $\omega = \lim_{k\to\infty}\omega_k(0)$. Furthermore, the convergence is C^1 in the variable t. If there is a critical metric, then there is convergence for all $t > 0$.*

The last part of the theorem is a consequence of the long time existence of the flow and the fact that when there is a critical metric, the J-flow converges towards this critical metric [SW08, Theorem 1.1, $(i) \Leftrightarrow (ii)$]. Thus the metrics involved in the J-flow belong to a compact set in the space of smooth Kähler metrics when there is a critical metric. The proof of Theorem 4.3.4 will occupy subsections 4.3.2.1, 4.3.2.2, 4.3.2.3, and 4.3.2.4.

4.3.2.1 First order approximation

We know that from any starting point $\omega = \omega_0$, there exists a solution

$$\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$$

to the J-flow for $t > 0$. We can write $\omega_t = c_1(h_t)$ where h_t is a sequence of hermitian metrics on the line bundle L_1. Furthermore, we can construct a natural sequence of Bergman metrics

$$\hat{h}_k(t) = FS(Hilb_\chi(h_t^k))^{1/k}$$

by pulling back the Fubini-Study metric using sections which are orthonormal with respect to the inner product

$$\frac{1}{k^n}\frac{1}{\gamma}\int_M h_t(.,.)^k \chi \wedge c_1(h)^{n-1}.$$

Using Proposition 3.1.1, we obtain the asymptotic behavior

$$\hat{h}_k(t) = \left(\frac{\gamma k^n c_1(h_t)^n}{\chi \wedge c_1(h_t)^{n-1}} + O\left(\frac{1}{k}\right)\right)^{1/k} h_t$$

for $k >> 1$. Thus, the sequence $\hat{h}_k(t)$ converges to h_t as $k \to \infty$.

On the other hand, the rescaled J-balancing flow provides a sequence of metrics $\omega_k(t) = c_1(h_k(t))$ which are solutions to (4.6). Note that by construction, we fix $h_k(0) = \hat{h}_k(0)$ for the starting point of the rescaled J-balancing flow.

In this section, we wish to evaluate the distance between the two metrics $h_k(t)$ and $\hat{h}_k(t)$. The techniques are similar to Section 3.3.1. Since we are dealing with algebraic metrics, we have the (rescaled) metric on Hermitian matrices given by $d_k(H_0, H_1) = \left(\frac{\mathrm{tr}\,(H_0-H_1)^2}{k^2}\right)^{1/2}$ on $\mathrm{Met}(H^0(L_1^k))$ which induces a metric on $\mathrm{Met}(L_1)$, that we denote by dist_k.

Proposition 4.3.1 ([Kel13])**.** *One has*

$$\mathrm{dist}_k(h_k(t), \hat{h}_k(t)) \leq \frac{C}{k},$$

for some constant $C > 0$ independent of k.

Proof. Let us consider $e^{\phi(t)}h_0$ a family of hermitian metrics with positive curvature, and denote

$$\omega_t = c_1(e^{\phi(t)}h_0).$$

The infinitesimal change at t in the L^2 inner product induced by this path and the induced volume form is given by

$$\hat{U}_{\alpha,\beta}(t) = \frac{1}{\gamma k^n}\int_M \langle s_\alpha, s_\beta\rangle \left(\left(k\dot{\phi}(t) + \Delta_{\omega_t}\dot{\phi}(t)\right)\chi\wedge\omega_t^{n-1} - \tilde{\Delta}_{\omega_t}\dot{\phi}(t)\omega_t^n\right)$$

where Δ_{ω_t} is the Laplacian with respect to ω_t and $\tilde{\Delta}_{\omega_t}$ is given by the Laplacian-type operator

$$\tilde{\Delta}_{\omega_t}u = \frac{1}{n}\omega_t^{k\bar{j}}\omega_t^{i\bar{l}}\chi_{i\bar{j}}\partial_k\partial_{\bar{l}}u.$$

Here $\{s_\alpha\}$ is an orthonormal basis of $H^0(L_1^k)$ with respect to the L^2-inner product

$$\frac{1}{\gamma k^n}\int_M e^{k\phi(t)}\chi\wedge\omega_t^{n-1}.$$

The formula is obtained by noticing that the variation occurs with respect to the fibrewise metric and the induced volume form. Now, if furthermore $\phi(t)$ is a solution to the J-flow, this infinitesimal change is given at $\hat{h}_k(t)$ as

$$\hat{U}_{\alpha,\beta}(t) = \frac{1}{k^n}\int_M \langle s_\alpha, s_\beta\rangle \left(k\left(1 - \frac{\chi\wedge\omega_t^{n-1}}{\gamma\omega_t^n}\right) + O(1)\right)\chi\wedge\omega_t^{n-1}$$

with $\{s_\alpha\}$ satisfy the same assumption as above.

On another hand, the tangent (at the same point $\hat{h}_k(t)$) to the rescaled J-balancing flow (4.6) is given by directly by the moment map $\mu^0_{k,\chi}$, and we write the infinitesimal change of the L^2 metric as

$$U_{\alpha,\beta}(t) = k\int_M \left(\frac{\delta_{\alpha\beta}}{N+1} - \frac{\langle s_\alpha, s_\beta\rangle}{\sum_{i=1}^{N+1}|s_i|^2}\right)\chi\wedge\omega_t^{n-1},$$

where s_i are L^2 orthonormal with respect to the L^2 inner product induced by $h(t)^k$ and $\chi\wedge\omega_t^{n-1}$. Again, from Proposition 3.1.1, one has asymptotically

$$U_{\alpha,\beta}(t) = \hat{U}_{\alpha,\beta}(t) + \frac{1}{k^n}\int_M \langle s_\alpha, s_\beta\rangle O(1)\,\chi\wedge\omega_t^{n-1}.$$

Here the term $O(1)$ stands implicitly for a (smooth) function which is bounded independently of the variables t and k. Thus, one has

$$\frac{\mathrm{tr}\ (\hat{U}_{\alpha,\beta}(t) - U_{\alpha,\beta}(t))^2}{k^2} = \langle \frac{1}{k}O(1), Q_k\left(\frac{1}{k}O(1)\right)\rangle_{L^2}.$$

We can use Theorem 3.1.3, Inequality (3.5) to obtain that

$$\frac{\mathrm{tr}\,(\hat{U}_{\alpha,\beta}(t) - U_{\alpha,\beta}(t))^2}{k^2} = O(k^{-2}).$$

This shows that $d_k(\hat{U}_{\alpha,\beta}(t), U_{\alpha,\beta}(t))) = O(1/k)$. If we denote by $\tilde{h}_k(t)$ the rescaled J-balancing flow passing through $\hat{h}_k(t_0)$ at $t = t_0$, we have just proved that $\tilde{h}_k(t)$ and $\hat{h}_k(t)$ are tangent up to an error term in $O(1/k)$ at $t = t_0$. On the other hand, it is clear that $\tilde{h}_k(t)$ and $h_k(t)$ are close when $t \to \infty$, because they are obtained through the gradient flow of the same moment map and this gradient flow is distance decreasing (see also [Che04, Theorem 1]). Thus $\mathrm{dist}(\tilde{h}_k(t), h_k(t)) = O(1/k)$. This finally proves the result. □

4.3.2.2 Higher order approximation

In this section, we shall only describe the main differences with Section 3.3.2. The key operator appearing in the linearization of the problem is actually (compare with (3.17)),

$$\eta \to \mathfrak{L}_t(\eta) = \frac{\partial \eta}{\partial t} - \tilde{\Delta}\eta.$$

We mean that it is sufficient to solve inductively equations of the form $\mathfrak{L}_t(\eta_i) = \gamma_{i,0}(\eta_1, .., \eta_{i-1})$ where $\gamma_{i,0}$ is smooth. By the standard parabolic theory, a smooth solution η_t of

$$\{\mathfrak{L}_t(\eta) = \xi, \eta(0) = 0, \xi \in C^\infty(M, \mathbb{R})\}$$

exists for all time $t \geq 0$. Using this remark, it is easy to modify the arguments of the previous chapter in order to obtain the following result.

Theorem 4.3.5 ([Kel13]). *Fix $T > 0$. Given solution ϕ_t for $t \in [0, T]$ to the Donaldson J-flow (4.1) and $k >> 0$, there exist functions $\eta_1, ..., \eta_m$, $m \geq 1$, such that the deformation of ϕ_t given by the potential*

$$\psi(k, t) = \phi_t + \sum_{j=1}^{m} \frac{1}{k^j}\eta_j(t)$$

satisfies

$$\mathrm{dist}_k(h_k(t), \overline{h}_k(t)) \leq \frac{C}{k^{m+1}}$$

and

$$\mathrm{dist}_k\left(\frac{\partial h_k(t)}{\partial t}, \frac{\partial \overline{h}_k(t)}{\partial t}\right) \leq \frac{C}{k^m}.$$

Here $\overline{h}_k(t) = FS(Hilb_\chi(h_0^k e^{k\psi(k,t)}))^{1/k} \in \mathrm{Met}(L_1)$ is the induced Bergman *metric from the potential ψ, $h_k(t) \in \mathrm{Met}(L_1)$ is the sequence of metric obtained by the rescaled J-balancing flow (4.6), and C is a positive constant independent of k and t.*

4.3.2.3 L^2 estimates in finite dimensional set up

These estimates work in a similar way to Section 3.3.3. We have an analog of Proposition 3.3.3, Lemma 3.3.1, and Corollary 3.3.1 where μ_Ω is replaced by the moment map $\mu_{k,\chi}$. This is because all the proof of these results depend only on the integrand of the expression for the moment map $\mu_{k,\chi}$ given in (4.5).

Fix

$$H_A = \sum_{i,j} A_{ij}(s_i, s_j) = \mathrm{tr}(A\mu_{FS}) \in C^\infty(M, \mathbb{R}),$$

where $A = (A_{ij})$ is a Hermitian matrix, $\{s_i\}$ is a basis of $H^0(L_1^k)$, and $(.,.)$ denotes the fibrewise Fubini-Study inner-product induced by the basis $\{s_i\}$.

Proposition 4.3.2 ([Kel13])**.** *There exists $C > 0$ independent of k, such that for any basis $\{s_i\}$ of $H^0(L_1^k)$ with R-bounded geometry in C^r and any hermitian matrix A,*

$$\|H_A\|_{\mathrm{C}^r} \leq C\|\mu_{k,\chi}(\iota)\|_{op}\|A\|$$

where ι is the embedding induced by $\{s_i\}$.

Lemma 4.3.1. *Let us fix $r \geq 2$. Assume that for all $t \in [0, T]$, the family of basis $\{s_i\}(t)$ of $H^0(L_1^k)$ have R-bounded geometry. Let us define by $h(t)$ the family of Bergman metrics induced by $\{s_i\}(t)$. Then the induced family of Fubini-Study metrics $\tilde{\omega}(t)$ satisfy*

$$\|\tilde{\omega}(0) - \tilde{\omega}(T)\|_{\mathrm{C}^{r-2}} < C \sup_t \|\mu_{k,\chi}(\iota(t))\|_{op} \int_0^T \mathrm{dist}(h(s), h(0))ds,$$

and also

$$\left\|\frac{\partial\tilde{\omega}}{\partial t}(0) - \frac{\partial\tilde{\omega}}{\partial t}(T)\right\|_{\mathrm{C}^{r-2}} < C^* \sup_t \|\mu_{k,\chi}(\iota(t))\|_{op} \int_0^T \mathrm{dist}(\frac{\partial h}{\partial s}(s), \frac{\partial h}{\partial s}(0))ds$$
$$+ C^* \sup_t \|d\mu_{k,\chi}(\iota(t))\|_{op} \int_0^T \mathrm{dist}(h(s), h(0))ds,$$

where C, C^* are uniform constants in k.

Corollary 4.3.1 ([Kel13])**.** *Let $\tilde{\omega}_k$ be a sequence of metrics with $R/2$-bounded geometry in C^{r+2} such that the norms $\|\mu_{k,\chi}(\tilde{\omega}_k)\|_{op}$ are uniformly bounded. Then, there is a constant $C > 0$ independent of k such that if $\tilde{\omega}$ has $\mathrm{dist}_k(\tilde{\omega}, \tilde{\omega}_k) < C$, then $\tilde{\omega}$ has R-bounded geometry in C^r.*

4.3.2.4 Projective estimates

Using the same arguments as in Section 3.3.3 and Section 4.3.2.3, we get the following proposition.

Proposition 4.3.3 ([Kel13]). *Let h be a hermitian metric on L with curvature $\omega = c_1(h) > 0$. Consider the sequence $h_k = FS(Hilb(h)) \in \mathrm{Met}(L_1^k)$ of Bergman metrics, approximating after renormalization h, thanks to Theorem 3.1.2. Let us call*

$$\mathfrak{I}_{k,\chi} = \frac{1}{\gamma}\int_M \langle s_i, s_j\rangle_{h^k}\chi\wedge\omega^{n-1},$$

where $\{s_i\}$ is a basis of holomorphic sections of $H^0(L_1^k)$ with respect to $Hilb(h)$. Then, when $k\to+\infty$,

$$\|\mu_{k,\chi}(h_k) - \mathfrak{I}_{k,\chi}\|_{op}\to 0$$

and the convergence is uniform for ω lying in a compact subset of Kähler metrics in $c_1(L)$.

Lemma 4.3.2. *For any pair of Hermitian matrices $A, B\in T_b\mathcal{B}$, denote ζ_A, ζ_B the induced vector field on $\mathbb{P}^N$. One has*

$$\mathrm{tr}(Bd\mu_{k,\chi}(A)) = \frac{1}{\gamma}\int_M (\zeta_A,\zeta_B)\chi\wedge\omega^{n-1} - \partial H_B\wedge\bar\partial H_A\wedge\chi\wedge\omega_{FS}^{n-1},$$

where $(.,.)$ denotes the Fubini-Study inner product induced on the tangent vectors.

Proof. We have, using the fact that μ_{FS} is a moment map,

$$\begin{aligned}\mathrm{tr}(Bd\mu_{k,\chi}(A)) =& \frac{1}{\gamma}\int_M \mathrm{tr}(Bd\mu_{FS}(A))\chi\wedge\omega_{FS}^{n-1}\\ &+\frac{1}{\gamma}\int_M \mathrm{tr}(B\mu_{FS})L_{\zeta_A}(\chi\wedge\omega_{FS}^{n-1})\\ =&\frac{1}{\gamma}\int_M(\zeta_A,\zeta_B)\chi\wedge\omega_{FS}^{n-1}\\ &-\frac{1}{\gamma}\int_M \mathrm{tr}(B\mu_{FS})\partial\bar\partial\mathrm{tr}(A\mu_{FS})\wedge\chi\wedge\omega_{FS}^{n-1}\\ =&\frac{1}{\gamma}\int_M(\zeta_A,\zeta_B)\chi\wedge\omega_{FS}^{n-1}-\partial\mathrm{tr}(B\mu_{FS})\wedge\bar\partial\mathrm{tr}(A\mu_{FS})\wedge\chi\wedge\omega_{FS}^{n-1}.\end{aligned}$$

□

By integration and using Lemmas 4.3.2 and 3.3.3, we get

Lemma 4.3.3. *For any hermitian matrices $A, B \in T_b\mathcal{B}$,*

$$\mathrm{tr}(Bd\mu_{k,\chi}(A)) + \langle H_A, H_B\rangle_{L^2_1(M,\frac{1}{\gamma}\chi\wedge\omega_{FS}^{n-1})} = \mathrm{tr}(AB\mu_{k,\chi}),$$

where the $L^2_1(M, \frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1})$-norm is computed with respect the volume form $\frac{1}{\gamma}\chi \wedge \omega_{FS}^{n-1}$ and the gradient induced by χ.

Similarly to Lemma 3.3.5, we have

Lemma 4.3.4. *For any hermitian matrix $A \in T_b\mathcal{B}$,*

$$\|H_A\|^2_{L^2_1(M,\frac{1}{\gamma}\chi\wedge\omega_{FS}^{n-1})} \leq \|A\|^2\|\mu_{k,\chi}\|_{op}.$$

Lemma 4.3.5. *For any Hermitian matrix $A \in T_b\mathcal{B}$,*

$$\|d\mu_{k,\chi}(A)\|_{op} \leq \|d\mu_{k,\chi}(A)\| \leq 2\|A\|\|\mu_{k,\chi}\|_{op}.$$

Finally, we obtain

Proposition 4.3.4 ([Kel13])**.** *Let $b_0, b_1 \in \mathcal{B}$. Then,*

$$\|\mu_{k,\chi}(b_1)\|_{op} \leq e^{2\mathrm{dist}_k(b_0,b_1)}\|\mu_{k,\chi}(b_0)\|_{op}.$$

We have now all the ingredients to proceed to the proof of Theorem 4.3.4. The only difference with the proof page 61 of Theorem 3.0.11 is that we need to estimate $\|\mathfrak{I}_{k,\chi}\|_{op}$ which is bounded from above by $\sup_M \frac{\chi\wedge\omega^{n-1}}{\gamma\omega^n}$ using [Don01b, Lemma 28]. This latter term is also bounded along the J-flow by maximum principle. The other main ingredient of the proof is the uniformity in the evolving metrics, which is ensured by the fact that we are working in finite time or that we have smooth convergence.

4.3.3 Convergence result for J-balanced metrics

The previous results are uniform if one considers that the J-flow is convergent (and so T can be chosen $T = +\infty$). Thus a direct corollary of Theorem 4.3.4, the long time existence and convergence of the J-flow is the following.

Corollary 4.3.2 ([Kel13])**.** *Consider (M, L_1, L_2) a polarized manifold by L_1, L_2 such that that there exists a critical metric solution of (4.2). Then for k sufficiently large, there exists a sequence of J-balanced metrics on $\mathrm{Met}(L_1^k)$ obtained as the limit of the balancing flow at time $t = +\infty$. Furthermore, the sequence of J-balanced metrics converges in smooth topology towards the critical metric when $k \to +\infty$.*

This is an analogue of the main result of [Don01b]. Of course a more direct proof inspired from [Don01b] could be used to derive Corollary 4.3.2.

This would involve to the operator obtained from linearizing the Bergman function close to the critical point ω_∞ i.e explicitly

$$\phi \mapsto \tilde{\Delta}_{\omega_\infty}\phi$$

This operator is a uniformly elliptic 2^{nd} order operator. Its kernel consists of constant functions.

4.4 Variational approach to the J-balancing flow

4.4.1 Convexity along geodesics

Let us consider the functional $J_\chi : \mathrm{Met}(L_1) \to \mathbb{R}$ on the space of smooth hermitian metrics with positive curvature on L_1 defined up to an additive function by

$$\frac{dJ_\chi(h_t)}{dt} = \frac{1}{\gamma}\int_M \dot{\phi}_t \, \chi \wedge c_1(h_t)^{n-1}, \tag{4.8}$$

where $h_t = e^{-\phi_t}h_0$ is a smooth path in $\mathrm{Met}(L_1)$. Setting

$$\omega_t = c_1(h_t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_t,$$

a direct computation gives

$$\frac{d^2 J_\chi(h_t)}{dt^2} = \frac{1}{\gamma}\int_M \ddot{\phi}_t \, \chi \wedge \omega_t^{n-1} - \dot{\phi}_t \tilde{\Delta}_{\omega_t}\dot{\phi}_t \, \omega_t^n + \dot{\phi}_t \Delta_{\omega_t}\dot{\phi}_t \, \chi \wedge \omega_t^{n-1}.$$

We shall use the same notation J_χ as above for the induced functional defined on $\mathrm{Met}(L_1^k)$ for $k > 0$. Consider now the following functional $I_{\mu^0_{k,\chi}} : \mathcal{B}_k \to \mathbb{R}$ on the Bergman space defined by

$$I_{\mu^0_{k,\chi}}(H) = J_\chi \circ FS(H) + \frac{Vol_{L_1}(M)}{N+1}\log\det(H),$$

where $H \in \mathcal{B}_k$. It is clear that the derivative of $J_\chi \circ FS$ at a point $H \in \mathcal{B}_k$ is given by

$$\frac{1}{\gamma}\sum_{i,j}\int_M (\delta H)_{i,j}\langle s_i, s_j\rangle_{FS(H)} \, \chi \wedge c_1(FS(H))^{n-1}$$

where $\{s_i\}$ is an orthonormal basis of holomorphic sections of L_1^k with respect to H. Thus a J-balanced metric H is a critical point of the functional $I_{\mu^0_{k,\chi}}$.

The functional $I_{\mu^0_{k,\chi}}$ is the integral of the moment map $\mu^0_{k,\chi}$, in the sense defined in Section 3.4.2. In particular it is decreasing along the J-balancing flow. Furthermore, due to Kempf-Ness theory, its properness on $SL(N+1)$

is equivalent to the existence of a J-balanced metric.

One can ask at that stage what is the analogue of $I_{\mu^0_{k,\chi}}$ for the infinite dimensional space of Kähler potentials. Let us consider $\omega, \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ two Kähler metrics in $c_1(L_1)$. We define the functional

$$I_{\mu_J}(\omega, \omega_\phi) = \int_0^1 \int_M \dot{\phi}_t \left(\frac{1}{\gamma} \chi \wedge \omega_{\phi_t}^{n-1} - \omega_{\phi_t}^n \right) dt,$$

for ω_{ϕ_t} a smooth Kähler path from ω to ω_ϕ. The functional I_{μ_J} is well defined and independent of the chosen path. Remark that this functional appeared also in [SW08] where it is called $\hat{J}$.

Lemma 4.4.1. *The functional J_χ is strictly convex on the* $\mathrm{C}^{1,1}$ *geodesics of the space* $\mathrm{Met}(L_1)$ *of Kähler potentials in* $c_1(L_1)$.

Proof. See [Che04, Proposition 2.1]. □

We sum up the main properties of I_{μ_J} in the next proposition.

Proposition 4.4.1 ([Kel13])**.** *The functional I_{μ_J} is strictly convex on the* $\mathrm{C}^{1,1}$ *geodesics of the space* $\mathrm{Met}(L_1)$ *of Kähler potentials in* $c_1(L_1)$*. Along Donaldson J-flow, the functionals I_{μ_J} and J_χ are equal and decreasing. The functionals I_{μ_J} satisfies the cocyclicity property*

$$I_{\mu_J}(\omega, \omega_{\phi_0}) + I_{\mu_J}(\omega_{\phi_0}, \omega_{\phi_1}) = I_{\mu_J}(\omega, \omega_{\phi_1})$$

for $\omega_{\phi_0}, \omega_{\phi_1}$ Kähler forms in the Kähler class $[\omega]$. I_{μ_J} is the integral of the moment map of μ_J defined by (4.3).

Proof. It is well known that the functional $I_{\mathrm{AYM}}(\omega, \omega_\phi) = \int_0^1 \int_M \dot{\phi}_t \omega_{\phi_t}^n \, dt$ (often called Aubin-Yau-Mabuchi energy) is affine along geodesics of the space of Kähler potentials. Therefore the convexity is just a consequence of Lemma 4.4.1. Moreover, I_{μ_J} and J_χ are equal since $\int_M \dot{\phi}_t \omega_{\phi_t}^n$ vanishes along the flow and decreasing by definition. The cocyclicity property can be proved following the lines of [Mab86, Theorem 2.3]. □

Lemma 4.4.2. *The functional $J_\chi \circ FS$ is convex along geodesics of $\mathcal{B}_k$.*

Proof. A formal way to see the result is to use Phong-Sturm approximation result of geodesics in the space of Kähler potentials by geodesics from the Bergman space $\mathcal{B}_k$ (see [PS06; Ber09; BK12]) and Lemma 4.4.1.

One can also proceed by a direct computation. A geodesic in $\mathcal{B}_k$ is just a line. Given A a hermitian matrix and a Hamiltonian function $H_A = \mathrm{tr}(A\mu_{FS})$ for the corresponding action and the 1-parameter group of embeddings $\iota_t = t^A \circ \iota$, one needs to evaluate the derivative with respect to t of

$$\frac{1}{\gamma} \int_M \iota_t^*(H_A) \chi \wedge \iota_t^*(\omega_{FS}^{n-1}),$$

but this is equal, up to the factor $\frac{1}{\gamma}$, to

$$\begin{aligned}
&= \int_M |\nabla h_A|^2 \chi \wedge \omega_{FS}^{n-1} - \int_M h_A \partial\bar{\partial} h_A \wedge \chi \wedge \omega_{FS}^{n-1}, \\
&= \int_M |\nabla h_A|^2 \chi \wedge \omega_{FS}^{n-1} - \int_M \partial h_A \wedge \bar{\partial} h_A \wedge \chi \wedge \omega_{FS}^{n-1}, \\
&= \int_M |\nabla h_A|^2 \chi \wedge \omega_{FS}^{n-1} - \int_M |\nabla h_A|^2_{TM} \wedge \chi \wedge \omega_{FS}^{n-1} \\
&\quad + \int_M |\nabla h_A|^2_{TM,\chi} \wedge \chi \wedge \omega_{FS}^{n-1} \\
&= \int_M |\nabla h_A|^2_{\perp} \wedge \chi \wedge \omega_{FS}^{n-1} + \int_M |\nabla h_A|^2_{TM,\chi} \wedge \chi \wedge \omega_{FS}^{n-1}, \\
&\geq 0.
\end{aligned}$$

In the computation we have decomposed the restriction of the vectors to M in two components: the component which is tangent to M plus the component which is perpendicular with respect to the obvious metrics. Therefore, we have obtained the required convexity. □

Corollary 4.4.1 ([Kel13]). *The functional $I_{\mu_J} \circ FS$ is convex along geodesics of $\mathcal{B}_k$.*

Proof. This is a consequence of the previous lemma and the fact that the functional $-I_{\mathrm{AYM}} \circ FS$ is convex along geodesics in the Bergman space, cf. [Don05b, Proposition 1]. □

Now, using the fact that $\log\det$ is linear on geodesics, we also get

Corollary 4.4.2 ([Kel13]). *The functional $I_{\mu^0_{k,\chi}}$ is convex along geodesics of $\mathcal{B}_k$. It has at most a critical point. A J-balanced metric is an absolute minimum of the functional $I_{\mu^0_{k,\chi}}$.*

4.4.2 Iterates of the maps $Hilb_\chi \circ FS$ and $FS \circ Hilb_\chi$

In this section, we investigate the iterates of the map $T_{k,\chi}$.

Lemma 4.4.3. *Consider $h_0 \in \mathrm{Met}(L_1)$, $h = e^{-\phi} h_0 \in \mathrm{Met}(L_1)$. Then*

$$\frac{1}{\gamma}\int_M \phi\, \chi \wedge c_1(h)^{n-1} \leq J_\chi(h) - J_\chi(h_0) \leq \frac{1}{\gamma}\int_M \phi\, \chi \wedge c_1(h_0)^{n-1}$$

Proof. If one defines $h_t = e^{-t\phi} h_0$, and

$$f(t) = \frac{1}{\gamma}\int_M t\phi\chi \wedge c_1(h_0)^{n-1} - (J_\chi(h_t) - J_\chi(h_0)),$$

then $f(0) = f'(0) = 0$ and furthermore along the considered path,

$$
\begin{aligned}
f''(t) &= -(n-1)\frac{1}{\gamma}\int_M \phi\chi \wedge c_1(h_t)^{n-2} \wedge \sqrt{-1}\partial\bar{\partial}\phi, \\
&= (n-1)\sqrt{-1}\frac{1}{\gamma}\int_M \partial\phi \wedge \bar{\partial}\phi \wedge \chi \wedge c_1(h_t)^{n-2},
\end{aligned}
$$

which is non-negative. Thus $f(t) \geq 0$ at $t = 1$ which provides one inequality. Using the symmetry, we get the result. One can also a direct computation. For instance, when $n = 2$, $J_\chi(h) - J_\chi(h_0)$ writes as $\frac{1}{2\gamma}\int_M \phi\chi \wedge (c_1(h_0) + c_1(h))$. □

Define for $h \in \mathrm{Met}(L_1^k)$, $H \in \mathrm{Met}(H^0(L_1^k))$,

$$
\hat{P}(h,H) = \log \sum_{i=1}^{N+1} \|S_i\|^2_{Hilb_\chi(h)} - \log(N+1) + \log\det(H) + \frac{N+1}{\mathrm{Vol}_{L_1}(M)} J_\chi(h)
$$

where $\{S_i\}$ form an orthonormal basis with respect to H. Then it is not difficult to check that

$$
\hat{P}(FS(H),H) = \frac{N+1}{\mathrm{Vol}_{L_1}(M)} I_{\mu^0_{\chi,k}}(H).
$$

Lemma 4.4.4. *For any metrics h, H, one has*

$$
\hat{P}(h,H) \geq \hat{P}(FS(H),H).
$$

Proof. One checks that if we define $h = e^{-\phi}FS(H)$, then

$$
\begin{aligned}
\hat{P}(h,H) - \hat{P}(FS(H),H) &= \log\left(\frac{N+1}{\gamma \mathrm{Vol}_{L_1}(M)}\int_M e^{-\phi}\chi \wedge c_1(h)^{n-1}\right) \\
&\quad + \frac{N+1}{\mathrm{Vol}_{L_1}(M)}\left(J_\chi(h) - J_\chi(FS(H))\right), \\
&\geq \frac{N+1}{\gamma \mathrm{Vol}_{L_1}(M)}\int_M -\phi\, \chi \wedge c_1(h)^{n-1} \\
&\quad + \frac{N+1}{\mathrm{Vol}_{L_1}(M)}\left(J_\chi(h) - J_\chi(FS(H))\right), \\
&\geq 0
\end{aligned}
$$

using Lemma 4.4.3. □

Lemma 4.4.5.

$$
\hat{P}(h,H) \geq \hat{P}(h, Hilb_\chi(h)).
$$

Proof. This is a consequence of the arithmetico-geometric inequality. □

Suppose there exist $h_{bal} \in \mathrm{Met}(L_1^k)$ a J-balanced metric and $H_{bal} \in \mathrm{Met}(H^0(L_1^k))$ the J-balanced metric on the Bergman space. Then for any $h \in \mathrm{Met}(L_1^k), H \in \mathrm{Met}(H^0(L_1^k))$ one has

$$\begin{array}{rcl} \hat{P}(h,H) & \geq & \hat{P}(FS(H),H) \\ & = & \dfrac{1}{\mathrm{Vol}_{L_1}(M)} I_{\mu^0_{\chi,k}}(H) \\ & \geq & \dfrac{1}{\mathrm{Vol}_{L_1}(M)} I_{\mu^0_{\chi,k}}(H_{bal}) \\ & = & \hat{P}(FS(H_{bal}),H_{bal}) \\ & = & \hat{P}(h_{bal},H_{bal}) \end{array}$$

In particular it gives that

$$I_{\mu^0_{\chi,k}}(H) \geq I_{\mu^0_{\chi,k}}(H_{bal}).$$

Moreover $\hat{P}(h, Hilb_\chi(h)) \geq \hat{P}(h_{bal}, Hilb_\chi(h_{bal}))$. Thus the functional on $\mathrm{Met}(L_1^k)$ defined by

$$\hat{I}_k(h) := \frac{\mathrm{Vol}_{L_1}(M)}{N+1}\hat{P}(h, Hilb_\chi(h)) = J_\chi(h) + \frac{\mathrm{Vol}_{L_1}(M)}{N+1}\log\det Hilb_\chi(h)$$

satisfies

$$\hat{I}_k(h) \geq \hat{I}_k(h_{bal}).$$

We will see soon that this new functional has a geometric interpretation. A direct consequence of Lemmas 4.4.4 and 4.4.5 is the following corollary.

Corollary 4.4.3 ([Kel13])**.** *In our setting, the following hold:*

1. *A J-balanced metric on* $\mathrm{Met}(L_1^k)$ *(resp.* $\mathrm{Met}(H^0(L_1^k))$*) is a minimum of the functional* $\hat{I}_k$ *(resp.* $I_{\mu^0_{\chi,k}}$*).*

2. *The map* $FS \circ Hilb_\chi$ *decreases the functional* $\hat{I}_k$ *while the map* $Hilb_\chi \circ FS$ *decreases the functional* $I_{\mu^0_{\chi,k}}$*.*

3. *The functional* $\hat{I}_k$ *is bounded from below if and only if the functional* $I_{\mu^0_{\chi,k}}$ *is bounded from below.*

Let us explain now the asymptotic behavior of the functional $\hat{I}_k$, by studying the term

$$\log\det Hilb_\chi(h_t)$$

where $h_t = he^{-k\phi_t}$ (with $\|\phi_t\|_{C^\infty} = O(1)$ when $k \to \infty$) is a path in $\mathrm{Met}(L_1^k)$. Then, we can write $Hilb_\chi(h_t) = \langle s_i, s_j\rangle_{Hilb(h_t)}$ where $\{s_i\}$ is an orthonormal basis with respect to $Hilb_\chi(h)$. Thus its derivative at $t = 0$ is given by the derivative of $\sum_i \|s_i\|^2_{Hilb_\chi(h_t)}$ and because of the variation of the volume form, this writes as

$$-\frac{1}{\gamma}\int_M k\dot{\phi}\sum_i |s_i|^2_h\,\chi\wedge c_1(h)^{n-1} + (n-1)\sum_i |s_i|^2_h\,\chi\wedge c_1(h)^{n-1}\wedge\sqrt{-1}\partial\bar{\partial}\dot{\phi},$$

Together with (4.7) and the fact that the second term in the integrand is negligible compared to the first one when $k\to+\infty$ (by uniformity of the Bergman expansion in C^2 topology), we obtain when $k\to+\infty$ that

$$\frac{d}{dt}_{|t=0}\left(\frac{\mathrm{Vol}_{L_1}(M)}{N+1}\log\det Hilb_\chi(he^{-k\phi_t})\right) = -k\int_M \dot{\phi}\,c_1(h)^n + O(1),$$

where we have used that $N=\mathrm{Vol}_{L_1}(M)k^n+O(k^{n-1})$. This leads to the following conclusion for $\hat{I}_k$.

Corollary 4.4.4. *Over compact subsets of* $\mathrm{Met}(L_1)$*, the functionals* $\hat{I}_k$ *and* I_{μ_J} *are equivalent, up to a normalization, i.e*

$$\frac{1}{k}\hat{I}_k(h^k) = I_{\mu_J}(h) + O(1/k)$$

for $h\in\mathrm{Met}(L_1)$*.*

Corollary 4.3.2 and our last result show that a critical metric solution of (4.2) is actually an absolute minimum of I_{μ_J}. Of course this fact is also a consequence of I_{μ_J} being the integral of the moment map μ_J.

Finally, following the techniques of [San06], we obtain that if there is a J-balanced metrics of order k, then the iterates of $Hilb_\chi\circ FS$ on $\mathrm{Met}(H^0(L_1^k))$ will converge towards this metric.

Theorem 4.4.1 ([Kel13])**.** *Assume that there exists* $H_\infty\in\mathrm{Met}(H^0(L_1^k))$ *J-balanced. For any* $H_0\in\mathrm{Met}(H^0(L_1^k))$*, denote from now*

$$H_l = Hilb_\chi\circ FS(H_{l-1}),$$

for $l\geq 1$*. Then, up to a positive constant* r*,*

$$H_l\to rH_\infty$$

as $l\to+\infty$*.*

For the sake of clearness, we give the details of the proof which consists in an easy modification of [San06], which is not surprising since it is a purely finite dimensional problem. We will decompose the proof in several lemmas starting with the following definition.

Definition 4.4.2. Let $\{s_i\}$ be a basis of $H^0(L_1^k)$. Using this basis, we can view elements of $\mathrm{Met}(H^0(L_1^k))$ as hermitian matrices $(N+1)\times(N+1)$. A subset $\mathcal{U}\subset\mathrm{Met}(H^0(L_1^k))$ is bounded if there exists a number $R>1$ satisfying the following. For any $H\in\mathcal{U}$, there exists a constant $\gamma_H>0$ so that the smallest and largest eigenvalues of H satisfy

$$\frac{\gamma_H}{R}\leq\min\frac{|H(\zeta)|}{|\zeta|}\leq\max\frac{|H(\zeta)|}{|\zeta|}\leq\gamma_H R.$$

With the notations of the the previous definition, we have an obvious proposition due to the fact that the closure of bounded sets are compact in finite dimension.

Proposition 4.4.2. *Any bounded sequence H_k has a subsequence H_{n_k} such that $\frac{1}{\gamma_{n_k}}H_{n_k}$ converges in $\mathrm{Met}(H^0(L_1^k))$.*

Lemma 4.4.6. *The set $\mathcal{U}$ is bounded if and only if there exists a number $R>1$ so that for any $H\in\mathcal{U}$, we have*

$$\frac{1}{R}\leq\min\frac{|\tilde{H}(\zeta)|}{|\zeta|}\leq\max\frac{|\tilde{H}(\zeta)|}{|\zeta|}\leq R,$$

where $\tilde{H}=\frac{1}{\det(H)^{\frac{1}{N+1}}}H$.

Proof. Without loss of generality, we can assume $H(s_i,s_j)$ is diagonal with entries e^{λ_i}, $\lambda_1\leq..\leq\lambda_{N+1}$. Assuming $\mathcal{U}$ bounded, we obtain $\gamma_H\leq Re^{\lambda_i}$ and $\gamma_H\geq\frac{1}{R}e^{\lambda_i}$. Thus, $e^{\lambda_{N+1}}\leq R^2e^{\lambda_i}$ and $e^{\lambda_1}\geq R^{-2}e^{\lambda_i}$ for all $i=1,..,N+1$, which gives

$$\det(H)^{-1/(N+1)}e^{\lambda_{N+1}}=\left(\prod_i e^{\lambda_{N+1}-\lambda_i}\right)^{1/(N+1)}\leq R^2.$$

Similarly, we obtain $\det(H)^{-1/(N+1)}e^{\lambda_1}\geq\frac{1}{R^2}$. □

Lemma 4.4.7. *Under the assumption of the theorem, if the sequence H_l is bounded in $\mathrm{Met}(H^0(L_1^k))$, then the sequence $\det(H_l)$ is convergent and $\det(H_{l+1}H_l^{-1})\to 1$ and $l\to+\infty$.*

Proof. From Lemma 4.4.5, we deduce that the sequence $\log\det(H_l)$ is decreasing. From Lemma 4.4.4, we deduce that the sequence $J_\chi\circ FS(H_l)$ is also decreasing. Since $I_{\mu^0_{\chi,k}}(H_l)$ is decreasing and bounded from below, $\log\det(H_l)$ is bounded and converges. □

Lemma 4.4.8. *Assume the sequence H_l is bounded in $\mathrm{Met}(H^0(L_1^k))$. Let $H\in\mathrm{Met}(H^0(L_1^k))$ and $\{s_i^l\}_i$ be an orthonormal basis with respect to H_l so* *that the matrix $H(s_i^l, s_j^l)$ is diagonal. Then,*

$$\lim_{l\to+\infty} \|s_i^l\|^2_{Hilb_\chi(FS(H_l))} = \frac{\mathrm{Vol}_{L_1}(M)}{N+1}.$$

Proof. Let us consider $\hat{s}_i^l$ another basis, orthonormal with respect to H_l and so that $H_{l+1}(\hat{s}_i^l, \hat{s}_j^l)$ is diagonal. From the previous lemma, we deduce that $\lim_{l\to+\infty} \det(H_{l+1}(\hat{s}_i^l, \hat{s}_i^l)) = 1$. We have always that

$$\mathrm{tr}(Hilb_\chi(FS(H))H^{-1}) = N+1,$$

so we get $\mathrm{tr}(H_{l+1}(\hat{s}_i^l, \hat{s}_i^l)) = N+1$ for all l. It is not difficult to check using the arithmetico-geometric inequality that it implies

$$H_{l+1}(\hat{s}_i^l, \hat{s}_i^l) \to 1$$

as $l \to +\infty$. Now, we write

$$s_i^l = \sum_{j=1}^{N+1} a_{ij}^l \hat{s}_j^l,$$

with $(a_{ij}^l) \in U(N+1)$. The matrix a_{ij}^l converges when $l \to +\infty$ up to taking a subsequence. For the limit $(a_{ij}^\infty) \in U(N+1)$ we have

$$H_{l+1}(s_i^l, s_j^l) \to \sum_{j=1}^{N+1} |a_{ij}^\infty|^2 = 1,$$

which means that

$$\lim_{l\to+\infty} \frac{N+1}{\gamma \mathrm{Vol}_{L_1}} \int_M |s_i^l|^2_{FS(H_l)}\, \chi \wedge c_1(FS(H_l))^{n-1} = 1,$$

as expected. □

Lemma 4.4.9. *If the sequence H_l is bounded, then for any $H \in \mathrm{Met}(H^0(L_1^k))$ and $\epsilon > 0$, we have*

$$I_{\mu^0_{\chi,k}}(H) \geq I_{\mu^0_{\chi,k}}(H_l) - \epsilon \tag{4.9}$$

for l sufficiently large.

Proof. Fix $\{s_i^l\}$ an orthonormal basis with respect to H_l such that $H(s_i^l, s_j^l)$ is diagonal with entries $e^{\lambda_i^l}$. Define $f(t) = I_{\mu^0_{\chi,k}}(H_t)$ where H_t is the matrix of entries $e^{t\lambda_i^l}$, so that $H_{|t=0} = H_l$ and $H_{|t=1} = H$. By convexity of $I_{\mu^0_{\chi,k}}$ in the Bergman space, we have $f(1) - f(0) \geq f'(0)$. By definition, one gets

$$\begin{aligned} f'(0) &= \int_M \frac{d}{dt}_{|t=0}(FS(H_t))\frac{\chi \wedge c_1(FS(H_t))^n}{\gamma} - \frac{\mathrm{Vol}_{L_1}(M)}{N+1}\sum_i {\lambda_i}^l \\ &= \int_M \sum_{i=1}^{N+1} \lambda_i^l |s_i^l|^2_{FS(H_l)} \frac{\chi \wedge c_1(FS(H_t))^n}{\gamma} - \frac{\mathrm{Vol}_{L_1}(M)}{N+1}\sum_i {\lambda_i}^l \end{aligned}$$

Let us assume that λ_i^l are bounded. Then we can apply Lemma 4.4.8 and obtain that $f'(0) \to 0$ when $l \to +\infty$, which provides eventually (4.9).
It remains to show that λ_i^l are all bounded when l varies. Using the fact that $\{e^{-\lambda_i^l/2} s_i^l\}_i$ is an orthonormal basis for H and Lemma 4.4.6, we have the existence of $R > 1$ such that

$$\frac{1}{R} < \frac{H_l(s_i^l, s_i^l)e^{-\lambda_i}}{\det(H_l)^{1/(N+1)}} = \frac{e^{-\lambda_i}}{\det(H_l)^{1/(N+1)}} < R$$

from which we deduce that λ_i^l is bounded if and only if $\det(H_l)$ is bounded. But this is the case by Lemma 4.4.7. □

Proof of Theorem 4.4.1. As we have already seen in the previous section, the J-balanced metric H_∞ is unique up to normalization. To normalize our metrics, we choose to work in the space

$$\{\tilde{H} = (\det H)^{-1/(N+1)} H,\ H \in \mathrm{Met}(H^0(L_1^k))\} \subset \mathrm{Met}(H^0(L_1^k)).$$

Furthermore, as integral of the moment map $\mu^0_{\chi,k}$, the functional $I_{\mu^0_{\chi,k}}$ is proper and bounded from below. The sequence $I_{\mu^0_{\chi,k}}(H_l)$ is decreasing, and thus $J_\chi((\det H_l)^{-1/(N+1)} FS(H_l))$ is also bounded. By properness, it comes that

$$\tilde{H}_l = (\det H_l)^{-1/(N+1)} FS(H_l)$$

is bounded. This forces this sequence to converge as we shall see now.

Otherwise, we can at least take a non convergent subsequence $\tilde{H}_{l_k}$ which always remain at a distance ϵ of the balanced metric H_∞. But $\tilde{H}_{l_k}$ is bounded and its image by $I_{\mu^0_{\chi,k}}$ converges to the minimum of the functional $I_{\mu^0_{\chi,k}}$, up to taking a subsequence that we denote $\tilde{H}_{l_{k_m}}$. As a matter of fact, we have obtained from the previous results that for any bounded sequence H_l, $I_{\mu^0_{\chi,k}}(H_l)$ converges to the minimum of the functional $I_{\mu^0_{\chi,k}}$, see Lemma 4.4.9. Therefore, $\tilde{H}_{l_{k_m}}$ converges and its limit is actually a balanced metric from Corollary 4.4.2. This is a contradiction with the fact that all the terms $\tilde{H}_{l_k}$ are at distance ϵ of H_∞.

From Lemma 4.4.7, we get that $\log(\det(H_l))$ is bounded and decreasing. This allows us to conclude that H_l is convergent to rH_∞. □

From Corollary 4.3.2 we obtain the following result.

Corollary 4.4.5 ([Kel13])**.** *Assume the existence of a critical metric solution ω_∞ of (4.1). Then for all $k >> 0$, the map*

$$Hilb_\chi \circ FS : Met(H^0(L_1^k)) \to \mathrm{Met}(H^0(L_1^k))$$

(respectively $T_{k,\chi} = FS \circ Hilb_\chi : \mathrm{Met}(L_1^k) \to \mathrm{Met}(L_1^k)$) defines a dynamical system that has a fixed (unique) attractive point, the J-balanced metric H_{bal} (respectively h_{bal} with $FS(H_{bal}) = h_{bal}$). Furthermore,

$$\left\|\omega_\infty - \frac{1}{k}c_1(FS(H_{bal}))\right\|_{C^\infty} = O\left(\frac{1}{k}\right).$$

Part III

Stability of projective bundles and existence of canonical metrics

Chapter 5

About Chow stability of projectivization of Gieseker stable bundles

5.1 Statement of the results

In this chapter, we investigate the connection between stability of a vector bundle E and stability of the projective bundle $\mathbb{P}(E)$ as a polarized manifold. The main results of this chapter are Theorems 5.1.1, 5.1.2 and Corollary 5.1.1.

Roughly speaking one expects that $\mathbb{P}(E)$ is stable, with respect to polarizations that make the fibres sufficiently small, if and only if E is a stable vector bundle over a base that is stable as a manifold. The first result along these lines is due to I. Morrison [Mor80] who showed that if E is a stable rank 2 bundle on a smooth Riemann surface B then the ruled surface $\pi\colon \mathbb{P}(E) \to B$ is Chow stable with respect to the polarization $\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{O}_B(k)$ for $k \gg 0$. Later, building on the work of E. Calabi, A. Fujiki, C. Lebrun and many others, V. Apostolov, D. Calderbank, P. Gauduchon and C. Tønnesen-Friedman have provided a complete understanding of the situation for higher rank bundles over a smooth Riemann surface. They show there is a constant scalar curvature Kähler metric in any Kähler class on $\mathbb{P}(E)$ if and only if the bundle E is Mumford polystable [Apo+08a]. Such metrics are related to stability through the Yau-Tian-Donaldson conjecture (see, for example, [RT06] for an account). In particular it implies through work of S.K. Donaldson [Don01b] that $\mathbb{P}(E)$ is asymptotically Chow stable, by which one means that if r is sufficiently large then the embedding of $\mathbb{P}(E)$ into projective space using the linear series determined by $(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*\mathcal{O}_B(k))^{\otimes r}$ is Chow stable (see also A. Della Vedova and F. Zuddas [DZ12] for a generalization).

There are at least two extensions to the case when the base B has higher dimension. First, a result of Y.-J. Hong [Hon99] states that if E is a Mumford-stable bundle of any rank over a smooth base B that has a discrete automorphism group and a cscK metric, then $\mathbb{P}(E)$ will also admit a cscK metric, again making the fibres small. (Once again, from [Don01b], this implies that $\mathbb{P}(E)$ is asymptotically Chow stable.) Second, a result of R. Seyyedali states that in fact under these conditions, $\mathbb{P}(E)$ is Chow stable with respect to L_k for $k \gg 0$, the novelty here being that stability is not taken asymptotically, which implies Morrison's result.

The purpose of this chapter is to relax the assumption that E is Mumford stable and instead consider bundles that are merely Gieseker stable. To state the theorems precisely, let B be a smooth polarized manifold carrying an ample line bundle L such that the automorphism group $\mathrm{Aut}(B, L)/\mathbb{C}^*$ is discrete. The projective bundle $\pi\colon \mathbb{P}(E) \to B$ carries a tautological bundle $\mathcal{O}_{\mathbb{P}E}(1)$, and the line bundle

$$\mathcal{L}_k := \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L^k$$

is ample for k sufficiently large.

Theorem 5.1.1 ([KelRos12])**.** *Suppose that E is Gieseker stable and its Jordan-Hölder filtration is given by subbundles, and assume there is a constant scalar curvature Kähler metric in the class $c_1(L)$. Then $(\mathbb{P}(E), \mathcal{L}_k)$ is Chow stable for k sufficiently large.*

Theorem 5.1.2 ([KelRos12])**.** *Suppose that E is a rank 2 bundle over a surface and F is a subbundle of E such that E/F is locally free. Suppose furthermore $\mu(F) = \mu(E)$ and*

$$4\,(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F)) + c_1(B)\,(c_1(E)/2 - c_1(F)) < 0,$$

where ch_2 denotes the degree 2 term in the Chern character. Then for k sufficiently large, $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-semistable and thus not asymptotic Chow stable.

These theorems should be compared to an observation of D. Mumford that a quartic cuspidal plane curve is Chow stable (as a plane curve), but not asymptotically Chow stable [Mum77, Section 3]. We consider the results here noteworthy insofar as it gives in Section 5.5 smooth examples of the same nature.

Corollary 5.1.1 ([KelRos12])**.** *There exists smooth polarized complex manifolds (X, L) such that (X, L) is Chow stable but not asymptotically Chow stable.*

When E is Mumford stable Theorem 5.1.1 is due to Seyyedali [Sey10] which in turn builds on work of Donaldson [Don01b]. Our proof will be along the same lines, the main innovation being to replace the Hermitian-Einstein metrics used by Seyyedali with the almost Hermitian-Einstein metrics on a Gieseker stable bundle furnished by N.C. Leung [Leu97]. Under the above assumptions, we construct a sequence of metrics that are balanced i.e make the Bergman function for $(\mathbb{P}(E), \mathcal{L}_k)$ constant. The proof of Theorem 5.1.2 consists of a calculation of the Donaldson-Futaki invariant similar to that of Ross-Thomas [RT06].

Conventions: If $\pi\colon E \to B$ is a vector bundle then $\pi\colon \mathbb{P}(E) \to B$ shall denote the space of complex *hyperplanes* in the fibres of E. Thus $\pi_* \mathcal{O}_{\mathbb{P}(E)}(r) = S^r E$ for $r \geq 0$.

5.2 About Gieseker stability and Bergman kernel endormorphism

Before discussing almost Hermitian Einstein metrics we recall some basic definitions. Let (B, L) be a polarized manifold with $b = \dim B$ and $E \to B$ a vector bundle. We say that E is *L-Mumford stable*[1] if for all proper coherent subsheaves $F \subset E$

$$\mu(F) < \mu(E)$$

where the slope $\mu(F) = \mu_L(F) = \deg_L F/\mathrm{rk}(F)$ is the quotient of the degree of F (with respect to L) by its rank $\mathrm{rk}(F)$. We say it is *Mumford semistable* if the same condition holds but with non-strict inequality. Finally E is *Mumford polystable* if it is the direct sum of Mumford stable bundles whose factors all have the same slope.

Any Mumford semi-stable bundle E has a Jordan-Hölder filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = E$ by torsion free subsheaves such that the quotients F_i/F_{i+1} are Mumford stable with $\mu(E) = \mu(F_i/F_{i+1})$. We say that E has a *Jordan-Hölder filtration given by subbundles* if it is Mumford semistable and all the quotients F_i/F_{i+1} are locally free (see [Leu97, Theorem 3]).

We say that E is *Gieseker stable* if for all proper coherent subsheaves $F \subset E$ one has the following inequality for the normalized Hilbert polynomials

$$\frac{\chi(F \otimes L^k)}{\mathrm{rk}(F)} < \frac{\chi(E \otimes L^k)}{\mathrm{rk}(E)} \quad \text{for } k \gg 0,$$

and *Gieseker semistability, Gieseker polystability* is defined analogously. It is known that if E is Gieseker stable then it is simple [Kob87], which means that $\mathrm{Ker}(\bar\partial) = \mathrm{Ker}(\partial) = \mathbb{C}\mathrm{Id}_E$ [Kob87; LT95].

These stability notions are related; using that $\mu_L(F)$ is the leading order term in k of $\chi(F \otimes L^k)/\mathrm{rk}(F)$ one sees immediately that

[1] We shall omit the polarization parameter when there is no possibility of confusion.

$$\begin{matrix}\text{Mumford} \\ \text{stable}\end{matrix} \Rightarrow \begin{matrix}\text{Gieseker} \\ \text{stable}\end{matrix} \Rightarrow \begin{matrix}\text{Gieseker} \\ \text{semistable}\end{matrix} \Rightarrow \begin{matrix}\text{Mumford} \\ \text{semistable}\end{matrix}.$$

5.2.1 Almost Hermitian-Einstein metrics and Gieseker stability

Now suppose L is equipped with a smooth hermitian metric h_L with curvature $\omega := c_1(h_L) > 0$.

Definition 5.2.1. We say that a sequence of hermitian metrics H_k on E is *almost Hermitian-Einstein* if for each $r \geq 0$ the curvature F_{H_k} is bounded in the C^r-norm uniformly with respect to k, and furthermore

$$[e^{F_{H_k}+k\omega \mathrm{Id}_E}\mathrm{Todd}(B)]^{(b,b)} = \frac{\chi(E\otimes L^k)}{\mathrm{rk}(E)}\mathrm{Id}_E\frac{\omega^b}{b!}.$$

In the above the (b,b) indicates taking the top order forms on the left hand since, and $\mathrm{Todd}(B) = 1 + c_1(B) + \frac{1}{2}(c_1(B)^2 + c_2(B)) + \cdots$ is the harmonic representative of the Todd class with respect to ω.

By a simple rearrangement this condition implies

$$\sqrt{-1}\Lambda_\omega F_{H_k} - \mu(E)\mathrm{Id}_E = T_0 + k^{-1}T_1 + \cdots \tag{5.1}$$

where $T_i \in \Gamma(\mathrm{End}(E))$ are hermitian endomorphisms depending on F_{H_k} and ω that are bounded uniformly over k in the C^r-norm. Moreover we can arrange so

$$T_0 = -\frac{\mathrm{scal}(\omega)}{2}\mathrm{Id}_E. \tag{5.2}$$

where $\mathrm{scal}(\omega)$ is the scalar curvature of ω.

Remark 5.2.1. By the C^r-norm above we mean the sum of the supremum norms of the first r derivatives taken using the pointwise operator norm with respect to a background metric on the bundle in question (that should be fixed once and for all). From now on we shall write $O_{\mathrm{C}^r}(k^i)$ to mean a sum of terms bounded in the C^r-norm by Ck^i for some constant C. Thus, in the above, $T_i = O_{\mathrm{C}^r}(k^0) = O_{\mathrm{C}^r}(1)$.

In [Leu97], Leung proved a Hitchin-Kobayashi type correspondence for Gieseker stable vector bundles.

Theorem 5.2.2 (Leung)**.** *Assume that the Jordan-Hölder filtration of E is given by subbundles. Then E is Gieseker stable if and only if E admits a sequence of almost Hermitian-Einstein metrics for $k \gg 0$.*

For simplicity we package together the following assumption:

($\mathcal{A}$) Let E be an Gieseker stable holomorphic vector bundle of rank $\mathrm{rk}(E)$ whose Jordan-Hölder filtration is given by subbundles.

We refer to Section 5.5 for examples of bundles that satisfy assumption ($\mathcal{A}$). From now on we will also assume that E is not Mumford stable, otherwise our results are direct consequences of [Sey10; Wan02; Wan05].

5.2.2 Balanced metrics

Suppose now in addition to our metric h_L on L we also have a smooth Hermitian metric H on E. These induce a hermitian metric $H \otimes h_L^k$ on $E \otimes L^k$ which determines an L^2-inner product on the space of smooth sections $\Gamma(E \otimes L^k)$ given by

$$\|s\|_{L^2}^2 = \int_B |s|^2_{H\otimes h_L^k} \frac{\omega^b}{b!}.$$

Associated to this data there is a projection operator $P_k \colon \Gamma(B, E \otimes L^k) \to H^0(B, E \otimes L^k)$ onto the space of holomorphic sections for each k. The *Bergman kernel* is defined to be the kernel of this operator which satisfies

$$P_k(f)(x) = \int_B B_k(x,y) f(y) \frac{\omega_y^b}{b!} \quad \text{for all } f \in \Gamma(E \otimes L^k),$$

(see [Wan02, Section 4]).

We wish to consider the Bergman kernel restricted to the diagonal, which by abuse of notation we write as $B_k(x) = B_k(x,x)$. Thus $B_k(x)$ lies in $\Gamma(\mathrm{End}(E))$ which we shall refer to as the *Bergman endomorphism* for $E\otimes L^k$, which of course depends on the data $(H \otimes h_L^k, \omega^b/b!)$.

Definition 5.2.3. We say that the metric $H \in \mathrm{Met}(E)$ is *balanced* at level k if the Bergman endomorphism $B_k(H \otimes h_L^k, \omega^b/b!)$ is constant over the base, i.e.

$$B_k = \frac{h^0(E \otimes L^k)}{\mathrm{rk}(E)\mathrm{Vol}_L(B)} \mathrm{Id}_E.$$

The connection with Gieseker stability is furnished by the following result of X. Wang [Wan02]:

Theorem 5.2.4 (Wang)**.** *The bundle E is Gieseker polystable if and only if there exists a sequence of metrics H_k on E such that H_k is balanced at level k for all $k \gg 0$.*

Taking E to be the trivial bundle, with the trivial metric, gives an important special case. Here the only metric that can vary is that on the line bundle L, and B_k becomes scalar valued. To emphasize the importance of this case we use separate terminology. We say the metric h_L is *balanced at* *level* k if its Bergman function $\rho_k = \rho_k(h_L, \omega^b/b!)$ is a constant function, i.e.

$$\rho_k = \frac{h^0(L^k)}{\mathrm{Vol}_L(B)}.$$

In this context, balanced metrics are related to stability of the base B as in the following result proved by Zhang [Zha96], Luo [Luo98], cf. Section 2.5 for details.

Remark 5.2.2. It will turn out that the assumption that $\mathrm{scal}(\omega)$ is constant is not strictly speaking necessary for our proof of Theorem 5.1.1. In fact, as will be apparent, a simple modification shows it is sufficient to assume that there is a sequence of metrics $h_{L,k}$ on L that are balanced at level k and whose associated curvatures ω_k are themselves bounded in the right topology. However we know of no examples of manifolds that admit such a sequence of metrics that do not admit a cscK metric, and thus this generalization does not give anything new.

5.2.3 Density of States Expansion

Through work of S.T. Yau [Yau86], G. Tian [Tia90], D. Catlin [Cat99], S. Zelditch [Zel98], X. Wang [Wan05] among others, one can understand the behavior of the Bergman endomorphism as k tends to infinity through the so-called "density of states" asymptotic expansion. We refer to [MM07] as a reference for this topic. The upshot is that for fixed $q, r \geq 0$ one can write

$$B_k = k^b A_0 + k^{b-1} A_1 + \cdots + k^{b-q} A_q + O_{\mathrm{C}^r}(k^{b-q-1}) \tag{5.3}$$

where $A_i \in \Gamma(\mathrm{End}(E))$ are hermitian endomorphism valued functions. The A_i depend on the curvature of the metrics in question, and when necessary will be denoted by $A_i = A_i(h, H)$; in fact

$$A_0 = \mathrm{Id}_E \quad \text{and} \quad A_1 = \sqrt{-1}\Lambda_\omega F_H + \frac{\mathrm{scal}(\omega)}{2}\mathrm{Id}_E.$$

Now a key point for our application is the observation that the above expansion still holds if the metrics on L and E are allowed to vary, so long as the *curvature* of the metric on E remains under control. This is made precise in the following proposition which is a slight generalization of [MM07, Theorem 4.1.1].

Proposition 5.2.1 ([KelRos12])**.** *Let $q, r \geq 0$ fixed as above. Let $h_{L,k} \in \mathrm{Met}(L)$ be a sequence of metrics converging in C^∞ topology to $h_L \in \mathrm{Met}(L)$ such that $\omega := c_1(h_L) > 0$. Let $H_k \in \mathrm{Met}(E)$ be a sequence of metrics such that the curvatures F_{H_k} are bounded independently of k in $\mathrm{C}^{r'}$ norm for some $r' \gg r, q$. Consider the Bergman endomorphism B_k associated* *to* $H_k \otimes h_{L,k}^k \in \mathrm{Met}(E \otimes L^k)$. *Then* B_k *satisfies the uniform asymptotic expansion* (5.3) *with* $A_i = A_i(h_{L,k}, H_k)$.

We only provide a sketch of the proof of this proposition by pointing out how to adapt Tian's construction of peak sections ([Tia90], [Wan05, Section 5]) to this setting. It is important for our application that we do not assume the H_k necessarily converge.

In order to modify a smooth peaked section to a holomorphic one, and control the L^2 norm of this change, one needs to apply Hörmander L^2 estimates for the $\bar\partial$ operator. But under our assumptions $\sqrt{-1}\Lambda F_{H_k} + k\mathrm{Id}_E$ is positive definite for k large enough so Hörmander's theorem (see [Dem96, Theorem (8.4)]) holds on $E \otimes L^k$.

Since the calculation of the asymptotics is local in nature, another key ingredient is a pointwise expansion of the involved metrics. Fix a point $z_0 \in B$. From [Dem97, Chapter V - Theorem 12.10], we know that there exists a holomorphic frame $(e_i)_{i=1,..,rk(E)}$ over a neighborhood of $z_0 \in B$ such that, with respect to this frame, the endomorphism $\mathbf{H}_k(z)_{ij} = H_k(e_i, e_j)$ associated to the metric H_k has the following expansion:

$$\mathbf{H}_k(z)_{ij} = \left(\delta_{ij} - \sum_{1 \le k,l \le n} (F_{H_k})_{i\bar{j}k\bar{l}}\, z_k \bar{z}_l + O\left(|z|^3\right) \right), \tag{5.4}$$

Furthermore, by induction one can show that the higher order terms of the expansions are given by derivatives of the curvature of the metric on E. For instance, at order 3, $\mathbf{H_k}(z)_{ij}$ has an extra term of the form

$$-\frac{1}{2}\left((F_{H_k})_{i\bar{j}a\bar{b},c} z_a \bar{z}_b z_c + (F_{H_k})_{i\bar{j}a\bar{b},\bar{c}}\right) z_a \bar{z}_b \bar{z}_c,$$

and thus in (5.4), $O\left(|z|^3\right) = O_{C^{r'-1}}(k^0)$ under our assumptions. Similarly, the higher order terms of this Taylor expansion are under control. Using this, one can follow line by line the arguments of [Wan05, Section 5] to obtain the proposition.

5.3 Construction of balanced metrics

In this section we construct metrics for $\mathbb{P}(E)$ that are almost balanced by perturbing the metrics on the bundle E. Then an application of the implicit function argument from [Sey10] will provide the required balanced metrics.

5.3.1 Relating the metric on the bundle to the metric on the projectivization

We first recall the techniques in [Sey10] that relate the Bergman endomorphism on $E \otimes L^k$ and the Bergman function on the projectivization

$(\mathbb{P}(E), \mathcal{L}_k := \mathcal{O}_{\mathbb{P}E}(1) \otimes \pi^* L^k)$.

Let V be a vector space equipped with a hermitian metric H_V. This induces in a natural way a Fubini-Study hermitian metric on $\mathcal{O}_{\mathbb{P}(V)}(1))$ which we denote by $\hat{h}_V$. Similarly given a hermitian metric H on E we get an induced metric $\hat{h}_E$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$. We denote by

$$\rho_k = \rho_k(\hat{h}_E \otimes \pi^* h_L^k)$$

the Bergman function on $(\mathbb{P}(E), \mathcal{L}_k)$ induced from the metric $\hat{h}_E \otimes \pi^* h_L^k$. The next results gives an asymptotic expansion for ρ_k in k (observe this is not the same as the usual density of states expansion, since we are not taking powers of a fixed line bundle).

Theorem 5.3.1 (Seyyedali [Sey10])**.** *There exists smooth endomorphism valued functions $\widetilde{B}_k = \widetilde{B}_k(H, h_L)$ such that*

$$\rho_k([v]) = \frac{1}{c_r} \mathrm{tr}\left(\frac{v \otimes v^{*_H}}{\|v\|_H^2} \widetilde{B}_k(H, h_L) \right) \quad \text{for } [v] \in \mathbb{P}(E), \tag{5.5}$$

where $c_r := \int_{\mathbb{C}^{r-1}} \frac{d\zeta \wedge d\bar{\zeta}}{(1+\sum_{j=1}^{r-1} |\zeta_j|^2)^{r+1}}$. Moreover $\tilde{B}_k$ has an asymptotic expansion of the form

$$\widetilde{B}_k(H, h_L) = k^b \mathrm{Id}_E + k^{b-1} \left(\sqrt{-1}[\Lambda_\omega F_H]^0 + \frac{\mathrm{rk}(E)+1}{2\mathrm{rk}(E)} \mathrm{scal}(\omega) \mathrm{Id}_E \right) + \cdots$$

where $[T]^0$ denotes the traceless part of the operator T.

We refer to $\tilde{B}_k$ as the *distorted Bergman endomorphism.* The proof of the previous results is obtained by relating $\tilde{B}_k$ to the Bergman endomorphism by an identity of the form

$$(\sum_{j=0}^{b} k^{j-b} \Psi_j) \widetilde{B}_k(H, h_L) = B_k(H \otimes h_L^k, \omega^b / b!),$$

for certain $(\Psi_j)_{j=0..b} \in \mathrm{End}(E)$ that depend only on the curvature of the metric $H \in \mathrm{Met}(E)$. In fact,

$$\Psi_j = \Lambda_\omega^{b-j} \left(F_H^{b-j} + P_1(H) F_H^{b-j-1} + \ldots + P_{b-j}(H) \right),$$

where $P_i(H) = P_i(C_1(H), \ldots, C_{b-j}(H))$ are polynomials of degree i in the k-th Chern forms $C_k(H)$ of H, $1 \leq k \leq b-j$ [Sey10, p 594]

Given this, the asymptotic expansion for $\tilde{B}_k$ follows from that of B_k. Thus, using Proposition 5.2.1 we see that Theorem 5.3.1 in fact holds uniformly if the metric h is allowed to vary in a compact set, and the metric

H is allowed to vary in such a way that the curvature F_H is bounded (as in the case for almost Hermitian-Einstein metrics).

5.3.2 Perturbation Argument

From now on let $E \to B$ be a vector bundle satisfying assumption ($\mathcal{A}$), equipped with a family of almost Hermitian-Einstein metrics $H_k \in \mathrm{Met}(E)$, and (L, h_L) a polarization of the underlying manifold B such that $\omega = c_1(h_L)$ is a cscK metric. We now show how to adapt the methods of [Sey10, Theorem 1.2], and prove the existence of metrics on $\mathcal{L}_k$ that are almost balanced, in the sense that the associated Bergman function is constant up to terms that are negligible for large k [Don01b].

The approach is to perturb both the Kähler metric ω and the almost Hermitian-Einstein metric H_k on E by considering

$$\omega'_k = \omega + \sqrt{-1}\partial\bar{\partial}(\sum_{i=1}^{q} k^{-i}\phi_i),$$
$$H'_k = H_k \left(\mathrm{Id}_E + \sum_{i=1}^{q} k^{-i}\Phi_i \right),$$

where ϕ_i are smooth functions on B and Φ_i are smooth endomorphisms of E. We will also denote the perturbed metric on L as $h'_L = h_L e^{-\sum_{i=1}^{q} k^{-i}\phi_i} \in \mathrm{Met}(L)$ which satisfies $\omega'_k = c_1(h'_L)$. The perturbations terms Φ_i and ϕ_i will be constructed iteratively to make the distorted Bergman endomorphism approximately constant. In fact it will be necessary for ϕ_i and Φ_i to themselves depend on k, but for fixed i they will be of order $O_{C^r}(k^0)$, and this will be clear from their construction.

Observe the metrics ω'_k lie in a compact set, and the curvature of the metrics H'_k are bounded over k. For the first step of the iteration, where $q = 1$, we can apply Theorem 5.3.1 to deduce

$$\tilde{B}_k(H'_k, h'_L) = k^b \mathrm{Id}_E + k^{b-1} A_1(H_k, \omega) + k^{b-2}(A_2(H_k, \omega) + \delta) + \cdots$$

where

$$A_1(H_k, \omega) = \sqrt{-1}[\Lambda_\omega F_{H_k}]^0 + \frac{\mathrm{rk}(E) + 1}{2\mathrm{rk}(E)} \mathrm{scal}(\omega)\mathrm{Id}_E$$

and we have defined

$$\delta = \delta(\Phi_1, \phi_1) := D(A_1)_{H_k,\omega}(\Phi_1, \phi_1),$$

where $D(A_1)$ is the linearization of A_1. Thus

$$D(A_1)_{H,\omega}(\Phi, \phi) = \frac{d}{dt}\Big|_{t=0} A_1(H(\mathrm{Id}_E + t\Phi), \omega + t\sqrt{-1}\partial\bar{\partial}\phi)$$

$$
\begin{aligned}
&= \frac{\mathrm{rk}(E)+1}{2\mathrm{rk}(E)}(\mathbb{L}\phi)\mathrm{Id}_E \\
&\quad + \sqrt{-1}\left[\Lambda_\omega \bar{\partial}\partial\Phi + \Lambda_\omega^2(F_H \wedge \sqrt{-1}\partial\bar{\partial}\phi) - \Delta_\omega\phi\Lambda_\omega F_H\right]^0
\end{aligned}
$$

where $\mathbb{L}$ denotes the Lichnerowicz operator with respect to ω.

Now from the definition of the almost Hermitian-Einstein metrics we have an expansion (5.1)

$$
T := \sqrt{-1}\Lambda_\omega F_{H_k} - \mu(E)\mathrm{Id}_E = T_0 + T_1 k^{-1} + T_2 k^{-2} + \cdots + T_{b-1}k^{b-1}, \tag{5.6}
$$

where the $T_i = O_{C^r}(k^0)$ and T_0 is constant, see (5.2).
The metric H_k on E induces a metric on $\mathrm{End}(E)$ (that we still denote by H_k in the sequel) and one has an operator $\partial_{\mathrm{End}(E)}$ (that we still denote ∂ in the sequel) as the $(1,0)$ part of the connection operator induced on $\mathrm{End}(E)$ from the Chern connection on E compatible with H_k. One can write the associated curvature on $\mathrm{End}(E)$ as $F_{\mathrm{End}(E),H_k} = F_{H_k} \otimes \mathrm{Id}_{E^*} + \mathrm{Id}_E \otimes F_{E^*,H_k^*}$.
Thus, we obtain a similar expansion as (5.6),

$$
\begin{aligned}
R &:= \sqrt{-1}\Lambda_\omega F_{\mathrm{End}(E),H_k} - \mu(\mathrm{End}(E))\mathrm{Id}_{\mathrm{End}(E)} \\
&= \sqrt{-1}\Lambda_\omega F_{\mathrm{End}(E),H_k} \\
&= R_0 + R_1 k^{-1} + R_2 k^{-2} + \cdots + R_{b-1}k^{b-1},
\end{aligned} \tag{5.7}
$$

where the $R_i = O_{C^r}(k^0)$ and R_0 is constant.
Using this, we rewrite the distorted Bergman endomorphism as

$$
\tilde{B}_k(H'_k, h'_L) = k^b\mathrm{Id}_E + k^{b-1}\tilde{A}_1 + k^{b-2}\tilde{A}_2 + \cdots
$$

where

$$
\begin{aligned}
\tilde{A}_1 &= \frac{\mathrm{rk}(E)+1}{2\mathrm{rk}(E)}\mathrm{scal}(\omega)\mathrm{Id}_E, \\
\tilde{A}_2 &= [T_1]^0 + A_2 + \delta(\Phi_1, \phi_1),
\end{aligned}
$$

since $[T_0]^0 = 0$.

Observe since $\mathrm{scal}(\omega)$ is constant, the top coefficient $\tilde{A}_1$ is also constant. The aim now is to find a perturbation that makes the lower order terms also constant. To this end it is convenient to define

$$
\tilde{R} = R - R_0 = O_{C^r}(1/k)
$$

and to rewrite the Bergman endomorphism once again, this time in the following way using (5.7)

$$
\tilde{B}_k(H'_k, h'_L) = k^b\mathrm{Id}_E + k^{b-1}\tilde{A}_1 + k^{b-2}(\tilde{A}_2 - [\tilde{R}\Phi_1]^0) + k^{b-3}(\tilde{A}_3 + [R_1\Phi_1]^0) + \cdots . \tag{5.8}
$$

Remark that since Φ_1 is $O_{C^r}(k^0)$, the same will be true of $[R_1\Phi_1]^0$. The reason for adding and subtracting this term arises when it comes to ensuring that the Φ_i we construct are hermitian operators, as in the next proposition which ensures that it is possible to find ϕ_1 and Φ_1 to make the k^{b-2} term constant.

Define $\mathrm{End}_0(E)$ to be the vector space of endomorphisms η of E such that $\int_B \mathrm{tr}\eta\, \frac{\omega^b}{b!} = 0$, $\mathrm{End}_0^0(E)$ the trace free elements of $\mathrm{End}_0(E)$ and $C_0^\infty(B,\mathbb{R})$ the space of smooth functions with null integral with respect to the volume form $\frac{\omega^b}{b!}$.

Proposition 5.3.1 ([KelRos12]). *Assume that $\mathrm{Aut}(B,L)/\mathbb{C}^*$ is discrete, ω is a cscK metric and that E satisfies assumption ($\mathcal{A}$). Then for any endomorphism $\zeta \in \mathrm{End}_0(E)$ there exists a unique couple $(\Phi_1,\phi_1) \in \mathrm{End}_0^0(E) \times C_0^\infty(B,\mathbb{R})$ such that*

$$\delta(\Phi_1,\phi_1) - [\tilde{R}\Phi_1]^0 = \zeta. \tag{5.9}$$

Furthermore, Φ_1 is hermitian with respect to H_k if and only if the same is true of ζ.

Finally if $r \geq 4$ and $\alpha \in (0,1)$ there is a $c_{r,\alpha}$ such that for all ζ,

$$\|\phi_1\|_{C^{r,\alpha}} + \|\Phi_1\|_{C^{r-2,\alpha}} \leq c_{r,\alpha}\|\zeta\|_{C^{r-4,\alpha}}.$$

We observe that $\tilde{R}$ is non-zero for all k since by assumption E is not Mumford stable and thus none of the H_k are Hermitian-Einstein. The proof of the previous proposition will depend on a number of Lemmas, the first of which is a consequence of Kähler identities.

Lemma 5.3.1. *Let H be a hermitian metric on E which induces a metric on $\mathrm{End}(E)$ that we still denote H. Then for any $\zeta \in \mathrm{End}(E)$,*

$$\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\zeta^{*_H} = (\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\zeta - [\sqrt{-1}\Lambda_\omega F_{\mathrm{End}(E),H},\zeta])^{*_H}.$$

Lemma 5.3.2 (Poincaré type inequality). *Assume that E is a simple holomorphic vector bundle. Then there is a constant C such that if $H \in \mathrm{Met}(E)$ and $\eta \in \mathrm{End}(E)$, we have the following inequality with respect to the metric induced on $\mathrm{End}(E)$,*

$$\|\eta\|^2_{L^2_H} \leq C\|\bar{\partial}\eta\|^2_{L^2_H} + \frac{1}{rk(E)\mathrm{Vol}_L(B)}\Big|\int_B \mathrm{tr}\eta\,\frac{\omega^b}{b!}\Big|^2.$$

Note that if we consider another reference metric H_0 and H such that $r\cdot H_0 > H > r^{-1}\cdot H_0$ with $r > 1$, then we can choose C depending only on (H_0,r).

Proof. This is standard from the fact $\bar{\partial}^*\bar{\partial}$ provides a positive elliptic operator and our simpleness assumption [Wan05, Section 3]. Here the constant C in the statement can be taken as the first positive eigenvalue of the elliptic operator. Note that for a varying metric in a bounded family of $\mathrm{Met}(E)$, since the $\bar{\partial}$-operator doesn't depend on the metric, we can choose the constant C uniformly. □

Lemma 5.3.3. *Assume that E is a simple holomorphic vector bundle. For k sufficiently large, given any $\zeta \in \mathrm{End}_0(E)$ there is a unique $\eta \in \mathrm{End}_0(E)$ such that*

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0 = \zeta \tag{5.10}$$

Furthermore η is hermitian (with respect to H_k) if and only if the same is true for ζ. Finally if $r \geq 2$ and $\alpha \in (0,1)$ there is a constant $c_{r,\alpha}$ such that

$$\|\eta\|_{\mathrm{C}^{r,\alpha}} \leq c_{r,\alpha}\|\zeta\|_{\mathrm{C}^{r-2,\alpha}}.$$

Proof. We use Fredholm alternative for elliptic equations. Firstly, $\tilde{R}$ is hermitian thus, the operator

$$\eta \mapsto \sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0$$

is hermitian and elliptic. To show existence of a solution, we need to show that this operator restricted on $\mathrm{End}_0(E)$ has trivial kernel. Let us assume that

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\eta - [\tilde{R}\eta]^0 = 0. \tag{5.11}$$

Let us fix a smooth hermitian metric on E which gives us a metric on $\mathrm{End}(E)$. Equation (5.11) implies, by Kähler identities and by taking inner product with η, that we have pointwise

$$\langle \partial\eta, \partial\eta\rangle - \langle [\tilde{R}\eta]^0, \eta\rangle = 0.$$

Now, this implies

$$\langle \bar{\partial}\eta^*, \bar{\partial}\eta^*\rangle - \langle [\underline{\tilde{R}}\eta]^0, \eta\rangle = \|\bar{\partial}\eta^*\|^2 - \langle [\tilde{R}\eta]^0, \eta\rangle = 0. \tag{5.12}$$

Using Cauchy-Schwartz inequality, we have $\langle \tilde{R}\eta, \eta\rangle \leq \|\tilde{R}\|\|\eta\|^2 \leq \|\tilde{R}\|\|\eta^*\|^2$ and $\langle \mathrm{Tr}(\tilde{R}\eta)\mathrm{Id}_E, \eta\rangle \leq \mathrm{rk}(E)\|\tilde{R}\|\|\eta\|^2$. By integration, we deduce, using that $\|\tilde{R}\|_{\mathrm{C}^0} = O_{\mathrm{C}^r}(1/k)$ and Lemma 5.3.2, that $\|\bar{\partial}\eta^*\|^2_{L^2}(1 - C/k) = 0$ from (5.12). Thus $\bar{\partial}\eta^* = 0$ if $k \gg 0$. But, since E is simple, this gives that $\eta = \alpha\mathrm{Id}_E$ for a constant α, see [LT95, Section 7.2]. Finally, the kernel of the operator $\sqrt{-1}\Lambda_\omega\bar{\partial}\partial \cdot -[\tilde{R}\cdot]$ on $\mathrm{End}_0(E)$ is trivial and we get uniqueness.

Let us show that we get a hermitian solution. From Lemma 5.3.1, one has that

$$\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\eta^* = \left(\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\eta - [\sqrt{-1}\Lambda_\omega F_{\mathrm{End}(E),H_k}, \eta]\right)^*$$

where now the adjoint is computed with respect to the almost Hermitian-Einstein metric H_k on E. Since $[R_0, \eta] = 0$ (any term of the form $\theta\mathrm{Id}_{End(E)}$

with θ a function is in the centre of the Lie algebra $\mathrm{End}(E \otimes L^k)$), one can rewrite this equation as

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \eta^* = \left(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \eta - [\tilde{R}, \eta]\right)^*.$$

After expansion, this is equivalent to

$$(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial - \tilde{R})\eta^* = \left((\sqrt{-1}\Lambda_\omega \bar{\partial}\partial - \tilde{R})\eta\right)^*$$

since $\tilde{R}$ is hermitian, and this can be rewritten as

$$(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \eta^* - [\tilde{R}\eta^*]^0) = \left(\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \eta - [\tilde{R}\eta]^0\right)^*.$$

Now, from the uniqueness we have shown previously, one gets that the solution is hermitian with respect to the metric H_k.

Let us denote $\mathrm{End}_0(E)^{r,\alpha}$ the Sobolev space of $\mathrm{C}^{r,\alpha}$ hermitian endomorphisms of $\mathrm{End}_0(E)$. For $k \gg 0$, $r \geq 2$, we have that $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial \cdot -[\tilde{R}\cdot]^0$ is an invertible linear differential operator of order 2 from $\mathrm{End}_0(E)^{r,\alpha}$ to $\mathrm{End}_0(E)^{r-2,\alpha}$ with uniformly bounded coefficients since we have the uniform control $\tilde{R} = O_{\mathrm{C}^r}(1/k)$. The eigenvalues of $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\cdot$ are strictly positive, while the eigenvalues of the hermitian operator (of order 0) $[\tilde{R}\cdot]^0$ tend to 0 as k becomes larger. Thus this operator is uniformly elliptic, we can apply Schauder theory of elliptic regularity [LT95, Section 7.3]. Note that we could also invoke the work of Uhlenbeck and Yau for the operator $\sqrt{-1}\Lambda_\omega \bar{\partial}\partial\cdot$ with a slight generalization. Finally, the inverse of this operator is bounded and we obtain the existence of a uniform constant $c > 0$ such that for any (η, ζ) satisfying (5.10),

$$\|\eta\|_{\mathrm{C}^{r,\alpha}} \leq c\|\zeta\|_{\mathrm{C}^{r-2,\alpha}}.$$

$\square$

Proof of Proposition 5.3.1. Obviously, we have the decomposition

$$\mathrm{End}_0(E) = \mathrm{End}_0^0(E) \oplus C_0^\infty(B, \mathbb{R})\mathrm{Id}_E.$$

First we deal with existence, by looking at the kernel of the operator on $\mathrm{End}_0(E)$ given by

$$D(A_1)_{H_k,\omega}(\Phi_1, \phi_1) - [\tilde{R}\Phi_1]^0 = 0$$

where $\Phi_1 \in \mathrm{End}_0^0(E)$ and $\phi_1 \in C_0^\infty(B,\mathbb{R})$. This is equivalent to ask that

$$\frac{\mathrm{rk}(E)}{2\mathrm{rk}(E)+1}\mathbb{L}\phi_1 = 0 \tag{5.13}$$

$$\left[\sqrt{-1}\left(\Lambda_\omega\bar{\partial}\partial\Phi_1 + \Lambda_\omega^2(F_{H_k}\wedge\sqrt{-1}\partial\bar{\partial}\phi_1) - \Delta_\omega\phi_1\Lambda_\omega F_{H_k}\right) - \tilde{R}\Phi_1\right]^0 = 0 \tag{5.14}$$

Equation (5.13) gives immediately that $\phi_1 = 0$ since the kernel of the Lichnerowicz operator consists of just the constant functions (see [Don01b]) thanks to the fact that $\mathrm{Aut}(B,L)/\mathbb{C}^*$ is discrete and since $\int_B \phi_1 \frac{\omega^b}{b!} = 0$. Now, since Φ_1 is trace free, Equation (5.14) reduces to

$$\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\Phi_1 - [\tilde{R}\Phi_1]^0 = 0$$

which admits only the trivial solution, from Lemma 5.3.3 ($\tilde{R} \neq 0$ since the vector bundle E is not Mumford stable). Thus, by Fredholm alternative, we can solve Equation (5.9). Moreover, we know that the terms $\frac{\mathrm{rk}(E)}{2\mathrm{rk}(E)+1}\mathbb{L}\phi_1$ and $\sqrt{-1}\Lambda_\omega^2(F_{H_k}\wedge\sqrt{-1}\partial\bar{\partial}\phi) - \sqrt{-1}\Delta_\omega\phi_1\Lambda_\omega F_{H_k}$ are hermitian. Hence, for the solution Φ_1 of (5.9), if ζ is hermitian, one can rewrite this equation as

$$\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\Phi_1 - [\tilde{R}\Phi_1]^0 = \zeta'$$

where ζ' is hermitian with respect to H_k. Then, applying Lemma 5.3.3, we get that Φ_1 is hermitian. Finally the regularity of the solution is a consequence of Lemma 5.3.3 and the fact that the Lichnerowicz operator is a strongly elliptic operator of order 4. □

Returning now to the construction of the almost balanced metrics, using Proposition 5.3.1, we obtain (Φ_1, ϕ_1) such that the second term of (5.8) satisfies

$$\tilde{A}_2 - [\tilde{R}\Phi_1]^0 = C_2\mathrm{Id}_E$$

or equivalently

$$\delta(\Phi_1,\phi_1) - [\tilde{R}\Phi_1]^0 = -A_2 - [T_1]^0 + C_2\mathrm{Id}_E$$

where C_2 is a topological constant. Note that we have used here the obvious fact that $\int_B \mathrm{tr}(C_2 - A_2)\,\omega^b = 0$.

For the next step of our iterative process, we perturb the metrics H_k and ω_k at the order $q = 2$ and try to find Φ_2, ϕ_2 such that the third term of (5.8) is constant. Now this third term can be written

$$A_3(H_k,\omega) + \delta(\Phi_2,\phi_2) - [\tilde{R}\Phi_2]^0 + [T_2]^0 + [R_1\Phi_1]^0 + b_{1,2}$$

with $b_{1,2}$ obtained from the deformation of A_2, and thus depends only on the $(H_k, \Phi_1, \omega, \phi_1)$ computed at the previous step of the iteration. We then use the same trick as before, introducing the term $[\tilde{R}\Phi_2]^0$ in order to obtain a hermitian solution, and see that Φ_2, ϕ_2 need to satisfy

$$\delta(\Phi_2, \phi_2) - [\tilde{R}\Phi_2]^0 = C_3 \mathrm{Id}_E - b_{1,2} - A_3(H_k, \omega) - [R_1\Phi_1]^0$$

where C_3 is a topological constant. Now solutions to this equation are guaranteed just as before using Proposition 5.3.1.

Repeating this iteration one sees that at each step one is led to solve the equation

$$\delta(\Phi_i, \phi_i) - [\tilde{R}\Phi_i]^0 = \zeta_i$$

where ζ_i is hermitian with respect to H_k and depends on the computations of the previous steps, i.e on the data $(H_k, \omega, \Phi_1, ..., \Phi_{i-1}, \phi_1, ..., \phi_{i-1})$ and $\int_B \mathrm{tr}\zeta_i\, \omega^b = 0$. Clearly then the metric that we construct with this process is hermitian. Thus we have the following result:

Theorem 5.3.2 ([KelRos12]). *Let E be a vector bundle that satisfies assumption ($\mathcal{A}$) on the projective manifold B with $\dim_{\mathbb{C}} B = b$, (L, h_L) a polarization on B with $\omega = c_1(h_L) > 0$. Assume that $\mathrm{Aut}(B, L)/\mathbb{C}^*$ is discrete and ω is a cscK metric. Consider an almost Hermitian-Einstein metric $H_k \in \mathrm{Met}(E)$. Then any fixed integers $q, r > 0$, and $k \gg 0$, the metrics H_k and h_L can be deformed to new metrics $H'_k \in \mathrm{Met}(E)$ and $h'_L \in \mathrm{Met}(L)$ such that the distorted Bergman endomorphism $\tilde{B}_k(H'_k, h'_L)$ satisfies*

$$\tilde{B}_k(H'_k, h'_L) = k^b \mathrm{Id}_E + \epsilon_k \in \mathrm{End}(E)$$

where $\epsilon_k = O_{C^r}(k^{b-q})$.

Next consider $\hat{h}'$ the metric induced on $\mathcal{O}_{\mathbb{P}(E)}(1)$ from $H'_k \in \mathrm{Met}(E)$. Then using (5.5) gives the following corollary.

Corollary 5.3.1 ([KelRos12]). *Under the same assumptions as in Theorem 5.3.2, for any fixed integers $q, r > 0$, and $k \gg 0$ each metric H_k and ω can be deformed to obtain a smooth hermitian metric $H'_k \in \mathrm{Met}(E)$ and a smooth and $h'_L \in \mathrm{Met}(L)$ such that the induced Bergman function $\rho_k(\hat{h}' \otimes \pi^* {h'_L}^k)$ on $\mathbb{P}(E)$ satisfies*

$$\rho_k(\hat{h}' \otimes \pi^* {h'_L}^k) = \hat{C} k^b + \hat{\epsilon}_k \in C^\infty(\mathbb{P}(E), \mathbb{R})$$

where $\hat{C}$ is a topological constant and $\hat{\epsilon}_k = O_{C^r}(k^{b-q})$.

Proof of Theorem 5.1.1. The rest of the proof is the same as [Sey10, Theorem 1.2] which shows how it is possible to perturb the almost balanced metrics above to obtain balanced metrics. Observe that all the estimates in sections 2,3 and 4 of [Sey10] only require E to be simple, which is the case since we are assuming it to be Gieseker stable. Note also that $\mathbb{P}(E)$ has no nontrivial holomorphic vector fields ([Sey10, Proposition 7.1]) since E is simple.

Finally, the existence of a balanced metric on $(\mathbb{P}(E), \mathcal{L}_k)$ implies the stability of the Chow point induced by $(\mathbb{P}(E), \mathcal{L}_k)$ [Luo98; Zha96] since there is no nontrivial automorphism, completing the proof. □

5.4 Computation of the Donaldson-Futaki invariant

We turn now to proving the instability result of Theorem 5.1.2 using the techniques of [RT06]. What is required is to consider one parameter degenerations (test configurations) of our manifold $\mathbb{P}(E)$ and these can be constructed rather naturally from subbundles.

Suppose that F is a subbundle of E such that $G := E/F$ is locally free. This gives rise to a family of bundles $\mathcal{E} \to X \times \mathbb{C} \to \mathbb{C}$ with general fibre E and central fibre $F \oplus G$ over $0 \in \mathbb{C}$. Moreover $\mathcal{E}$ admits a $\mathbb{C}^*$ action that covers the usual action on the base $\mathbb{C}$, and whose restriction to $F \oplus G$ scales the fibres of F with weight 1 and acts trivially on G. (One can see this in a number of ways, for instance if $\xi \in H^1(F \otimes G^*)$ represents the extension determined by E then this action takes ξ to zero as $\lambda \in \mathbb{C}^*$ tends to zero.)

Setting $\mathcal{X} = \mathbb{P}(\mathcal{E}) \to \mathbb{C}$ and $\tilde{\mathcal{L}}_k = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* L^k$ we thus have a flat family of polarized varieties with $\mathbb{C}^*$ action whose general fibre is $(\mathbb{P}(E), \mathcal{L}_k)$ (i.e. a test-configuration as introduced in [Don02]).

The goal is to calculate the sign of a certain numerical invariant DF_1, the Donaldson-Futaki invariant defined as in Section 2.5. We use the convention that if $DF_1 < 0$ then $\mathbb{P}(E)$ is K-unstable, which is known to imply that it is asymptotically Chow unstable.

To make the computations more palatable we restrict to the case that $\mathrm{rank}(E) = 2$ over a smooth polarized base (B, L) of complex dimension $b \geq 2$, and assume that F and G are locally free (although the computation is essentially the same without this assumption, see [RT06, Section 5.4]). We denote by ch_2 the second Chern character, so $\mathrm{ch}_2(F) = c_1(F)^2/2$ and $\mathrm{ch}_2(E) = c_1(E)^2/2 - c_2(E)$.

We work initially over a base of complex dimension b since this adds no significant difficulties, although the reader may wish to set $b = 2$ which will be all that is necessary for our applications. To ease notation set $\omega = c_1(L)$ and if $\alpha_i \in H^{2d_i}(B)$ with $d_1 + \cdots d_r = b$ we write $\alpha_1.\alpha_2 \cdots \alpha_r = \int_X \alpha_1 \wedge \cdots \wedge \alpha_r$.

Proposition 5.4.1 ([KelRos12]). *The Donaldson-Futaki invariant of the* *test configuration $\mathcal{T} = (\mathcal{X}, \bar{\mathcal{L}}_k)$ defined above is*[2]

$$DF_1(\mathcal{T}) = C_1 k^{2b-1} + C_2 k^{2b-2} + O(k^{2b-3})$$

where

$$\begin{aligned}
C_1(E,F) =& \frac{\omega^b}{3b!(b-1)!}\left(\mu(E) - \mu(F)\right),\\
C_2(E,F) =& \frac{\omega^b}{12b!(b-2)!}(c_1(E)/2 - c_1(F))c_1(B).\omega^{b-2}\\
&+ \frac{\omega^b}{3b!(b-2)!}(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F)).\omega^{b-2}\\
&+ \frac{1}{12(b-1)!^2}\left(2c_1(E).\omega^{b-1} - c_1(B).\omega^{b-1}\right)\right)(\mu(E) - \mu(F)).
\end{aligned}$$

Proof of Proposition 5.4.1. Recall $\pi_* \mathcal{L}_k^r = S^r E \otimes L^{rk}$ for $r \geq 0$, so from the Riemann-Roch theorem, we get

$$\begin{aligned}
\chi(\mathcal{L}_k^r) =& \chi(S^r E \otimes L^{kr}) = \int_B e^{rk\omega}\mathrm{ch}(S^r E) Td(B),\\
=& \frac{r^b k^b \omega^b}{b!}\mathrm{rank}(S^r E)\\
&+ \frac{r^{b-1}k^{b-1}}{(b-1)!}\omega^{b-1}\left(\mathrm{rank}(S^r E)\frac{c_1(B)}{2} + c_1(S^r E)\right)\\
&+ \frac{r^{b-2}k^{b-2}}{(b-2)!}\omega^{b-2}\left(\mathrm{rank}(S^r E)\mathrm{Todd}_B^{(2)} + \frac{c_1(S^r E).c_1(B)}{2} + \mathrm{ch}_2(S^r E)\right)\\
&+ O(k^{b-3}),
\end{aligned}$$

where $\mathrm{Todd}_B^{(2)}$ denotes the second Todd class of B, and we use the convention that $O(k^{b-3})$ vanishes if $b = 2$. Now, using the splitting principle, it is elementary to check that

$$\begin{aligned}
\mathrm{rank}(S^r E) &= r + 1,\\
c_1(S^r E) &= r(r+1)c_1(E)/2,\\
\mathrm{ch}_2(S^r E) &= r^3[c_1(E)^2/12 + \mathrm{ch}_2(E)/6] + r^2\mathrm{ch}_2(E)/2 + O(r).
\end{aligned}$$

Thus for $r \gg 0$,

$$p(r) := h^0(\mathbb{P}(E), \mathcal{L}_k^r) = a_0 r^{b+1} + a_1 r^b + O(r^{b-1}),$$

[2]This corrects an error in the lower order term of [RT06, Prop. 5.23]

where

$$\begin{aligned}
a_0 =& \frac{k^b\omega^b}{b!} + \frac{k^{b-1}\omega^{b-1}c_1(E)}{2(b-1)!} + \frac{k^{b-2}\omega^{b-2}}{(b-2)!}\left(\frac{1}{12}c_1(E)^2 + \frac{1}{6}\mathrm{ch}_2(E)\right) \\
&+ O(k^{b-3}), \\
a_1 =& \frac{k^b\omega^b}{b!} + \frac{k^{b-1}\omega^{b-1}}{2(b-1)!}.\left(c_1(B)+c_1(E)\right) \\
&+ \frac{k^{b-2}\omega^{b-2}}{(b-2)!}\left(\frac{\mathrm{ch}_2(E)}{2} + \frac{c_1(E).c_1(B)}{4}\right) + O(k^{b-3}).
\end{aligned}$$

Turning to the central fibre $\mathbb{P}(F \oplus G)$, we have a splitting

$$\begin{aligned}
H^0(\mathbb{P}(F\oplus G), \tilde{\mathcal{L}}_k^r) &= H^0(B, S^r(F\oplus G)\otimes L^{kr}) \\
&= \bigoplus_{i=0}^{r} H^0(B, F^i \otimes G^{r-i}\otimes L^{kr}),
\end{aligned}$$

Moreover this is the eigenspace decomposition for the action, with the i-th space having weight i. Let $w(r)$ be the sum of the eigenvalues of the action on this vector space, so

$$w(r) = \sum_{i=0}^{r} i h^0(B, F^i \otimes G^{r-i} \otimes L^{kr}).$$

Now since $\tilde{\mathcal{L}}_k$ is relatively ample, the higher cohomology groups vanish, and thus pushing forward to B we have that the higher cohomology groups of $F^i \otimes G^{r-i} \otimes L^{kr}$ vanish for $r \gg 0$. Thus from Riemann-Roch again, $h^0(F^i \otimes G^{r-i} \otimes L^{kr})$ equals

$$\begin{aligned}
&\frac{k^b r^b \omega^b}{b!} + \frac{k^{b-1}r^{b-1}\omega^{b-1}}{(b-1)!}\left(\frac{c_1(B)}{2} + ic_1(F) + (r-i)c_1(G)\right) \\
&\quad + \frac{k^{b-2}r^{b-2}\omega^{b-2}}{(b-2)!}\left(\frac{(ic_1(F)+(r-i)c_1(G))^2}{2} + Td_B^{(2)}\right) \\
&\quad + \frac{k^{b-2}r^{b-2}\omega^{b-2}}{(b-2)!}\left(\frac{c_1(B)(ic_1(F)+(r-i)c_1(G))}{2}\right) + O(r^{b-3}).
\end{aligned}$$

Now an elementary calculation gives $w(k) = b_0 r^{b+2} + b_1 r^{b+1} + O(r^b)$, where

$$\begin{aligned}
b_0 =& \frac{k^b\omega^b}{2b!} + \frac{k^{b-1}\omega^{b-1}c_1(F)}{3(b-1)!} + \frac{k^{b-1}\omega^{b-1}c_1(G)}{6(b-1)!} \\
&+ \frac{k^{b-2}\omega^{b-2}}{2(b-2)!}\left(\frac{c_1(F)^2}{4} + \frac{c_1(F)c_1(G)}{6} + \frac{c_1(G)^2}{12}\right) + O(k^{b-3}),
\end{aligned}$$

$$
\begin{aligned}
b_1 =& \frac{k^b\omega^b}{2b!} + \frac{k^{b-1}\omega^{b-1}c_1(F)}{2(b-1)!} + \frac{k^{b-1}\omega^{b-1}c_1(B)}{4(b-1)!} \\
&+ \frac{k^{b-2}\omega^{b-2}c_1(B)}{2(b-2)!}\left(\frac{c_1(F)}{3} + \frac{c_1(G)}{6}\right) + \frac{k^{b-2}\omega^{b-2}c_1(F)^2}{4(b-2)!} + O(k^{b-3}).
\end{aligned}
$$

The definition of the Donaldson-Futaki invariant is $DF_1(\mathcal{T}) = b_0a_1 - b_1a_0$, and putting this all together gives the result as stated. $\square$

Proposition 5.4.2 ([KelRos12]). *Suppose B is a surface, and $\chi(F\otimes L^k) = \chi(E\otimes L^k)/2$ for all k. Suppose also that either $c_1(B) = 0$ or $\omega = \pm c_1(B)$. Then $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-polystable for k sufficiently large.*

Proof. The previous computations can be extended to show that for a surface one can write the Donaldson-Futaki invariant as $DF_1(\mathcal{T}) = C_1k^3 + C_2k^2 + C_3k + C_4$ with C_1, C_2 given by Proposition 5.4.1 that vanish and

$$
\begin{aligned}
48C_3(E,F) =& \left(8\deg_L E - 4c_1(L)c_1(B)\right)\left(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F)\right) \\
&+ 2c_1(E)^2(\deg_L(E)/2 - \deg_L F) \\
&+ 2\deg_L(F)c_1(E)c_1(B) - 2\deg_L(E)c_1(B)c_1(F) \\
144C_4(E,F) =& c_1(E)^2\left[(c_1(E)/2 - c_1(F)).c_1(B) + 6(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F))\right] \\
&- 4c_1(E).c_1(B)(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F)) \\
&+ 2(c_1(E).c_1(B)\mathrm{ch}_2(F) - c_1(F).c_1(B)\mathrm{ch}_2(E))
\end{aligned}
$$

It is now a computation to check that under our assumptions, the terms C_3 and C_4 vanish. This can be seen easily if one rewrites these terms using Notation 6.0.1 (i.e the terms appearing in the difference of the normalized Hilbert polynomials of E and F) and the reformulation given by Proposition 6.0.1. Also observe that the degeneration used above is not a product test configuration, so $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-polystable for k large enough. $\square$

Proof of Theorem 5.1.2. Suppose that E is a rank 2 vector bundle that is Gieseker stable but not Mumford stable and $\mu(F) = \mu(E)$. From Proposition 5.4.1, the term C_1 vanishes and the Donaldson-Futaki invariant of the test configuration associated to F is

$$
DF_1(\mathcal{T}) = \frac{k^2}{24}\left(4\left(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F)^2\right) + c_1(B)\left(c_1(E)/2 - c_1(F)\right)\right) + O(k).
$$

Thus by hypothesis, $DF_1(\mathcal{T}) < 0$ for $k \gg 0$, proving that $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-semistable for $k \gg 0$ as claimed. $\square$

5.5 Examples of Chow stable but not asymptotically Chow stable manifolds

We end by constructing examples of polarized surfaces (B, L) and vector bundles E over B that satisfy the assumptions of Theorems 5.1.1 and 5.1.2. To do so we start with a base B with trivial automorphism group that has an abundance of cscK metrics.

Fix a rank 2 Mumford stable bundle V over a complex projective curve C of genus $g \geq 2$ and define $B = \mathbb{P}(V)$. As is well known, using the Narasimhan-Seshadri Theorem [NS65] one can prove there exists a cscK metric in each Kähler class of B. Moreover as V is simple and $g \geq 2$, there are no infinitesimal automorphisms of B [Sey10, Proposition 7.1].

We seek a suitable vector bundle E over B which is Gieseker stable and not Mumford stable. In order to do so, we fix some notations and describe the ample cone of B. The Néron-Severi group of B can be identified with $\mathbb{Z} \times \mathbb{Z}$, with generators the class $\mathfrak{b}$ of $\mathcal{O}_B(1)$ and the class $\mathfrak{f}$ of a fibre over C. We have $\mathfrak{b}^2 = \deg(V)$, $\mathfrak{f}^2 = 0$ and $\mathfrak{b} \cdot \mathfrak{f} = 1$ while the anti-canonical divisor is given by $-K_B = 2\mathfrak{b} + 2(1-g)\mathfrak{f}$. To ease the computations we may as well take $\deg V = 0$. Then, from [Tak72, Proposition 3.1], or [Fri98, Proposition 15], we know that a class $x\mathfrak{b} + y\mathfrak{f}$ is ample if $x > 0$ and $y > 0$.

Following the ideas of [Tak72, Proposition 3.9], consider a rank 2 vector bundle E_1 obtained as an extension

$$0 \to \mathcal{O}_B \to E_1 \to F_1 \to 0,$$

where F_1 has class $-\mathfrak{b}+(m+1)\mathfrak{f}$ for some large positive m. To ensure that we can take such an extension that does not split, we need that $Ext^1(\mathcal{O}_B, F_1^*) = H^1(F_1^*)$ is non trivial. But this follows easily from Riemann-Roch since $\chi(F_1^*) = h^0(F_1^*) - h^1(F_1^*) + h^2(F_1^*) \geq -h^1(F^*)$ and

$$\begin{aligned}
\chi(F_1^*) =& c_1(F_1^*)^2 + \frac{c_1(B)}{2} c_1(F_1^*) + \mathrm{Todd}_2(B), \\
=& -2(m+1) + (-(m+1) + (1-g)) + (1-g), \\
=& -3(m+1) + 2(1-g) < 0.
\end{aligned}$$

Over B, we take the polarization $L_{m+1} = \mathfrak{b} + (m+1)\mathfrak{f}$ and one checks easily that

$$\mu(F_1) = \mu(E_1) = 0.$$

We claim that E_1 is in fact Mumford semi-stable. A priori, we need to check stability with respect to any rank 1 torsion free subsheaf $\mathcal{F}$ of E_1 but since we are working with a rank 2 bundle on a surface, $\mathcal{F}^{**}$ is a reflexive rank 1 sheaf on B and thus a line bundle. So $\mathcal{F} = \mathcal{O}(D) \otimes \mathcal{I}$ where $\mathcal{O}(D)$ is a line bundle and $\mathcal{I}$ is an ideal sheaf with 0-dimensional support, so $c_1(\mathcal{F}) = c_1(\mathcal{F}^{**}) = c_1(\mathcal{O}(D))$. Since $E_1 = E_1^{**}$, it is now clear that it is sufficient to consider stability with respect to subbundles of E. But, for any rank 1 subbundle $\mathcal{O}(D)$ of E_1, either $\mathcal{O}(D) \hookrightarrow \mathcal{O}$ or $F_1 \otimes \mathcal{O}(-D)$ is effective. In the first case it is immediate that $\mathcal{O}(D)$ does not destabilize E_1. In the second case if we write the first Chern class of $\mathcal{O}(D)$ as $x_D\mathfrak{b} + y_D\mathfrak{f}$ we see by intersecting with ample line bundles that $x_D \leq -1$ and $y_D \leq m+1$. Hence $\mu(\mathcal{O}(D)) \leq \mu(F_1) = \mu(E_1)$ and E_1 is Mumford semi-stable with respect to L_{m+1} as claimed.

In order to construct a Gieseker stable bundle which is not Mumford stable, we tensor the previous extension by a line bundle F_2 with first Chern class $c_1(F_2) = -\mathfrak{b} + (g-3-m)\mathfrak{f}$, resulting in a non-trivial extension

$$0 \to F_2 \to E \to F_1 \otimes F_2 \to 0.$$

Observe that $\mu(F_2) = \mu(E)$ and so $\{0\} \subset F_2 \subset E$ is the Jordan-Hölder filtration of the Mumford semistable bundle E.

We claim that E is in fact Gieseker stable. As before, let $\mathcal{F} = \mathcal{O}(D) \otimes \mathcal{I}$ is a rank 1 torsion free subsheaf of E, and taking the double dual $\mathcal{O}(D)$ is a subbundle of E. Then either $\mathcal{O}(D) \hookrightarrow F_2$ or $F_1 \otimes F_2 \otimes \mathcal{O}(-D)$ is effective. In the first case by writing $c_1(\mathcal{O}(D)) = x_D\mathfrak{b} + y_D\mathfrak{f}$ one checks that if $(x_D, y_D) \neq (-1, g-3-m)$ then $\mu(\mathcal{O}(D)) < \mu(F_2) = \mu(E)$ while if $(x_D, y_D) = (-1, g-3-m)$ then $\mu(\mathcal{O}(D)) = \mu(E)$ and

$$\frac{\mathrm{ch}_2(E)}{2} - \mathrm{ch}_2(\mathcal{O}(D)) + \frac{c_1(B)}{2}\left(\frac{c_1(E)}{2} - c_1(\mathcal{O}(D))\right) = \frac{1}{2} > 0.$$

Thus by Riemann-Roch we conclude $\frac{1}{2}\chi(E \otimes L_{m+1}^p) > \chi(\mathcal{O}(D) \otimes L_{m+1}^p)$ for $p \gg 0$ and so $\mathcal{O}(D)$ does not Gieseker destabilize. Moreover this inequality only improves if $\mathcal{O}(D)$ is replaced by $\mathcal{F}$ since $c_2(\mathcal{F})$ is the length of the support of $\mathcal{I}$ and thus is non-negative. In the second case, in which $F_1 \otimes F_2 \otimes \mathcal{O}(-D)$ is effective, one deduces $x_D \leq -2$ and $y_D \leq g-2$ with at least one inequality being strict, and so $\mu(\mathcal{O}(D)) < \mu(F_1 \otimes F_2) = \mu(E)$. Hence E is Gieseker stable with respect to L_{m+1} as claimed.

So Theorem 5.1.1 can be applied in this setting and $(\mathbb{P}(E), \mathcal{L}_k)$ is Chow stable for k sufficiently large where $\mathcal{L}_k = \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L_{m+1}^k$. To apply Theorem 5.1.2, we compute

$$4(\mathrm{ch}_2(E)/2 - \mathrm{ch}_2(F_2)) + c_1(B).(c_1(E)/2 - c_1(F_2)) = -m - g + 2 < 0.$$

Hence $(\mathbb{P}(E), \mathcal{L}_k)$ is not K-semistable, thereby proving Corollary 5.1.1.

Chapter 6

Extra results about Chow, Hilbert and K-stability of ruled manifolds over surfaces

In the case of smooth curves of genus g, it is known through the classical work of Mumford that the polarized manifold (X, L) is stable (in the sense of Chow) as long as $\deg L > 2g$, $g \geq 1$. However in higher dimensions the question of stability may depend on the polarization chosen and rather little is known about the shape of the space of stable polarizations. The purpose of this chapter is to observe some phenomena that can occur in the special case of ruled manifolds over surfaces. The main results are Theorems 6.1.1, 6.2.1 and Corollaries 6.3.1, 6.3.2.

So suppose E is a Mumford polystable holomorphic vector bundle over a base projective manifold B, polarized by an ample line bundle L. Then by the Hitchin-Kobayashi correspondence E admits a Hermitian-Einstein metric. Suppose also that L is a line bundle on B and there exists a constant scalar curvature Kähler metric in the class $c_1(L)$. In [Hon99; Hon02; Hon08], Y-J. Hong proved that there exist cscK metrics on $\mathbb{P}E$ for the polarizations that make the fibres sufficiently small as we stressed in the previous chapter. More precisely Hong proves, among other things, that if E is simple and there is no non trivial holomorphic vector field on B, then there exists a cscK metric in the Kähler class defined by the polarization

$$\mathcal{L}_{r,m} = \mathcal{O}_{\mathbb{P}E}(r) \otimes \pi^* L^m$$

when m is very large. Such a result is quite natural, since one can expect for sufficiently small fibres that the geometry of the projectivization is governed by the one from the base. From the work of S.K. Donaldson, J. Stoppa, and T. Mabuchi, one had the algebro-geometric consequence that under the above assumptions the polarized manifold $(\mathbb{P}E, \mathcal{L}_{r,m})$ is K-stable for

$m >> 0$.

Now there are two natural ways that one can change the polarization $\mathcal{L}_{r,m}$. First one may be able to change the polarization L so as to make E unstable, in which case $(\mathbb{P}E, \mathcal{L}_{r,m})$ will be unstable for m large [RT06]. The second way is to fix L but consider the case of small m. In this chapter we shall show that there are examples of triples (B, L, E) that satisfy the assumptions of Hong's theorem, but such that for smaller values of m the polarizations $\mathcal{L}_{r,m}$ are unstable. This is Theorem 6.1.1. Such phenomena does not happen when the base is a curve. Furthermore, building on the same techniques, this chapter provides some new examples of asymptotically Hilbert or Chow semistable polarizations that are not asymptotically Hilbert or Chow stable, see Section 6.3, and extend some results of the previous chapter (Theorem 6.2.1).

Let us consider the setup of Section 5.4 of the previous chapter. We have a described in Section 5.4 a certain type of test configurations $\mathcal{T}$.

Notations 6.0.1. *Let us assume that the vector bundle E has rank $rk(E) = 2$ and B is a surface. We set*

$$\delta_L = \mu_L(E) - \mu_L(F)$$
$$\Delta = \frac{\mathrm{ch}_2(E)}{2} - \mathrm{ch}_2(F) + \frac{1}{2}\delta_{K_B^*}$$

so that one can write $\mathcal{P}_E(k) - \mathcal{P}_F(k) = k\delta_L + \Delta$.

In the following proposition, we give another formulation of Proposition 5.4.1 for expressing the Donaldson-Futaki invariant for the polarization $\mathcal{L}_{r,m}$ associated to the test configuration $\mathcal{T}$.

Proposition 6.0.1 ([Kel14b])**.** *The Donaldson-Futaki invariant of the test configuration $\mathcal{T}$ for a rank 2 vector bundle E over a polarized surface (B, L) induced by the deformation to the normal cone of $\mathbb{P}F$ where F is a subbundle of E is given by*

$$DF_1(\mathcal{T}) = \frac{r^6}{36}(\delta_{K_B^*})^2 - \frac{r^4}{72}\Gamma_1\delta_{K_B^*} + \frac{r^3}{24}\Gamma_2\left(m\delta_L + r\mathit{\Delta}\right),$$

with

$$\begin{aligned}\Gamma_1 =& r^2(c_1(E)^2 - 4c_1(F)^2) + 3c_1(F^r \otimes L^m)^2 + 4r^2\Delta + 12rm\delta_L \\ &- 3rc_1(B)c_1(F^r \otimes L^m), \\ \Gamma_2 =& (rc_1(E) + 2mc_1(L))^2 - 2rc_1(F^r \otimes L^m)c_1(B).\end{aligned}$$

Proof. It was proved in the previous chapter that

$$e_{4,3}(\mathcal{T}) = DF_1(\mathcal{T}) = C_1 r^3 m^3 + C_2 r^4 m^2 + C_3 r^5 m + C_4 r^6 \tag{6.1}$$

where

$$
\begin{aligned}
C_1(E,F) =& \frac{c_1(L)^2}{6}\left(\mu_L(E)-\mu_L(F)\right),\\
C_2(E,F) =& \frac{c_1(L)^2}{48}(c_1(E)-2c_1(F))c_1(B)\\
&+\frac{c_1(L)^2}{12}(\mathrm{ch}_2(E)-2\mathrm{ch}_2(F))\\
&+\frac{1}{12}\left(2c_1(E)c_1(L)-c_1(B)c_1(L)\right)\left(\mu_L(E)-\mu_L(F)\right),\\
C_3(E,F) =& -\frac{1}{12}\deg_L(E)c_1(F)^2+\frac{1}{12}\deg_L(E)\mathrm{ch}_2(E)\\
&+\frac{1}{48}\deg_L(E)c_1(E)^2-\frac{1}{24}\deg_L(F)c_1(E)^2\\
&+\frac{1}{24}c_1(L)c_1(B)\cdot c_1(F)^2-\frac{1}{24}c_1(L)c_1(B)\cdot\mathrm{ch}_2(E)\\
&+\frac{1}{24}\deg_L(F)c_1(E)c_1(B)-\frac{1}{24}\deg_L(E)c_1(B)c_1(F),\\
C_4(E,F) =& \frac{1}{288}c_1(E)^2\cdot c_1(B)c_1(E)-\frac{1}{144}c_1(E)^2\cdot c_1(B)c_1(F)\\
&+\frac{1}{48}c_1(F)^2\cdot c_1(E)c_1(B)-\frac{1}{72}\left(c_1(B)c_1(F)+c_1(E)c_1(B)\right)\mathrm{ch}_2(E)\\
&+\frac{1}{48}c_1(E)^2\left(\mathrm{ch}_2(E)-c_1(F)^2\right).
\end{aligned}
$$

By a simple algebraic manipulation one obtains from (6.1) the expected result. □

Remark 6.0.1. We remark that the Donaldson-Futaki invariant DF_1 we are computing does not depend on $c_2(B)$ (unlike the Chow weight associated to this test-configuration).

6.1 Examples of unstable projectivizations of stable bundles

6.1.1 An elementary example over an abelian surface

6.1.1.1 Construction

Let C_1, C_2 be two elliptic curves. Let P_i' the principal polarization on C_i, $i = 1,2$ of degree 1. We consider $B = C_1 \times C_2$ and $P_i = \pi_i^* L_i'$ for π_i the projection $B \to C_i$. Of course, it is well known that there exists a Ricci-flat Kähler metric in any Kähler class of B. We define $L_i = P_i^{\alpha_i} \otimes P_2^{\beta_i}$ for $i = 1,2$ and the vector bundle E of rank 2 by the exact sequence

$$0 \to L_1 \to E \to L_2 \to 0$$

Then

$$\begin{aligned} c_1(E) &= (\alpha_1 + \alpha_2)c_1(P_1) + (\beta_1 + \beta_2)c_1(P_2), \\ c_2(E) &= c_1(L_1)c_1(L_2) = \beta_1\alpha_2 + \alpha_1\beta_2, \\ ch_2(E) &= \alpha_1\beta_1 + \alpha_2\beta_2. \end{aligned}$$

The product of elliptic curves is an abelian variety and thus Todd$(B) = 1$. To see that the extension does not split it is sufficient to check that $Ext^1(\mathcal{O}_S, A) = H^1(S, A)$ is non trivial, where $A = L_1 \otimes L_2^*$. Note that $\chi(A) = c_1(A)^2/2 = \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_1\beta_2 - \beta_1\alpha_2 = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2)$.
From [Tak72, Lemma 3.8], if $\alpha_1 > \alpha_2$ and $\beta_2 > \beta_1$ and $gcd(\alpha_1 - \alpha_2, \beta_2 - \beta_1) = 1$ then the extension does not split and E is Mumford stable for a choice of a good polarization obtained using Bezout's theorem.
If $\alpha_i > 0$ and $\beta_i > 0$ for $i = 1, 2$, then E is ample from [Laz04, Proposition 6.1.13]. Therefore the line bundle $\mathcal{L}_{r,m}$ is ample for any $r, m > 0$.
In our computation of the Donaldson-Futaki invariant, it is important to pick up a subbundle (and not just a coherent subsheaf of E). We consider the subbundle $F = L_1$ and the deformation to the normal cone associated to $\mathbb{P}(F)$. We apply Proposition 6.0.1. Since $c_1(E)^2 > 0$, we see that the sign of $C_4(E, F)$ is determined by the sign of $\mathrm{ch}_2(E) - c_1(F)^2 = \alpha_2\beta_2 - \alpha_1\beta_1$. Therefore, in order to get a destabilizing test configuration, we are looking for integers α_i, β_i satisfying

$$\begin{cases} \alpha_1 > \alpha_2 > 0, \\ \beta_2 > \beta_1 > 0, \\ gcd(\alpha_1 - \alpha_2, \beta_2 - \beta_1) = 1, \\ \alpha_2\beta_2 - \alpha_1\beta_1 < 0. \end{cases}$$

One solution among others is given by $\alpha_1 = 3, \alpha_2 = 1, \beta_1 = 2, \beta_2 = 5$ which provides a polarization $\mathcal{L}_{r,m}$ for $r >> 0$ which has negative Donaldson-Futaki invariant and so is K-unstable, while E is Mumford stable and B is a Kähler Ricci-flat manifold. The only issue with this construction is that B has holomorphic vector fields, and we have not been able to apply Hong's results [Hon02, Corollaries A and B]. This drives us to the next step.

6.1.1.2 Blowing up to remove infinitesimal automorphisms

One can obtain examples in which B does not have holomorphic vector fields by blowing up at a sufficiently large number of well chosen points. To describe the details suppose we are in the general situation of having a rank 2 bundle E that admits a subbundle F with $C_4(E, F) < 0$.

Let us call d the dimension of the space of holomorphic vector fields on B. For a choice of d points p_i in a dense open subset of the product manifold B^d, the blow up at $p_1, ..., p_d$ has a trivial automorphisms group.

From results of [AP09], we know the existence of an integer $d_0 > d$ such that for all $d' \geq d_0$, there exists a non empty open subset $U_{d'} \in B^{d'}$ such that for all $(p_1, ..., p_{d'}) \in U_{d'}$, the blow up $\widetilde{B}$ at the points $(p_1, ..., p_{d'})$ admits a family of cscK metrics.

Let us denote $\sigma : \widetilde{B} \to B$ this blowup map. More precisely, the results of [AP09] are giving the existence of cscK metrics in all the ample classes $c_1(\widetilde{L}_\epsilon)$ where

$$\widetilde{L}_\epsilon := \sigma^* L - (\alpha_{1,\epsilon}[E_1] + ... + \alpha_{d',\epsilon}[E_{d'}])$$

with $\epsilon > 0$ small enough. Here E_i stand for exceptional divisors of the blow up at p_i. The numbers $\alpha_{i,\epsilon} > 0$, which are related together by a condition of G.I.T stability, can be chosen all rational and one can impose $\max_i \alpha_{i,\epsilon}$ as small as wished by fixing ϵ, cf. [AP09, Section 1.4].

Now the bundle

$$\tilde{E} := \sigma^* E$$

is also Mumford stable with respect to all rational polarizations

$$\sigma^* L - \left(\frac{1}{N_1}[E_1] + ... + \frac{1}{N_{d'}}[E_{d'}]\right)$$

for positive integers N_i sufficiently large, see [FM88, Theorem 5.5] and [Bru90, Theorem 4]. Therefore, up to choosing ϵ smaller enough, one can get both the vector bundle $\sigma^* E$ $\widetilde{L}_\epsilon$-Mumford stable and a cscK metric in the class $c_1(\widetilde{L}_\epsilon)$ over the smooth surface $\widetilde{B}$ that has no nontrivial holomorphic vector fields.

We need to check how this change of polarization has affected our computation of the Donaldson-Futaki invariant. Observe that as E is ample the bundle $\tilde{E}$ is nef [Laz04, Proposition 6.2.12] and so

$$\widetilde{\mathcal{L}}_{r,m} := \mathcal{O}_{\mathbb{P}(\tilde{E})}(r) \otimes \pi^* \widetilde{L}_\epsilon^m$$

is ample for $m > 0$, $r > 0$. Now from Proposition 6.0.1, we can compute as before the Donaldson-Futaki invariant for $(\tilde{E}, \sigma^* F, \widetilde{\mathcal{L}}_{r,m})$ by noticing that $c_1(\sigma^* E)$ and $\sigma^* F$ are trivial along each E_i and so

$$C_4(\tilde{E}, \sigma^* F) = C_4(E, F) < 0.$$

Thus, for large enough r we have that $(\mathbb{P}(\tilde{E}), \widetilde{\mathcal{L}}_{r,m})$ is K-unstable as well.

We sum up our work with the following theorem.

Theorem 6.1.1. *There exist a ruled threefold $P(\tilde{E})$ given as the projectivisation of a Mumford stable bundle $\tilde{E}$ over the blow up of a surface endowed with a cscK metric such that some Kähler classes of $P(\tilde{E})$ admit a cscK metric (and are K-stable) and some other classes do not admit a cscK metric (are are K-unstable).*

6.1.2 An elementary example over $\mathbb{CP}^2$

Consider the projective plane $B = \mathbb{P}^2$ endowed with the Fubini-Study metric which is a Kähler-Einstein metric. Let us fix $L = -K_{\mathbb{P}^2}$. From general results of the classification of Fano threefolds (see [SW08] and [Ish97, Section 4]) we know the existence of a L-Mumford stable rank 2 vector bundle E on $\mathbb{P}^2$ given by the exact sequence

$$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to E(1) \to 0$$

with $c_1(E) = 0$, $ch_2(E) = -2$ and $c_2(E) = 2$. The anticanonical bundle of $\mathbb{P}E$ is given by $\mathcal{L}_{r=2,m=1}$, and $\mathbb{P}E$ is actually a Fano threefold (note that it is unknown if it enjoys a Kähler-Einstein metric).

Non-trivial subsheaves of a rank 2 vector bundle on a manifold are associated to sections of twists of the bundle in question. Actually, we may assume that the subsheaf we wish to consider is reflexive so it is associated to an embedding of an invertible sheaf (or line bundle) $\mathcal{L} \to E$, which by tensorizing with $\mathcal{L}^{-1}$ yields an embedding $\mathcal{O} \to E \otimes \mathcal{L}^{-1}$, and hence a holomorphic section of $E \otimes \mathcal{L}^{-1}$. From Riemann-Roch, we get $h^0(E \otimes \mathcal{O}(1)) = 4$.

Therefore, we can consider $F = \mathcal{O}(-1)$ that has $\deg_L(F) = -3 < \deg_L(E)/2 = 0$. By a direct computation, we obtain for the Donaldson-Futaki invariant of the polarization $\mathcal{L}_{r,m}$,

$$DF_1(\mathcal{T}) = \frac{9}{2} m^3 r^3 - \frac{27}{8} m^2 r^4 + \frac{9}{8} m r^5 - \frac{1}{12} r^6$$

which is negative for large r. Note that however, for $r = 2, m = 1$, $DF_1(\mathcal{T}) > 0$, suggesting that the anticanonical polarization is K-stable.

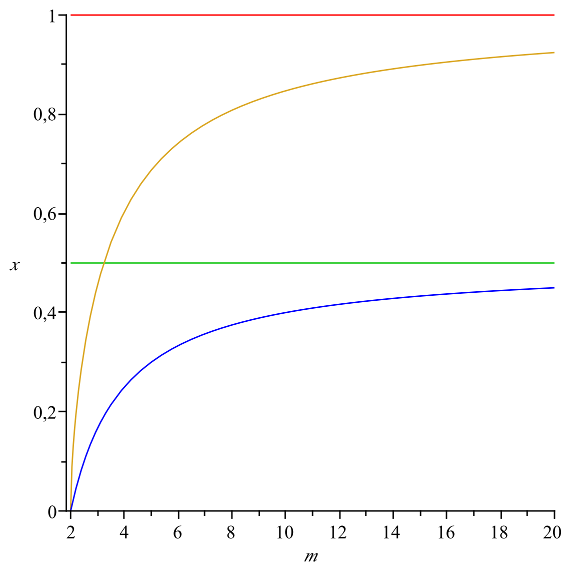
6.1.3 Kähler cone and K-stability

Consider the Fano surface $B = \mathbb{CP}^1 \times \mathbb{CP}^1$. We denote H_1, H_2 the standard basis of $Pic(B) \simeq \mathbb{Z}^2$ that satisfy $H_1^2 = H_2^2 = 0$ and $H_1 H_2 = 1$. Let us consider the general case of a non split extension

$$0 \to \mathcal{O}_B(\alpha H_1 + \beta H_2) \to E \to \mathcal{O}_B \to 0$$

where $\alpha > 0$ and $\beta < -1$. A polarization on $\mathbb{P}E$ has the form $\mathcal{L}_{r,m} = \mathcal{O}_{\mathbb{P}E}(r) \otimes \pi^* L^m$ with L an ample line bundle on the base B. One can write $L = \mathcal{O}_B(xH_1 + yH_2)$, $x > 0$, $y > 0$ and without loss of generality, we can assume that $r = 1$, $x + y = 1$ and x, y, m are both rational positive numbers. We are interested in describing in the Kähler cone of $\mathbb{P}E$ the polarizations that are K-stable.

From the extension, we get $c_1(E) = 2\alpha\beta = 2\mathrm{ch}_2(E)$, $\deg(E) = \alpha y + \beta x$. We consider the subbundle $F = \mathcal{O}_B(\alpha H_1 + \beta H_2)$ and the induced test-configuration $\mathcal{T}$ to the normal cone $\mathbb{P}(F)$. We expect this test configuration $\mathcal{T}$ is the maximal destabilizing test-configuration for $(\mathbb{P}E, \mathcal{L}_{1,m})$, i.e $DF_1(\mathcal{T}) \leq DF_1(\mathcal{T}')$ for any test-configuration $\mathcal{T}'$. The Donaldson-Futaki invariant $DF_1(\mathcal{T})$ is a real polynomial of order 3 in the variable x with leading coefficient $\frac{1}{6}m^3(\beta - \alpha) < 0$. The 3 roots r_1, r_2, r_3 of this polynomial expression depend on m, α, β and may not be all real. For the choice $\alpha = 2, \beta = -2$, the computations turn out to be pretty easy and the roots are $r_1 = \frac{m-2-\sqrt{m^2-2m}}{2m}$, $r_2 = \frac{m-2}{2m}$ and $r_3 = \frac{m-2+\sqrt{m^2-2m}}{2m}$. If $m \leq 2$, $DF_1(\mathcal{T}) \leq 0$ and $(\mathbb{P}E, \mathcal{L}_{1,m})$ cannot be K-stable. If $m > 2$ then $DF_1(\mathcal{T})$ is positive for $r_2 < x < r_3$. Note that this set contains the set of points for which E is L-Mumford stable ($\frac{1}{2} < x < 1$, $m >> 0$). In the plot below, we draw the curves of $r_2(m), r_3(m)$ and their asymptotics. Remark that for $m > 2$, $x = r_3$ we have $DF_1(\mathcal{T}) = 0$ which suggests the existence of a wall for K-stability but at an irrational class within the ample cone. Such phenomena for moduli spaces of vector bundles were discussed in [Sch00].



6.2 Chow weights and Hilbert weights of certain test configurations

We extend now the result of Proposition 6.0.1.

Proposition 6.2.1 ([Kel14b]). *In the same setting as in Proposition 6.0.1 and with Notations 6.0.1, the Chow weight associated to the test configura-*

tion $\mathcal{T}$ is given by

$$\begin{aligned}\mathrm{Chow}_s(\mathcal{T}) = e_4(s) =& \frac{sr^4\,(rs-1)\,(rs+1)}{36}{\delta_{K_B^*}}^2\\ &- \frac{sr^2\,(rs+1)}{72}A_1\delta_{K_B^*}\\ &+ \frac{sr^2\,(rs+1)}{24}A_2\,(m\delta_L + r\Delta)\end{aligned}$$

with

$$\begin{aligned}A_1 &= sr\Gamma_1 - A_1'\\ A_2 &= s\Gamma_2 - 4r\mathrm{Todd}_2(B).\end{aligned}$$

where we set $A_1' = \Gamma_1 + 3c_1(F^r \otimes L^m)^2 + 3rc_1(B)c_1(F^r\otimes L^m) + 6\mathrm{Todd}_2(B)$. *Moreover,*

$$\mathrm{Chow}_s = s^3 DF_1(\mathcal{T}) + s^2 DF_2(\mathcal{T}) + sDF_3(\mathcal{T})$$

with higher Futaki invariants $DF_2(\mathcal{T})$, $DF_3(\mathcal{T})$ *given by*

$$\begin{aligned}DF_2(\mathcal{T}) &= \left(\frac{1}{r}DF_1(\mathcal{T}) + rDF_3(\mathcal{T})\right),\\ DF_3(\mathcal{T}) &= -\frac{1}{36}r^4(\delta_{K_B^*})^2 + \frac{1}{72}r^2A_1'\delta_{K_B^*} - \frac{1}{6}r^3\mathrm{Todd}_2(B)\,(m\delta_L + r\Delta)\,,\end{aligned}$$

with $\mathrm{Todd}_2(B)$ *the second Todd class of* B.

Proof. Writing the weight of the action as $w(s) = \sum_{l=0}^{n+1} b_l s^{n+1-l}$ and $P(s) = \dim H^0(\mathbb{P}E, \mathcal{L}^s_{r,m}) = \sum_{l=0}^{n} a_l s^{n-l}$ with $n=3$ and s large enough (see Section 2.5), we get

$$e_4(s) = \sum_{l=1}^{3}(b_0a_l - a_0b_l)s^{4-l} - a_0b_4.$$

In the case we are considering, we have

$$\begin{aligned}a_0 =& \frac{1}{2}rm^2c_1(L)^2 + \frac{1}{2}mr^2\deg_L(E) + \frac{1}{6}r^3\mathrm{ch}_2(E) + \frac{1}{12}r^3c_1(E)^2,\\ a_1 =& \frac{r^2}{4}c_1(E)c_1(B) + \frac{m^2}{2}c_1(L)^2 + \frac{rm}{2}(c_1(L)c_1(B) + \deg_L(E)) + \frac{r^2}{2}\mathrm{ch}_2(E),\\ a_2 =& -\frac{r}{12}c_1(E)^2 + r\mathrm{Todd}_2(B) + \frac{r}{4}c_1(E)c_1(B) + \frac{m}{2}c_1(L)c_1(B) + \frac{r}{3}\mathrm{ch}_2(E),\\ a_3 =& \mathrm{Todd}_2(B),\end{aligned}$$

and

$$b_0 = \frac{r^4}{24}c_1(E)^2 + \frac{r^4}{12}c_1(F)^2 + \frac{m^2r^2}{4}c_1(L)^2 + \frac{mr^3}{6}(\deg_L(E) + \deg_L(F)),$$

$$\begin{aligned}
b_1 =& \frac{r^3}{4}c_1(F)^2 + \frac{r^3}{12}c_1(F)c_1(B) + \frac{r^3}{12}c_1(E)c_1(B) + \frac{rm^2}{4}c_1(L)^2 \\
&+ \frac{mr^2}{4}(c_1(L)c_1(B) + 2\deg_L(F)), \\
b_2 =& \frac{r^2}{2}\mathrm{Todd}_2(B) + \frac{r^2}{6}c_1(F)^2 - \frac{r^2}{24}c_1(E)^2 + \frac{r^2}{4}c_1(F)c_1(B) \\
&+ \frac{rm}{3}\deg_L(F) - \frac{rm}{6}\deg_L(E) + \frac{rm}{4}c_1(L)c_1(B), \\
b_3 =& \frac{r}{2}\mathrm{Todd}_2(B) + \frac{r}{6}c_1(F)c_1(B) - \frac{r}{12}c_1(E)c_1(B), \\
b_4 =& 0.
\end{aligned}$$

Note that the computation of the terms a_l, b_l is done using Hirzebruch-Riemann-Roch theorem.

□

We dress now some easy consequences of the Propositions 6.0.1 and 6.2.1. We get the following theorem which strengthens Proposition 5.4.2.

Theorem 6.2.1 ([Kel14b]). *Consider E an irreducible rank 2 holomorphic vector bundle on a polarized surface (B, L) with $c_1(B)$ proportional to $c_1(L)$.*

1. *Assume that E is strictly Gieseker semistable and F is a subbundle of E with $\mathcal{P}_F = \mathcal{P}_E$ with respect to L. Then all the tensor powers of the polarization $\mathcal{L}_{r,m}$ are not Chow polystable, $\mathcal{L}_{r,m}$ is not asymptotically Chow polystable and not K-polystable.*

2. *Assume that E is not Gieseker semistable and F is a destabilizing subbundle. Then $\mathcal{L}_{r,m}$ is not K-semistable and thus not asymptotically Chow semistable for $m \gg 0$.*

3. *If $\mathcal{L}_{r,m}$ is K-stable (resp. K-polystable, resp. K-semistable) for all $m \gg 0$ then E is Gieseker stable (resp. Gieseker polystable, resp. strictly Gieseker semistable) with respect to L.*

Proof. For (1), we consider the test configuration $\mathcal{T}$ of the deformation to the normal cone of $\mathbb{P}F$ described as before. From our assumption of Gieseker semistability we have $\delta_L = \Delta = 0$ while the assumption on the first Chern class gives $\delta_{K_B^*} = 0$ since $c_1(B) = 0$ or $c_1(B) = \lambda c_1(L)$. Therefore from Propositions 5.4.1 and 6.2.1, one has $DF_1(\mathcal{T}) = \mathrm{Chow}_s(\mathcal{T}) = 0$ while the test configuration $\mathcal{T}$ is not a product test configuration. The point (2) can be treated in a similar way using the proof of Proposition 5.4.1. Actually the destabilizing subbundle leads to $C_1 = 0$ and $C_2 < 0$ or $C_1 < 0$ and thus $DF_1(\mathcal{T}) < 0$. Remark that (2) strengthens a result of [RT06, Theorem 5.12] where it is shown that if E is not Mumford stable then $\mathcal{L}_{r,m}$ is not K-semistable.
Note that under the assumptions of (1) or (2), there is no Kähler metric with

constant scalar curvature in the class $c_1(\mathcal{L}_{r,m})$ as a consequence of [Mab08b; Sto09; Don01b].
Now let us assume that $\mathcal{L}_{r,m}$ is K-stable. Then $C_1 \geq 0$ in the proof of Proposition 5.4.1 for all subbundles F of E. If the inequality is strict for any subbundle then E is Mumford stable. Actually, for a rank 2 bundle over a surface, it is sufficient to test stability with respect to subbundles. For any rank 1 torsion free subsheaf $\mathcal{F}$ of E, $\mathcal{F}^{**}$ is a reflexive rank 1 sheaf on the surface B and thus a line bundle. Now, if $C_1 = 0$ for a subbundle F of E, one has necessarily $C_2 \geq 0$. If $C_2 > 0$ then $\mathcal{P}_E > \mathcal{P}_F$. Now given $\mathcal{F}$ rank 1 torsion free subsheaf of E, one has $\mathcal{F} = F \otimes \mathcal{I}$ where F is a line bundle and $\mathcal{I}$ is an ideal sheaf with 0-dimensional support, the inequality $\mathcal{P}_E > \mathcal{P}_F$ only improves if F is replaced by $\mathcal{F}$ since $c_2(\mathcal{F})$ is the length of the support of $\mathcal{I}$ and thus is non-negative. Eventually if the inequality $C_2 > 0$ holds for all subbundles of E, then we have obtained that E is Gieseker stable. Consider now that $C_2 = 0$. Then we have $\delta_L = \delta_{K_B^*} = \Delta = 0$ and by Proposition 5.4.1, $DF_1(\mathcal{T})$ vanishes. But the test configuration is not trivial so this leads to a contradiction. Therefore one has necessarily $C_2 > 0$ and we obtain Gieseker stability. The case of K-semistability is obtained by contraposition of (2).
In the case of K-polystability, the only case for which $C_2 = 0$ is when the rank 2 bundle E splits as a direct sum of two line bundles of same slope so is necessarily Mumford polystable. Since $C_3 \geq 0$, one has moreover Gieseker semistability. □

Remark that the case of K-unstability in (3) cannot be included since the base manifold B may be K-unstable which would induce a destabilizing test configuration for the projectivization $\mathbb{P}E$.

Non simple semi-homogeneous rank 2 vector bundles over an abelian surface are Gieseker semistable and thus provide concrete examples of applications of our theorem, see [Muk78, Section 6]. We formulate the following conjecture.

Conjecture 1. Consider E an irreducible rank 2 holomorphic vector bundle on a K-stable polarized surface (B, L) with $c_1(B)$ proportional to $c_1(L)$. For $m \gg 0$, the polarization $\mathcal{L}_{r,m}$ is K-stable (resp. K-polystable, resp. K-semistable) if and only E is Gieseker stable (resp. Gieseker polystable, resp. Gieseker semistable).

Our conjecture is wrong if one removes the assumption on the first Chern class of B: in the previous chapter it is constructed an example of a Gieseker stable bundle with $\mathcal{L}_{1,m}$ not K-semistable for $m \gg 0$. The hard sense of the conjecture is true under stronger assumption: on a surface with a constant scalar curvature Kähler metric and no non trivial holomorphic vector field, a Mumford stable bundle gives rise to a polarization $\mathcal{L}_{r,m}$ that admits a constant scalar curvature Kähler metric and thus is K-stable, see [Hon99; Hon02; Hon08].

One can now wonder when the Donaldson-Futaki invariant as computed in Proposition 5.4.1 may vanish. We cannot say much for a fixed couple (r, m) but at the fiber or base limit we obtain the following result.

Proposition 6.2.2 ([Kel14b]). *Let (B, L) be a polarized surface such that its first Chern class satisfies $c_1(B) = 0$ or $c_1(B)c_1(L) \neq 0$ and E a rank 2 holomorphic vector bundle on B. Then, for the test configuration as in Proposition 5.4.1,*

- *the Donaldson-Futaki invariant $DF_1(\mathcal{T})$ vanishes for all $m \gg 0$ (or all $r \gg 0$) if and only if the Chow weight $\mathrm{Chow}_s(\mathcal{T})$ vanishes for all $m \gg 0$ and any fixed $s > 0$ (or all $r \gg 0$ and $s \gg 0$).*
- *the Donaldson-Futaki invariant $DF_1(\mathcal{T})$ is positive for all $m \gg 0$ if and only if the Chow weight $\mathrm{Chow}_s(\mathcal{T})$ is positive for all $m \gg 0$ and $s \gg 0$.*

Proof. This comes from the computations of the Donaldson-Futaki invariant and Chow weight. Imposing $C_1 = C_2 = C_3 = 0$ in Proposition 5.4.1 implies firstly that $\delta_L = 0$, then $\Delta = \frac{1}{4}\delta_{K_B^*}$ and finally $\delta_{K_B^*}c_1(L)c_1(B) = 0$. Under our assumptions one gets in all the cases

$$\delta_L = \delta_{K_B^*} = \Delta = 0. \tag{6.2}$$

This forces obviously the Chow weight to vanish, see Proposition 6.2.1. Conversely, if the Chow weight vanishes seen as a polynomial in the variables m, one gets from Proposition 6.2.1 that $\Delta = \frac{kr-2}{4kr}\delta_{K_B^*}$ and $\delta_{K_B^*}c_1(L)c_1(B) = 0$ and thus (6.2) holds which implies the vanishing of the Donaldson-Futaki invariant. Computations in the variables r are similar but slightly more involved. The second part of the result is using the same reasoning. □

Next we compute the Hilbert weight for the test configuration $\mathcal{T}$ for the deformation to the normal cone of $\mathbb{P}F$ where F is a subbundle of E. We remark that the Hilbert weight has a similar expression to the Chow weight and the Donaldson-Futaki invariant.

Proposition 6.2.3 ([Kel14b]). *In the same setting as in Proposition 5.4.1 and with Notations 6.0.1, the Hilbert weight associated to the test configuration $\mathcal{T}$ is given by*

$$\begin{aligned}\mathrm{Hilb}_{s,k}(\mathcal{T}) =& \frac{r(rs-1)(rk+1)}{36}\beta_1(s,r)\delta^2_{K_B^*} \\ &+ \frac{1}{72}(\beta_1(s,r)B_1 - \beta_2(s,r)A_1)\delta_{K_B^*} \\ &+ \left(\frac{\beta_2(s,r)}{24}A_2 - \frac{(rk+2)\beta_1(s,r)}{6}\mathrm{Todd}_2(B)\right)(m\delta_L + r\Delta)\end{aligned}$$

with $\beta_1(s,r) = rks(rs+1)(k-s)(rk+1)$, $\beta_2(s,r) = rs^3(rs+1)^2(k-s)$, *and*

$$\begin{aligned} B_1 =& kr^2(c_1(E)^2 + 2c_1(F)^2 + 4\Delta + 6\mathrm{Todd}_2(B)) \\ &+ 6krm\deg_L(E) + 6km^2c_1(L)^2 \\ &+ r(-c_1(E)^2 + 6\mathrm{Todd}_2(B) + 8\Delta + 6c_1(F)c_1(B) + 4c_1(F)^2) \\ &+ 6mc_1(L)c_1(B) \end{aligned}$$

Proof. The result is obtained by a computation of the weight $\mathrm{Hilb}_{s,k}(\mathcal{T}) = \tilde{w}(s,k)$ using (2.4) and the computations of a_i, b_i in Proposition 6.2.1. □

Proposition 6.2.2 can also be extended to Hilbert weights. We have also another obvious consequence.

Proposition 6.2.4 ([Kel14b])**.** *In the same setting as in Proposition 5.4.1, let us assume that* $c_1(B) = 0$*. Then the Chow weight* $\mathrm{Chow}_s(\mathcal{T})$ *and the Hilbert weight* $\mathrm{Hilb}_{s,k}$ *are proportional to the Donaldson-Futaki invariant* $DF_1(\mathcal{T})$*, and have same sign when one takes* $k, s > 0$ *large enough.*

Proof. This comes from the fact that when $c_1(B) = 0$ one has $\delta_{K_B^*} = 0$ and both quantities Γ_2 and A_2 do not depend on the bundle F. □

6.3 Strictly asymptotically semistable threefolds

Inspired from [BD88], we construct a new example of a threefold which is Asymptotically Chow semistable and not Asymptotically Chow stable.

Let (B, L) be a polarized surface such that $c_1(L)$ admits a Kähler metric with constant scalar curvature and $Aut(B, L)/\mathbb{C}^\times$ is trivial and assume that the torus $\mathrm{Pic}^0(B) = H^1(B, \mathcal{O})/H^1(B, \mathbb{Z})$ parametrizing line bundles with trivial first Chern class is not trivial. Consider $E_0 = G_1 \oplus G_2$ a direct sum of two line bundles with $c_1(G_1) = c_1(G_2)$ over B. Then E_0 is Mumford polystable. On the polarized ruled manifold

$$(X_0, \mathcal{L}^0_{r,m}) = (\mathbb{P}E_0, \mathcal{O}_{\mathbb{P}E_0}(r) \otimes \pi_0^* L^m)$$

there exists under our assumptions a Kähler metric with constant scalar curvature for all $m \gg 0$. Actually, the Futaki character associated to the Lie algebra $Lie(Aut(E_0)/\mathbb{C}^\times)$ vanishes thanks to Proposition 5.4.1, and one can apply [Hon02, Corollary B]. Therefore, $(X_0, \mathcal{L}^0_{r,m})$ is K-polystable for all $m \gg 0$ from the work of Donaldson, Stoppa and Mabuchi [Mab08b; Sto09; Don01b].

Next, we do a small deformation of the trivial line bundle $T_0 = \mathbb{C} \times B$ in order to obtain a line bundle T over B such that T^2 is non trivial. We can consider the following induced extension

$$0 \to G_1 \otimes T \to E \to G_2 \otimes T^* \to 0. \tag{6.3}$$

Using Riemann-Roch formula we have $h^0(B, G_1 \otimes G_2^* \otimes T^2) - h^1(B, G_1 \otimes G_2^* \otimes T^2) + h^2(B, G_1 \otimes G_2^* \otimes T^2) = \mathrm{Todd}_2(B)$ since $c_1(G_1) = c_1(G_2)$. Now, if we assume $\mathrm{Todd}_2(B) < 0$, the space $Ext^1(G_2 \otimes T^*, G_1 \otimes T) = H^1(B, G_1 \otimes G_2^* \otimes T^2)$ has positive dimension and our extension (6.3) does not split. The ruled manifold

$$(X, \mathcal{L}_{r,m}) = (\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(r) \otimes \pi^* L^m)$$

is not K-polystable for $m \gg 0$. Actually for the choice $F = G_1 \otimes T$ one checks that the Donaldson-Futaki invariant $DF_1(\mathcal{T})$ associated to the test configuration to the normal cone of $\mathbb{P}F$ vanishes for $m \gg 0$. Furthermore one obtains $\delta_L = \delta_{K_B^*} = \Delta = 0$. These relationships impose that the Chow weight Chow_s vanishes by Proposition 6.2.1. Therefore, $(X, \mathcal{L}_{r,m})$ cannot be asymptotically Chow stable.
On another hand, from the fact that the higher Futaki invariants $DF_2(\mathcal{T})$, $DF_3(\mathcal{T})$ vanish simultaneously we can apply Mabuchi's result in [Mab05] (see also [DZ12, Proposition 3.2, Theorem 3.5]). Then, one concludes that $(X_0, \mathcal{L}^0_{r,m})$ is asymptotically Chow polystable. By openness of the semistability condition in G.I.T, its small deformations are asymptotically Chow semistable and consequently $(X, \mathcal{L}_{r,m})$ is asymptotically Chow semistable.
Finally, in order to construct base manifolds that satisfy the assumptions as above, it is sufficient to consider for B a ruled surface as the projectivization of a rank 2 Mumford stable bundle over a curve of genus > 1, as in the previous chapter. We have proved the following result.

Corollary 6.3.1 ([Kel14b])**.** *There are some ruled threefolds (projectivization of rank 2 bundles over a surface endowed with a constant scalar curvature Kähler metric) that are asymptotically Chow semistable, but not asymptotically Chow stable.*

One can also compare Corollary 6.3.1 with [Wan04, Section 5] where other examples of non asymptotically Chow stable threefolds are discussed.

Since $(X_0, \mathcal{L}^0_{r,m})$ is asymptotically Chow polystable, for the test configurations that have positive Chow weight asymptotically, the main result of [Mab08a] shows that they have also positive Hilbert weight asymptotically. Thanks to our assumptions on B, the product test configurations that have vanishing Chow weight Chow_s for $s \gg 0$ are associated to the splitting of E_0 and the deformation to the normal cone of $\mathbb{P}G_1$ or $\mathbb{P}G_2$. Thus one gets in both case for $m \gg 0$ that $\delta_L = \Delta = \delta_{K_B^*} = 0$. Proposition 6.2.3 shows that the Hilbert weight also vanishes. Consequently, $(X_0, \mathcal{L}^0_{r,m})$ is asymptotically Hilbert polystable and thus its small deformation $(X, \mathcal{L}_{r,m})$ is also asymptotically Hilbert semistable. On another hand, considering the subbundle $F = G_1 \otimes T$ of E, one has for the test configuration associated to the deformation to the normal cone of $\mathbb{P}F$ that $\delta_L = \delta_{K_B^*} = \Delta = 0$ and so $\mathrm{Hilb}_{s,k} = 0$ for all s, k. Finally, $(X, \mathcal{L}_{r,m})$ for $m \gg 0$ cannot be asymptotically Hilbert stable since $\mathcal{T}$ is not a product test configuration.

Corollary 6.3.2 ([Kel14b]). *There are some ruled threefolds (projectivization of rank 2 bundles over a surface endowed with a constant scalar curvature Kähler metric) that are asymptotically Hilbert semistable, but not asymptotically Hilbert stable.*

We introduce the following conjecture.

Conjecture 2. Consider E a holomorphic vector bundle on a base manifold B polarized by L with $c_1(B) = 0$ or $c_1(B)c_1(L) \neq 0$. Then for $m \gg 0$, the following assertions are equivalent:

- the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is asymptotically Hilbert semistable,
- the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is asymptotically Chow semistable,
- the polarization $\mathcal{L}_{r,m}$ on $\mathbb{P}E$ is K-semistable.

Note that our conjecture is true if the base manifold is a curve of genus $g > 1$.

Chapter 7

Canonical metrics over projectivization of unstable bundles

In this chapter we investigate the existence of Kähler metrics with special curvature properties on ruled manifolds given as projectivization of bundles which are not Mumford stable bundles. Contrarily to Chapters 5 and 6 where we were mainly interested in algebro-geometric properties, we wish now to make appear some canonical metrics on the considered ruled manifolds. The problem is up to our knowledge very open in complete generality. In this chapter, we essentially study the particular case of a ruled surface given by the projectivization of a Mumford semistable rank 2 bundle (which is not stable) over a Riemann surface of genus $g \geq 2$. Some partial generalizations are also given for higher dimensional base. The main results of this chapter are Theorems 7.2.2, 7.3.1, 7.3.4.

7.1 Almost constant scalar curvature metric and K-semistability

7.1.1 Definition of an almost cscK metric

Let (X, L) be a polarized manifold. Let us denote $\hat{s}_L$ the average of the scalar curvature in the class $c_1(L)$, which is a topological invariant. For ω a Kähler metric in $c_1(L)$, we denote $\mathrm{scal}(\omega)$ its scalar curvature. From the main result of [Don05a], we know that if there exists a sequence of metric ω_ϵ such that

$$\|\mathrm{scal}(\omega_\epsilon) - \hat{s}_L\|_{L^2} \to 0,$$

then the manifold is K-semistable. The converse is an open question as far as we know.

Definition 7.1.1 ([Kel14a]). Given (X, L) a projective manifold, we say that there exists an almost cscK metric in the class $c_1(L)$ in C^r topology ($r \in \mathbb{N}$) if there is a family of Kähler metrics $\omega_\epsilon \in c_1(L)$ such $\|\mathrm{scal}(\omega_\epsilon) - \hat{s}_L\|_{\mathrm{C}^r} \to 0$ when $\epsilon \to 0$.

In the case of the anticanonical class, this definition appeared first in [Ban87] where it is related to the existence of a lower bound for the Mabuchi K-energy. Obviously, from Donaldson's result, a manifold (X, L) endowed with an almost cscK metric is K-semistable.

7.1.2 Construction of almost cscK metric

Let E be an irreducible Mumford semistable vector bundle of rank 2 over a polarized manifold (B, L_B), given by a non-split exact sequence of line bundles

$$0 \to L_1 \to E \to L_2 \to 0,$$

with $c_1(L_1) = c_1(L_2)$. Let us assume that h_1, h_2 are projectively flat metrics on L_1, L_2 satisfying $F_{h_1} = \deg_L(L_1)\omega = F_{h_2}$ with ω a cscK metric in the class $c_1(L_B)$. Consider the holomorphic structure on E that has the following form

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{L_1} & \alpha \\ 0 & \bar{\partial}_{L_2} \end{pmatrix}$$

where α is a smooth section of $\Omega^{0,1}(Hom(L_1, L_2))$, see [Kob87, Chapter I, Section 6]. Then one has for the curvature of E, and denoting $\mu(E)$ the slope of E,

$$\begin{aligned} \|F_E - \mu(E)\mathrm{Id}_E\omega\|_{\mathrm{C}^r} \leq & \|F_{h_1} - \deg(L_1)\omega\|_{\mathrm{C}^r} + \|F_{h_2} - \deg(L_2)\omega\|_{\mathrm{C}^r} \\ & + 2\|\alpha\|^2_{\mathrm{C}^r} + 2\|\bar{\partial}^*\alpha\|^2_{\mathrm{C}^r}. \end{aligned}$$

We can do a gauge change of the form $g = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and we obtain

$$g(\bar{\partial}_E) = \begin{pmatrix} \bar{\partial}_{L_1} & \xi^{-2}\alpha \\ 0 & \bar{\partial}_{L_2} \end{pmatrix}.$$

For any any $\epsilon > 0$ and for any $r > 0$, we can find the gauge transformation ξ such that

$$2\xi^{-2}(\|\alpha\|^2_{\mathrm{C}^r} + \|\bar{\partial}^*\alpha\|^2_{\mathrm{C}^r}) < \epsilon.$$

This provides a structure h_E (depending on the parameters ϵ, r) such that

$$\|F_{E,h_E} - \mu(E)\mathrm{Id}_E\omega\|_{\mathrm{C}^r} < \epsilon.$$

Note that fixing the holomorphic structure with variation of the metric, or fixing the metric with variation of the holomorphic structure is geomet-

rically equivalent in this setup. Therefore, we have obtained an approximate Hermitian-Einstein structure in the sense of [Kob87, Chapter IV], from which we deduce an almost cscK metric on $\mathbb{P}(E)$ using the next lemma.
From now we assume that E is ample, that is $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a positive line bundle (without loss of generality we can tensorize E by a sufficiently ample line bundle L_C, use the identification $\mathbb{P}(E \otimes L_C) \simeq \mathbb{P}(E)$ and the induced approximate Hermitian structure).

Lemma 7.1.1. *From h_E hermitian metric on the bundle E, one can define a metric $\hat{h}_E$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ which curvature is denoted $\hat{\omega}_E$ and is a Kähler form. Then at $v \in \mathbb{P}(E)$, with $\pi(v) = x \in C$, one has pointwise*

$$\hat{\omega}_E = \pi^* \left(\frac{\sqrt{-1}}{\|v\|^2_{h_E}} \langle F_{E,h_E}(v), v \rangle_{h_E} \right) + \omega_{FS|\mathbb{P}(E)_x}$$

and $\omega_{FS|\mathbb{P}(E)_x}$ is the Fubini-Study metric at $\mathbb{P}(E)_x$.

We refer to [Dem97, Chapter V, §15.C] for a proof. A direct consequence of the previous lemma and the existence of an approximate Hermitian-Einstein structure with respect to a cscK metric is the following proposition.

Proposition 7.1.1 ([Kel14a])**.** *Let $E \to B$ be an ample Mumford semistable rank 2 vector bundle induced by a non-split exact sequence of projectively flat line bundles as above over a cscK polarized manifold (B, L). Then for any $r \in \mathbb{N}$, there is an almost cscK metric on the ruled surface $\pi\colon \mathbb{P}(E) \to B$ in C^r topology.*

If the base manifold is a curve of genus $g > 1$, line bundles are automatically projectively flat, the exact sequence does not split for L_1 not isomorphic to L_2 since $h^1(C, L_2 \otimes L_1^*) = g - 1 > 0$ and there exists a cscK metric on the base manifold. Thus we obtain the next result.

Corollary 7.1.1 ([Kel14a])**.** *Consider E a rank 2 vector bundle on a curve C of genus ≥ 2. Assume that E is Mumford semistable. Then for any $r \in \mathbb{N}$, there is an almost cscK metric on the ruled surface $\pi\colon \mathbb{P}(E) \to C$ in C^r topology.*

Note that if E is not irreducible, then it is actually a direct sum of line bundles and thus we know the existence of a genuine cscK metric on the projectivization, see for instance [AT06].

7.1.3 Computation of the Donaldson-Futaki invariant

Proposition 7.1.2 ([Kel14a])**.** *Consider E an ample irreducible Mumford semistable vector bundle which is not stable over a curve of genus $g > 1$. Then $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ is not K-polystable and not asymptotically Chow polystable.*

Proof. This is a consequence of [RT06, Theorem 5.13], where it is done a computation of the Donaldson-Futaki invariant for the test configuration induced by a deformation to the normal cone of $\mathbb{P}(F)$ where F is any subbundle of E. This computation shows that the Donaldson-Futaki invariant for such a test-configuration is a multiple of the differences of slopes $\mu(E) - \mu(F)$. Remark that with [DZ12, Proposition 4.1, Theorem 4.5], it is also proved that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not asymptotically Chow polystable. □

Proposition 7.1.3 ([Kel14a]). *Assume that $E \to B$ an ample Mumford semistable rank 2 vector bundle induced by a non-split exact sequence of projectively flat line bundles over a cscK polarized manifold (B, L), as constructed in Section 7.1.2. Then $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ is not K-polystable and not asymptotically Chow polystable.*

Proof. Let us denote $b = \dim_{\mathbb{C}} B$. We compute the Donaldson-Futaki invariant DF_1 for the test configuration induced by a deformation to the normal cone of $\mathbb{P}(L_1)$. Note that $\mu(L_1) = \mu(E)$. As explained in [RT06] (see also [KelRos12]), $DF_1 = a_1 b_0 - a_0 b_1$ where one has defined

$$p(r) = h^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(r)) = a_0 r^{b+1} + a_1 r^b + ...$$
$$w(r) = \sum_{i=0}^{r} i h^0(B, L_1^i \otimes L_2^{r-i}) = b_0 r^{b+2} + b_1 r^{b+1} + ...$$

Now, under our assumptions, using the fact that $c_1(E) = 2c_1(L_1)$, the polynomials p and w are proportionals. This shows that DF_1 vanishes and a similar reasoning can be done to get that the Chow weight associated to this test configuration also vanishes. □

Corollary 7.1.2 ([Kel14a]). *There exist examples (X, L) of polarized manifolds such that L is K-semistable and not K-polystable. In particular, there are examples of non convergent sequence of almost cscK metrics. For any irreducible Mumford semistable bundle E (not Mumford stable) of rank 2 over a curve of genus $g \geq 2$, there are integral classes on the ruled surface $X = \mathbb{P}(E)$ that are K-semistable and not K-polystable.*

Proof. We fix a bundle as in the statement and apply Corollary 7.1.1 to produce a sequence of an almost cscK metric. The existence of such metric implies in turn that $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)))$ is K-semistable from [Don05a]. On another hand, the automorphism group $Aut(\mathbb{P}(E))$ is actually trivial, see [Sey10]. Therefore if the sequence of almost cscK metric was convergent, it would converge towards a cscK metric and (X, L) would be K-stable by [Don01b; Sto09]. This would contradict Proposition 7.1.2. □

Corollary 7.1.2 gives a positive answer to a conjecture of J. Stoppa [Sto08]. Our construction can be easily modified to produce examples of

K-semistable not K-stable manifolds in any dimension using vector bundles of higher rank over a curve (using an induction argument on their Harder-Narasimhan filtration, see [Kel05a]) or using Proposition 7.1.3. Note that Fano examples of K-semistable but not K-polystable threefolds have been found by G. Tian by considering small deformations of the Mukai-Umemura threefold.

7.2 Almost balanced metric and Asymptotic Chow semistability

7.2.1 Definition of almost balanced metric

Let us recall (cf Section 2.4) that one can define for X a submanifold of $\mathbb{P}^N$ the center of mass of X as

$$\mu(X) = \int_X \frac{zz^*}{|z|^2} d\mu_{FS} - \frac{\mathrm{Vol}(X)}{N+1}\mathrm{Id} \in \sqrt{-1}Lie(SU(N+1))$$

considering $\mathbb{P}^N$ as a co-adjoint orbit in the Lie algebra of $SU(N+1)$. The Chow weight of X with respect to A, hermitian matrix, is

$$FCh(A,X) = \mathrm{tr}(\mu(X)\cdot A) = \int_X \frac{z^*Az}{|z|^2} d\mu_{FS} - \frac{\mathrm{Vol}(X)}{N+1}\mathrm{tr}(A)$$

and the definition can be extended to any algebraic cycles. It is a classical fact, based on Kempf-Ness theory, that $FCh(A, e^{tA}\cdot X)$ is an increasing function of t and we refer [Don05a, Proposition 5] for details. In particular for $\overline{X}$ the limiting Chow cycle of $e^{tA}\cdot X$ as $t\to -\infty$, we get

$$FCh(A,X) \geq FCh(A,\overline{X}).$$

This provides the inequality

$$\|\mu(V)\|_2 \, \|A\|_2 \geq -FCh(A,\overline{X}) \tag{7.1}$$

where one has defined the norm $\|T\|_2^2 = \sum |\lambda_i|^2$ for λ_i eigenvalues of the hermitian matrix T, taking into account their multiplicities. This is the finite dimensional analogue of the main theorem of [Don05a] that we used previously. Let us now introduce a notion of almost-balanced metrics.

Definition 7.2.1 ([Kel14a])**.** Given (X,L) a projective manifold, we say that there exists a sequence of almost balanced metrics if for all $k >> 0$ and all $\epsilon > 0$, there exists a hermitian metric $h_{k,\epsilon}$ on L^k such that the Bergman function satisfies

$$\left\| \rho(h_{k,\epsilon}) - \frac{N_k+1}{\mathrm{Vol}_L(X)} \right\|_{C^0} \leq \epsilon.$$

Note that

$$\rho(h_{k,\epsilon}) = \sum_{i=1}^{N_k+1} |s_i|^2_{h_{k,\epsilon}} \in C^\infty(X, \mathbb{R}_+)$$

for $N_k + 1 = h^0(L^k)$ and $\{s_i\}_{i=1,..,N_k+1}$ an orthonormal basis of $H^0(L^k)$ with respect to the L^2-inner product induced by $h_{k,\epsilon}$. The existence of an almost balanced metric for (X, L) implies, using (7.1) and Hilbert-Mumford criterion, that (X, L) is asymptotically Chow stable since the Chow weight of the limiting Chow cycle along a test-configuration cannot be strictly negative (this appears also clearly in [Don05a, Equation (16)] where the lower order terms are the higher Chow weights associated to the one-parameter subgroup action).

7.2.2 Construction of almost balanced metric

Let us consider $\hat{h}_\epsilon$ a hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with the same notations as in the previous section. Then the Bergman function for $(\mathcal{O}_{\mathbb{P}(E)}(k), \hat{h}_\epsilon^k)$ has an asymptotic expansion

$$\rho(\hat{h}_\epsilon^k) = k^r + k^{r-1}\frac{\mathrm{scal}(\omega_\epsilon)}{2} + k^{r-2}a_2 + ... + k^{r-q}a_q \tag{7.2}$$

where r is the rank of the bundle $E \to C$ and $\omega_\epsilon = c_1(\hat{h}_\epsilon)$. The writing of (7.2) means the following inequality holds in C^0-topology (it will be sufficient to work in that topology in the sequel)

$$\left\|\rho(\hat{h}_\epsilon^k) - \left(k^r + k^{r-1}\frac{\mathrm{scal}(\omega_\epsilon)}{2} + k^{r-2}a_2 + ... + k^{r-q}a_q\right)\right\|_{\mathrm{C}^0} \leq C_q(\hat{h}_\epsilon)k^{n-q-1}. \tag{7.3}$$

The terms a_i involve at most the $(2i-2)$-th first covariant derivatives of the curvature ω_ϵ.

Lemma 7.2.1. *If the metric ω_ϵ is bounded from below and bounded in C^{2q} norm by a constant δ with some reference metric, then the constant $C_q(\hat{h}_\epsilon)$ in Equation (7.3) depends actually only on q and the constant δ.*

This is well known, see for instance [Don01b, Proposition 6].

We are now coming back to the setup of Section 7.1.2 and shall construct a sequence of almost-balanced metric. Since we work in the smooth category, it is not difficult to adapt the reasoning in order to obtain an approximate Hermitian-Einstein structure h_ϵ^∞ in the following sense

$$\|F_{E,h_\epsilon^\infty} - \mu(E)\mathrm{Id}_E\,\omega\|_{\mathrm{C}^\infty} < \epsilon. \tag{7.4}$$

Furthermore we can assume that h_E is real analytic. If h_E is not real analytic we may use a slight generalization of Tian's result [Tia90] of approximation

of a positive hermitian metric by a sequence of Bergman type metrics in smooth topology (for a discussion on the smooth convergence see [Rua98]). Actually, we can pull-back the canonical metric on the universal bundle $U_{(2)}$ over the Grassmannian $Gr(2, H^0(B, E \otimes L^s))$ for $s >> 1$, which provides a sequence of real analytic metrics. This sequence is convergent towards the metric h_E in smooth topology thanks to the asymptotic result for the Bergman kernel of $E \otimes L^s$ that can be found in [Wan05].
On $X = \mathbb{P}(E)$, the curvature ω_ϵ^∞ of the associated real analytic metric $\hat{h}_\epsilon^\infty$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is bounded in C^∞ norm and is positive. This can be seen by expressing the curvature of $\hat{h}_\epsilon^\infty$ in terms of the curvature terms of h_ϵ^∞, see Lemma 7.1.1. We can apply Lemma 7.2.1 and for $0 < \epsilon < \epsilon_0$ we get a uniform expansion of the Bergman function of $\hat{h}_\epsilon^\infty$ with constant depending only on the maximum order of the expansion and ϵ_0, and hence we denote the constant in Equation (7.3) by $C_q(\epsilon_0)$.

Lemma 7.2.2. *In the above setting, there is a constant $C_\infty > 0$ depending only on ϵ_0 such that $C_q(\epsilon_0) \leq C_\infty^q$. In other words, the growth of the error constant in* (7.3) *when taking higher order expansion is at most exponential.*

Proof. This is a consequence of the techniques used in the proof of [LL11, Theorem 1.3] (see also Theorem 1.2 of the associated announcement paper). We shall use the notations of the quoted paper. The Bergman function is given by the sum of an orthonormal basis of sections that can be taken as the union of a peaked section and vanishing sections at x that have been orthonormalized. In order to orthonomalize these sections, it is necessary to inverse a matrix formed of the inner products $\langle S, S\rangle_{L^2}$ and $\langle S, T\rangle_{L^2}$ as it is done in [LL11, Section 5]. Since we deal with analytic metrics, we can do the Taylor expansion of the involved metrics (on the line bundle and the volume form) and we get an expansion of the L^2 norm $\langle S, S\rangle_{L^2}$ of the peak section S of the form

$$\langle S, S\rangle_{L^2} = \frac{1}{k^n}(1 + \sum_{i\geq 1} \beta_i k^{-i}).$$

By convergence of this Taylor expansion, one has for a certain uniform constant $C > 0$ that depends on the metric, $|\beta_i| \leq C^i$ for $i \geq 1$. In [LL11, Theorem 4.1 (3)], it is proved a uniform bound on the L^2-inner product $\langle S, T\rangle_{L^2}$ between a holomorphic section S peaked at the point x and sections T that vanish at order $p' > 0$ at x. Together these two uniform controls provide the expected growth on the error term C_q of the expansion of the Bergman function. In particular we have shown that in (7.3), one has the control

$$|a_i| < \frac{c_0}{\gamma^i}$$

for a uniform constant $\gamma > 0$. ∎

Furthermore, the terms a_i enjoy the property of being polynomial expressions in the curvature and its covariant derivatives. For any integer q, we can find a metric $\hat{h}^{\infty}_{\epsilon,q}$ such that the term

$$\left(k^r + k^{r-1}\frac{\mathrm{scal}(\omega_\epsilon)}{2} + k^{r-2}a_2 + ... + k^{r-q}a_q\right),$$

which satisfies a similar property, is also constant up to an error term of the form $\epsilon/2$ in C^0 norm while we are are still under the assumptions of Lemma 7.2.1. This is a consequence of the uniform control in C^∞ topology of the curvature of ω^∞_ϵ using (7.4).
Furthermore, using Lemma 7.2.2 and taking $k > C_\infty$, we can impose

$$k^{r-q-1}C_q(\epsilon_0) \leq k^{r-1}\left(\frac{C'_\infty}{k}\right)^q < \epsilon/2$$

in Equation (7.3) by choosing q large enough. Hence, for $k >> 0$, we have obtained a metric $\hat{h}_{k,\epsilon} = (\hat{h}^\infty_{\epsilon,q})^k$ which is almost balanced in the sense of Definition 7.2.1.
Eventually, with the examples considered in Section 7.1.2 and Proposition 7.1.2, we have proved the following result.

Theorem 7.2.2 ([Kel14a]). *There exist examples (X, L) of smooth polarized manifolds such that L is asymptotically Chow semistable and not K-polystable. For instance, for any irreducible Mumford semistable bundle E (not Mumford stable) of rank 2 over a curve of genus $g \geq 2$, any Kähler integral class on the ruled surface $X = \mathbb{P}(E)$ is asymptotically Chow semistable, not asymptotically Chow polystable and not K-polystable.*

Note that Chow semistability implies K-semistability, see [Tho06]. Therefore Theorem 7.2.2 implies straightforward Corollary 7.1.2. We chose to present both results in order to stress how one could expect to find examples of K-semistable but non Chow semistable manifolds. Actually, one could try to find a sequence of almost cscK metrics in C^0 norm such that some high order covariant derivatives of the Riemann curvature tensor are unbounded.

7.3 Parabolic structures and conic Kähler metrics

7.3.1 Construction of stable parabolic structure

Given a vector bundle E over a curve, we provide an elementary construction that allows to construct a stable parabolic structure on E.

7.3.1.1 From an unstable to a semistable parabolic structure

We suppose that E is a holomorphic vector bundle that has rank 2 and is not semistable. Then we have an exact sequence

$$0 \longrightarrow F_1 \longrightarrow E \longrightarrow F_2 \longrightarrow 0$$

such that
(i) F_i are line bundles,
(ii) $\deg(F_1) > \mu(E) = \frac{\deg(F_1)+\deg(F_2)}{2} > \deg(F_2)$.

We set
$$A := \deg(F_1) - \mu(E) = \mu(E) - \deg(F_2) > 0. \tag{7.5}$$

We take a large integer N such that $2A/N < 1$ and fix some distinct points $p_i \in X$ for $i = 1, \ldots, N$. We take subspaces V_i of $E_{|p_i}$ $(i = 1, \ldots, N)$ of dimension 1 such that $V_i \not\subset F_{1|p_i}$.
We define the parabolic filtration $\mathcal{F}_j(p_i)$ at p_i by

$$\mathcal{F}_0 = E_{|p_i} \supset \mathcal{F}_1 = V_i \supset \mathcal{F}_2 = \{0\}$$

with weight 0 for $\mathcal{F}_0$ and $2A/N$ for $\mathcal{F}_1$. The, the degree of the parabolic bundle E_* is

$$\operatorname{par}\deg(E_*) = \deg(E) + \sum_{i=1}^{N} 2A/N = \deg(E) + 2A.$$

Hence, $\operatorname{par}\mu(E_*) = \mu(E) + A$. The degree of the induced parabolic bundle F_{1*} is
$$\operatorname{par}\deg(F_{1*}) = \deg(F_1) = \mu(E) + A = \operatorname{par}\mu(E_*).$$

If $F \subset E$ is a subsheaf of rank one such that $F \not\subset F_1$, then there exists a non-trivial morphism $F \longrightarrow F_2$. Hence, we have $\deg(F) \leq \deg(F_2)$ and thus, using (7.5),

$$\begin{array}{rcl} \operatorname{par}\mu(F_*) & = & \operatorname{par}\deg(F_*) \\ & \leq & \deg(F) + 2A \\ & \leq & \mu(E) + A \\ & = & \operatorname{par}\mu(E_*). \end{array}$$

Hence, we have proved that E_* is parabolic semistable.

7.3.1.2 From a semistable to a stable parabolic structure

Let E_* be semistable (not stable) parabolic bundle of rank 2 over (X, D), where D is a finite subset of X. We wish to modify the parabolic structure so that the new parabolic bundle is stable. One of the following holds:

1. $E_* \simeq F_{0*} \otimes \mathbb{C}^2$,
2. $E_* \simeq F_{1*} \oplus F_{2*}$,
3. there exists a non-split exact sequence

$$0 \longrightarrow F_{1*} \longrightarrow E_* \longrightarrow F_{2*} \longrightarrow 0,$$

where F_{i*} are parabolic bundles of rank one.

We choose $p'_1, p'_2, p'_3 \in X \setminus D$ and subspaces $V_i \subset E_{|p'_i}$ of dimension 1 with the following property: if $F_* \subset E_*$ with $\operatorname{par}\deg(F_*) = \operatorname{par}\mu(E_*)$, then $F_{|p'_i} = V_i$ may happen for at most one i.
Let us choose a real number $1 > \epsilon > 0$. We consider the parabolic filtration $\mathcal{F}'_j(p'_i)$ at p'_i by

$$\mathcal{F}'_0 = E_{|p'_i} \supset \mathcal{F}'_1 = V_i \supset \mathcal{F}'_2 = \{0\}$$

with weight 0 for $\mathcal{F}'_0$ and weight ϵ for $\mathcal{F}'_1$. We also consider the same parabolic structure at the points of D. We obtain a parabolic bundle $E_*^{(\epsilon)}$. We have

$$\operatorname{par}\mu(E_*^{(\epsilon)}) = \operatorname{par}\mu(E_*) + 3\epsilon/2.$$

For $F_* \subset E_*$ such that $\operatorname{par}\deg(F_*) = \operatorname{par}\mu(F_*) = \operatorname{par}\mu(E_*)$, we have

$$\operatorname{par}\deg(F_*^{(\epsilon)}) \leq \operatorname{par}\deg(F_*) + \epsilon < \operatorname{par}\mu(E_*^{(\epsilon)}).$$

If ϵ is sufficiently small, then

$$\operatorname{par}\mu(F_*^{(\epsilon)}) < \operatorname{par}\mu(E_*^{(\epsilon)})$$

for any F_* such that $\operatorname{par}\mu(F_*) < \operatorname{par}\mu(E_*)$. Finally, with the new induced parabolic structure, $E_*^{(\epsilon)}$ is parabolic stable over the rank 2 bundle E. We explain now how to deal with higher rank bundles.

7.3.1.3 Higher rank and corollaries

By induction on the rank of the bundle, we have a generalization of our reasoning on any vector bundle over a curve. Let $r \geq 2$ be the rank of E. Since the degrees of the subbundles of E are bounded from above, let choose F_2 the maximal destabilizing subbundle of E. Then $F_1 = E/F_2$ is a vector bundle and we have the exact sequence

$$0 \to F_1 \to E \to F_2 \to 0.$$

Let F a subbundle of E. If $F \subset F_1$, then by induction, we can find a stable parabolic stable structure on F_1 that we denote $(F_1)_{*_1}$ and thus $\operatorname{par}\mu(F_{*_1}) <$

$\mathrm{par}\mu((F_1)_{*_1})$. Let us assume that $A = \mathrm{par}\mu((F_1)_{*_1}) - \mathrm{par}\mu(E_{*_1})$ is non negative. It remains to show that one can refine the considered parabolic structure $*_1$ so that $\mathrm{par}\mu(F_{*_2}) < \mathrm{par}\mu((F_1)_{*_2}) \leq \mathrm{par}\mu(E_{*_2})$ with respect to a new structure $*_2$. This is done as in the previous subsection by choosing an adapted filtration so that

$$\begin{aligned}
\mathrm{par}\deg(E_{*_2}) &= \mathrm{par}\deg(E_{*_1}) + rA,\\
\mathrm{par}\deg((F_1)_{*_2}) &= \mathrm{par}\deg((F_1)_{*_1}),\\
\mathrm{par}\deg((F_2)_{*_2}) &\leq \mathrm{par}\deg((F_2)_{*_1}) + rA.
\end{aligned}$$

If $F \not\subset F_1$, then there is a non trivial morphism $F \longrightarrow F_2$ and eventually $\mathrm{par}\mu(F_{*_2}) \leq \mathrm{par}\mu(E_{*_2})$. We get the parabolic semistability of (E_{*_2}). We apply the same reasoning as in the previous subsection to derive the following result.

Theorem 7.3.1. *Given E a holomorphic vector bundle over a curve, one can find sufficiently many points p_i and sufficiently small weights $\beta > 0$ such that the associated parabolic structure E_* is Mumford parabolic stable. If E is Mumford semistable and has rank 2, it is sufficient to consider 3 points.*

Theorem 7.3.1 restricted to rational weights and the main result of [Rol13] provide a new proof of an old result of C. Lebrun and M. Singer [LS93, Theorem 3.11]. Note that the assumption on the genus is made to kill the non trivial holomorphic vector fields.

Corollary 7.3.1. *Consider a ruled surface S over a curve C of genus $g \geq 2$. The blow up of S at sufficiently many points admits a cscK metric.*

Of course, a different proof can be given using the work of C. Arezzo- F. Pacard. The ruled surface is birationnally equivalent to the product $C \times \mathbb{P}^1$, which means that a blow-up of S is also a blow-up of $C \times \mathbb{P}^1$, on which there exists a cscK metric. Then it is sufficient to apply the main result of [AP06]. Our construction has the advantage to be more constructive.

7.3.2 Construction of constant scalar curvature Kähler metric with conic singularities

Let's start with E a rank 2 semistable bundle over a curve C. From the previous section (Theorem 7.3.1), we have obtained a stable parabolic structure E_* along the points $\mathcal{P} = \{p_1, ..., p_{m_\mathcal{P}}\}$, with $m_\mathcal{P} \geq 3$ and weight less than $\beta_0 > 0$. We shall see that there is an Hermitian-Yang-Mills metric with respect to a Kähler-Einstein metric with conic singularity along the associated points to the parabolic structure E_*. Firstly, let us recall the notion of Kähler metric with conical singularity, focusing on the complex dimensional one case.

Definition 7.3.2. Let C be a complex curve. Let $\mathcal{P} = \{p_j\}_{j=1,..,m_{\mathcal{P}}} \subset C$ be a finite set of points. Let $\beta = (\beta_1, .., \beta_{m_{\mathcal{P}}})$ with $0 < \beta_i \leq 1$ be the cone angles. Given a point $p_i \in \mathcal{P}$, label a local chart (V_{p_i}, z_1) centered at p_i as local cone chart. A Kähler metric with conical singularity and cone angle $2\pi\beta_i$ along p_i (in short a conical Kähler metric) is a closed positive (1,1) current and a smooth Kähler metric on $C \setminus \mathcal{P}$ such that in a local cone chart V_{p_i} its Kähler form is quasi-isometric to the cone flat metric

$$\omega_{cone} = \frac{\sqrt{-1}}{2}\beta_i^2 |z_1|^{2(\beta_i - 1)} dz_1 \wedge d\bar{z}_1. \tag{7.6}$$

Over a complex curve, the notions of Kähler class and a pointwise conformal class are equivalent. We can apply the work of M. Troyanov [Tro91] and therefore fixing at any $m_{\mathcal{P}}$ points, with at least $m_{\mathcal{P}} \geq 3$, a curve C admits a conical Kähler-Einstein metric ω_β along these fixed points for any angle between 0 and 1. Let us consider a model cone metric of the form

$$\tilde{\omega}_\beta = \frac{\sqrt{-1}}{c_\beta} \sum_{i=1}^{m_{\mathcal{P}}} \partial\bar{\partial} |\sigma_i|^{2\beta} + \omega$$

where ω is a smooth Kähler form on C, $\beta < \beta_0$, the sections σ_i vanish exactly at p_i and $c_\beta > 0$ is large enough so that $\tilde{\omega}_\beta$ is positive over the curve. A direct computation shows that actually $\tilde{\omega}_\beta$ satisfies the previous definition. Now, the behavior of conical Kähler-Einstein metrics are pretty well understood and it has been derived some Laplacian estimates for it. From [Bre13, Theorem 1] (see also [CGP13, Theorem A], [GP13, Section 5.2] and [Yao13, Theorems 1 and 2]), we know that the conical Kähler-Einstein metric is uniformly equivalent to the model cone metric for small enough angle, i.e there exists a certain constant $\delta > 0$ such that

$$\frac{1}{\delta}\tilde{\omega}_\beta < \omega_\beta < \delta\tilde{\omega}_\beta. \tag{7.7}$$

We shall explain now how to apply Simpson's theory [Sim88] in our context. Firstly, it is well known that a conical Kähler metric has finite volume. Moreover there is an exhaustion function φ of C such that $\Delta_{\omega_\beta}\varphi$ is bounded, and Sobolev inequality holds with respect to ω_β. Both facts are checked in [Li00, Proposition 4.1] where the arguments use only the local expression of the metric (7.6) and thus are still valid for ω_β. Moreover, in dimension one, analytic stability and parabolic stability coincide. In [Li00, Section 3], it is constructed a metric K_0 on E_* such that its curvature satisfies $|\Lambda_{\tilde{\omega}_\beta} F_{K_0}|_{K_0}$ is bounded on C for small enough angle $\beta > 0$. Therefore $|\Lambda_{\omega_\beta} F_{K_0}|_{K_0}$ is still bounded using (7.7). By stability of the parabolic structure E_*, Simpson's theorem [Sim88, Theorem 1] imply the existence of a Hermitian-Yang-Mills metric H_E on E satisfying the Hermitian-Einstein equation

$$F_{H_E} = \frac{\mathrm{tr} F_{H_E}}{rk(E)} \mathrm{Id}_E\, \omega_\beta = \mathrm{par}\mu(E) \mathrm{Id}_E\, \omega_\beta \tag{7.8}$$

over $C \setminus \mathcal{P}$.

Let us now consider the ruled manifold $X = \mathbb{P}E$ and $\pi : X \to C$ the projection on the base manifold. We denote $D_{\mathcal{P}} \subset \mathbb{P}(E)$ the induced divisor from the preimage of $\mathcal{P}$. The metric H_E on E induce a metric $\hat{H}_E$ on $\mathcal{O}_X(1)$. Actually for any point $p \in C$, $a, b \in E_{|p}$, and $\gamma \in E^*_{|p}$, one can define locally the metric $\hat{H}_E$ by $\hat{H}_E(\hat{u}, \hat{v}) = \frac{\gamma(u)\overline{\gamma(v)}}{\|\gamma\|^2_{H_E}}$.
The curvature of $\hat{H}_E$ on X is denoted $\hat{\omega}_E$. Since its restriction to the fibre is the Fubini-Study, it is non degenerate on $\mathbb{P}E_{|p}$ for $p \in C \setminus \mathcal{P}$. From Equation (7.8), it satisfies

$$\hat{\omega}_E = \mathrm{par}\mu(E) \pi^* \omega_\beta,$$

see [Fuj92, Section 1]. Let us introduce a generalization of Definition 7.3.2.

Definition 7.3.3. Let X be a compact Kähler manifold of dimension n. Let $D = \sum_{i=1}^{m_D} \alpha_i V_i$ be a normal crossing effective $\mathbb{R}$-divisor, in which the V_i are irreducible hypersurfaces on X. Set $\beta_i = 1 - \alpha_i$, for $1 \le i \le m_D$. Given a point $p \in D$, label a local chart (U_p, z_i) centered at p as local cone chart where $z_1, ..., z_k$ are the local defining functions of the hypersurfaces where p locates. A Kähler cone metric ω of cone angle $2\pi\beta_i$ along V_i $(1 \le i \le m_D)$ is a smooth Kähler metric on $X \setminus D$ whose Kähler form is quasi-isometric to the cone metric

$$\omega_{cone} = \frac{\sqrt{-1}}{2} \left(\sum_{i=1}^{k} \beta_i^2 |z_i|^{2(\beta_i - 1)} dz_i \wedge d\bar{z}_i + \sum_{i=k+1}^{n} dz_i \wedge d\bar{z}_i \right),$$

in the cone chart U_p of the point $p \in \cap_{i=1}^k V_i \setminus \cup_{i=k+1}^n V_i$. Furthermore, it is said to have constant scalar curvature if the function $\mathrm{scal}(\omega)$ is pointwise constant outside D.

For any m large enough and from the properties of ω_β, the metric $\hat{\omega}_E + m\pi^*\omega_\beta$ satifies Definition 7.3.3, i.e is a closed positive current, a Kähler metric with conical singularities along $D_{\mathcal{P}}$ and has constant scalar curvature on $X \setminus D_{\mathcal{P}}$.

Theorem 7.3.4 ([Kel14a]). *Given E a rank 2 vector bundle over a curve C, there exists a smooth divisor $D_{\mathcal{P}} \subset \mathbb{P}(E)$ and a constant scalar curvature Kähler on $\mathbb{P}(E)$ with conical singularity along $D_{\mathcal{P}}$. If E is semistable, $D_{\mathcal{P}}$ can be given by the preimage of 3 points of C.*

We address now some remarks. By changing m, this theorem provides in particular examples of cscK cone metric in the class of $[\hat{\omega}_E + m\pi^*\omega_\beta]$ which are not Kähler-Einstein. In [RS05], it is explained in details how to obtain scalar-flat Kähler metrics with orbifold singularities when one restricts to the particular case of a ruled surface and the parabolic structure has rational weights, using Mehta-Seshadri theorem. Also, remark that from the point of view of extremal Kähler metrics, some conical metrics are constructed in [Li12] using the formalism developed by Apostolov et al in [Apo+08b], while G. Székelyhidi studied the geometric splitting of a ruled surface given by a non stable manifold under the Calabi flow in [Szé09]. From the point of view of stability, we expect that Theorem 7.3.4 provides examples of log-K-stable manifolds in the sense of [Don12].

Part IV

Applications: some numerical approximations of canonical metrics

Chapter 8

Numerical metrics on moduli spaces of Calabi-Yau manifolds

Research on differential geometry of complex manifolds has reached some impressive results on the existence of solutions to difficult non-linear PDE's. Yau's proof of Calabi's conjecture and Donaldson-Uhlenbeck-Yau's proof of the existence of Hermite-Einstein metrics on stable holomorphic vector bundles, are main examples of such theorems. Although only very rarely does one expect to find explicit formulae for the solutions, one can explore the geometry of the solutions using numerical methods. During the last years, several techniques that approximate Kähler-Einstein and Hermite-Einstein metrics have appeared in the literature, mainly due to [Don09].

In this chapter we begin numerical work to study Weil-Petersson metrics on moduli spaces of such solutions. The main focus is on moduli spaces of complex structures on polarized Calabi-Yau manifolds. We recall that a Calabi-Yau manifold is a compact Kähler manifold with trivial canonical bundle. By Yau's theorem, they can be equipped with Kähler Ricci-flat metric. First, we develop a fast algorithm that computes the metric by building on the theoretical work by Tian [Tia87] and Todorov [Tod89]. Secondly, by using Donaldson's quantization link between infinite and finite G.I.T quotients, we introduce a sequence of natural Kähler metrics. Finally, we introduce an algorithm that computes such metrics and discuss an example. We hope this work illuminates the techniques and difficulties that appear when approximating Weil-Petersson metrics on more general moduli spaces.

Motivation for this work can be found in different sources. For instance, we find specially motivating the program by Douglas et. al. [Dou+08], building on the work by Donaldson [Don09], to numerically compute Kähler metrics that appear in Calabi-Yau compactifications of string theory. Other source of motivation comes from the study of global Weil-Petersson geometry on moduli spaces of Calabi-Yau manifolds, [DL06]. In this case, one of the algorithms introduced in this chapter should allow to estimate Weil-Petersson volumes of moduli spaces in a sensible amount of time and with reasonable precision.

8.1 Motivation

Many approaches to unify particle physics attempt to describe known physics by considering a simple field theory defined on a higher dimensional space, and taking four-dimensional limits. The idea, today known as compactification of a field theory, has inspired much work in the interface between geometry and physics. Determining the action functional for fields, in four-dimensional limits, and for a large family of compactifications, is the main mathematical motivation for this work.

Remark 8.1.1. For the purpose of this introduction, by a *field theory* we mean a functional space $\mathfrak{D}$ of geometric data on a manifold Y (such as Riemannian metrics, connections on a principal bundle on Y, sections of vector bundles, ...), with an action functional $\mathcal{S}\colon \mathfrak{D} \to \mathbb{R}$ defined on it.

A compactification is then a field theory on a D-dimensional space-time which is the product of the 4-dimensional space-time $\mathbb{R}^4$ with a m-dimensional manifold X, the compactification manifold, carrying a Riemannian metric and other geometric structure corresponding to other fields in the theory. These must solve the Euler-Lagrange equations associated to $\mathcal{S}$, and preserve four dimensional Poincaré invariance. The most general metric ansatz for a Poincaré invariant compactification is

$$g_{IJ} = \begin{pmatrix} f\eta_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}$$

where the tangent space indices are $0 \leq I < 4 + m = D$, $0 \leq \mu < 4$, and $1 \leq i, j \leq m$. Here $\eta_{\mu\nu}$ is the Minkowski metric, g_{ij} is a metric on X, and f is a real valued function on X. As the simplest example, consider the D-dimensional Hilbert-Einstein action for general relativity. In this case, Einstein's equations reduce to Ricci flatness of g_{IJ} . Given our metric ansatz, this requires f to be constant, and the metric g_{ij} on X to be Ricci flat.

Typically, if a manifold admits a Ricci-flat metric, it will not be unique; rather there will be a moduli space of such metrics. Physically, one then expects to find solutions in which the choice of Ricci-flat metric on X is slowly varying in four dimensional space-time. General arguments imply that such variations must be described by variations of 4-dimensional fields, governed by an EFT. For simplicity, by this *Effective Field Theory* (EFT) we mean a four dimensional field theory that emerges in the small radius limit of X, when the geometric data on $\mathbb{R}^4 \times X$ restricted to X satisfies the Euler Lagrange equations. Thus, the action functional of the EFT is defined on a functional space of geometric data on $\mathbb{R}^4$.

Given an explicit parametrization of the family of metrics, say $g_{ij}(t_a)$ for some parameters t_a, the EFT could be computed explicitly by promoting the parameters $\{t_a\}$ to 4-dimensional fields $\{t_a(x)\}$, substituting this parametrization into the D-dimensional action, and expanding in powers of the 4-dimensional derivatives. For the Hilbert-Einstein action, we find the four-dimensional effective action functional

$$\begin{aligned}
\mathcal{S}^{GR}_{EFT} &= \int_{\mathbb{R}^4\times X} d^{(10)}\mathrm{Vol}(X)\ \mathrm{scal}(g_{IJ}) \\
&= \int_{\mathbb{R}^4\times X} d^4x d^m y \sqrt{\det g(t)}\mathrm{scal}(g_{ij}) + \\
&\quad \int_X d^m y \sqrt{\det g(t)} g^{ik}(t) g^{jl}(t) \frac{\partial g_{ij}}{\partial t_a}\frac{\partial g_{kl}}{\partial t_b} \times \int_{\mathbb{R}^4} d^4x \partial_\mu t_a(x)\partial^\mu t_b(x) \\
&\quad + \ldots
\end{aligned} \tag{8.1}$$

where y^i denotes a local coordinate chart on X, x^μ a local coordinate chart on $\mathbb{R}^4$, and $\mathrm{scal}(g)$ is the scalar curvature associated to the D dimensional metric. In general, a direct computation of (8.1) is impossible. This becomes especially clear when one learns that the Ricci-flat metrics g_{ij} are not explicitly known for the examples of interest.

An interesting class of compactifications come from the field theory limit of string theories, where the space-time dimension is $D = 10$. Requiring $\mathcal{N} = 1$ supersymmetry on the four dimensional EFT and the vanishing of torsion elements, fixes X to be a Calabi-Yau threefold. In this case, computing the four dimensional action functional for the $\{t_a(x)\}$ fields (8.1) involves to know the Weil-Petersson metric on the moduli space of Kähler Ricci-flat metrics on X.

These theories contain other objects besides the space-time metric. For instance, in a heterotic string theory , the geometric content also involves a principal $E_8 \times E_8$-bundle endowed with a gauge connection A; E_8 denotes the Cartan's exceptional simple Lie group of dimension 248. In a Poincaré invariant compactification, one defines the theory on a principal $E_8 \times E_8$-bundle $P \to \mathbb{R}^4 \times X$. For every point x on X, the restriction of the principal bundle P to $\mathbb{R}^4 \times x$ is trivial, i.e. $P|_{\mathbb{R}^4\times x \hookrightarrow \mathbb{R}^4 \times X}$ is equivalent to $E_8 \times \mathbb{R}^4$.

In the small radius limit of X one obtains an effective gauge theory on $\mathbb{R}^4$ with gauge group H, by expanding the Yang-Mills functional around a background reducible connection A_0 on $P \to \mathbb{R}^4 \times X$. For simplicity, one considers a subgroup G of E_8 and takes A_0 to be a connection on a principal G-subbundle of $P \to \mathbb{R}^4 \times X$. The gauge group H of the effective theory on $\mathbb{R}^4$ is the commutant of $G \hookrightarrow E_8$. In many applications G is the special unitary group $SU(r)$, with $2 < r < 6$. The Euler-Lagrange equations associated to the Yang-Mills functional require A_0 to be a Hermite Yang-Mills unitary connection.

As in the case of the Kähler Ricci-flat metric, if the bundle admits a Hermite Yang-Mills connection, it will not be unique; rather there will be a moduli space of E_8 connections on P with G-holonomy. Although a general description of such moduli spaces is not explicitly known for examples of interest, it is interesting enough to work with the space of local deformations around a particular A_0. Such space is in one to one correspondence with the null space of the Dirac operator on X, coupled to A_0, that acts on spinors which are sections of an associated vector bundle to P, adjoint representation of E_8.

Thus, in order to find the action functional that governs the dynamics of such particles on $\mathbb{R}^4$, one has to expand the 10-dimensional action functional in the small radius limit of X, for small perturbations of A_0 that preserve the linearized Yang-Mills equations on $P \to X$, and Poincaré invariance on $\mathbb{R}^4$.

More precisely, given a local coordinate chart $\{z^i, \bar{z}^{\bar{j}}\}_{i,j=1}^3$ on X, $\{x^\mu\}_{\mu=1}^4$ on $\mathbb{R}^4$, and a trivialization of P one can expand the gauge connection A around A_0 as

$$A(z,x) = A_{0,i}dz^i + A_{0,\bar{j}}d\bar{z}^{\bar{j}} + A_\mu(x)dx^\mu + t_p^*(x)\frac{\partial A_{\bar{j}}}{\partial \bar{t}_p}d\bar{z}^{\bar{j}} + t_p(x)\frac{\partial A_i}{\partial t_p}dz^i + \ldots$$

Here, $A_\mu dx^\mu$ is the 4-dimensional H-gauge connection and $\{t_p\}$ is a local coordinate chart on the space of infinitesimal deformations of the connection A_0 that preserve the linearized Yang-Mills equations. By the ellipsis, we denote higher order corrections in t and, also, corrections by terms which do not preserve the linearized Yang-Mills equations; one can assume that both corrections are irrelevant in low energy physics. If we expand the pure Yang-Mills action in 10 dimensions assuming our Poincaré invariant ansatz, we find

$$\begin{aligned} \mathcal{S}^{YM}_{EFT} &= \int_{\mathbb{R}^4\times X} d^{(10)}\mathrm{Vol}(X)\ \mathrm{Tr}\left(F_{IJ}F^{IJ}\right) \\ &= \int_X d^{(6)}\mathrm{Vol}(X)\ \mathrm{Tr}\left(\frac{\partial A_i}{\partial t_p}\frac{\partial A_{\bar{j}}}{\partial \bar{t}_p}\right) g^{i\bar{j}} \times \int_{\mathbb{R}^4} d^4x\, \partial_\mu t_p \partial^\mu t^*_{\bar{q}} + \ldots \end{aligned} \tag{8.2}$$

Here, we are using the usual Einstein's conventions for summation.

Hence, an understanding of the effective action for the t fields, known as charged matter and eventually related to particles such as electrons, quarks, etc. , requires to compute generalized Weil-Petersson metrics (as in (8.2)) on the moduli space of E_8 connections on P with G-holonomy. The numerical tools that we introduce in this chapter, should be useful in the case when $G = SU(r)$ and the principal $SU(r)$-subbundle underlies a family of stable holomorphic vector bundles $E \to X$ (with $c_1(E) = 0$, $\mathrm{rank}\,(E) = r$). In this case one can use balanced embeddings to approximate the Hermite Yang-Mills connections, identify the space of infinitesimal deformations of the background connection with sheaf cohomology groups, and approximate the Weil-Petersson metrics using the ideas of this chapter.

Outline of the chapter

In Section 8.2 we review some general results on moduli spaces of polarized Calabi-Yau manifolds, and define their corresponding Weil-Petersson metrics. We explain a first method to numerically compute the Weil-Petersson metric in Section 8.3. By combining formulae of local deformations of the holomorphic top form under diffeomorphisms, and Monte Carlo integration techniques, we evaluate the Weil-Petersson metric in a particular example (the Calabi-Yau quintic threefold in $\mathbb{P}^4$). In Section 8.4, we review basic concepts on moduli spaces of polarized varieties from the point of view of Geometric Invariant Theory. After defining a natural sequence of Kähler metrics, we provide another algorithm to compute the Weil-Petersson on the moduli space of constant scalar curvature Kähler metrics. We prove numerically on the quintic threefold that it actually converges to the Weil-Petersson metric on the moduli space of Kähler Ricci-flat metrics.

8.2 Weil-Petersson metrics on moduli of polarized manifolds

In the sequel, X will denote a smooth projective Calabi-Yau manifold of complex dimension n. Let ν, with $\mathrm{span}(\nu) = H^0(X, \mathcal{K}_X)$, be the corresponding holomorphic n-form, and $\mathcal{L}$ the defining polarization. We denote by ω the Kähler form, with $[\omega] = c_1(\mathcal{L})$. By $g_{i\bar{j}}$ we mean the compatible Riemannian metric on X, and by h the compatible Hermitian metric on $\mathcal{L}$ whose curvature is $c_1(h) = \omega$.

A *holomorphic family* of compact polarized Kähler manifolds (X_t, g_t) parametrized by $t \in \mathcal{T}$ is a complex manifold $\mathcal{X}$ together with a proper holomorphic map $\pi\colon \mathcal{X} \to \mathcal{T}$ which is of maximal rank. This means that the differential of π is surjective everywhere, and that $\pi^{-1}(t)$ is compact for any $t \in \mathcal{T}$.

Given a base point $0 \in \mathcal{T}$ we say that $\pi^{-1}(t) = X_t$ is a deformation of X_0. Locally, $\mathcal{X}$ is a trivial fiber product $\mathcal{X}|_{\mathcal{U}} \simeq \mathcal{U} \times X_t$. If $T_t\mathcal{T}$ denotes the holomorphic tangent space to $\mathcal{T}$ at t, we can define the infinitesimal deformation or Kodaira-Spencer map:

$$\rho_t \colon T_t\mathcal{T} \longrightarrow H^1(X_t, TX_t).$$

where $H^1(X_t, TX_t)$ can be identified with the harmonic representatives of

(0,1) forms with values in the holomorphic tangent bundle $TX_t = T^{1,0}X_t$; in other words $H^1(X_t, TX_t) \sim H^{0,1}_{\bar\partial}(TX_t)$. We know that the Kähler metric g_t induces a metric on $\Lambda^{0,1}(TX_t)$. Thus, for $v_1, v_2 \in T_t\mathcal{T}$, we can define a Kähler metric at $t \in \mathcal{T}$,

$$G(v_1, v_2) = \int_{X_t} \langle \rho_t(v_1), \rho_t(v_2)\rangle_{g_t} d\mathrm{Vol}(g_t). \tag{8.3}$$

Note that G is possibly degenerate. If ρ_t is injective and g_t satisfy an Einstein type condition, one says that G is a Weil-Petersson metric on the Kuranishi space.

8.2.1 Weil-Petersson metric for Calabi-Yau's manifolds

Suppose now that $\mathcal{X} \to \mathcal{T}$ is a family of polarized Calabi-Yau manifolds $(X, \mathcal{L})$, naturally equipped with a unique Ricci-flat Kähler metric in a given Kähler class. We can identify the tangent space at $t \in \mathcal{T}$, $T_t\mathcal{T}$ with $H^{0,1}_{\bar\partial}(TX_t)$. This allows us to define the Weil-Petersson metric on $\mathcal{T}$, the local moduli space of $(X, \mathcal{L})$, as follows.

Definition 8.2.1. Let $v_1,\, v_2 \in T_t\mathcal{T} \simeq H^{0,1}_{\bar\partial}(TX_t)$, then

$$\langle v_1,\, v_2\rangle_{\text{W-P}} := \int_{X_t} v^i_{1\bar k}\overline{v^j_{2\bar l}} g_{i\bar j} g^{l\bar k}\, \mathrm{dVol_t}$$

with $g = g_t$.

In this particular case Tian and Todorov proved the following

Theorem 8.2.2. *(Tian-Todorov, [Tia87; Tod89]) Let $\pi\colon \mathcal{X} \to \mathcal{T} \ni 0$, $\pi^{-1}(0) = X_0$, be the Kuranishi family of X_0, then $\mathcal{T}$ is a non-singular complex analytic space such that*

$$\dim_{\mathbb{C}} \mathcal{T} = \dim H^1(X_t,\, TX_t) = \dim H^1(X_t,\, \Omega^{n-1}_{X_t}),$$

where TX_t denotes the sheaf of holomorphic vector fields on X_t, and $\Omega^{n-1}_{X_t}$ the sheaf of holomorphic $(n-1)$ forms.

There is a correspondence between $H^{0,1}_{\bar\partial}(TX_t)$ and $H^1(X_t,\, \Omega^{n-1}_{X_t})$ given by the interior product and the global holomorphic n-form on X_t. Then, one can evaluate the Weil-Petersson metric in terms of the standard cup product on $H^{n-1,1}(X_t)$. Indeed, if we denote by ν_t the global holomorphic three-form on X_t, then

$$\Psi(t, \bar t) = (-1)^{\frac{n(n-1)}{2}} i^{n-2} \log\left(\int_{X_t} \nu_t \wedge \bar\nu_t\right) \tag{8.4}$$

is the local Kähler potential for the Weil-Petersson metric. This is an important formula; for instance, if we fix the differential structure on X and consider variations of the complex structure in the holomorphic top form ν_t, one can evaluate $\partial\bar{\partial}\Psi$ by computing differentials $\frac{\partial \nu_t}{\partial t_a}$, with $\frac{\partial}{\partial t_a}$ a basis for $T_t\mathcal{T} = H^1(X_t, \Omega^{n-1}_{X_t})$. This idea will play an important role in the next section, where we will perform a direct calculation of $\partial\bar{\partial}\Psi$. Also, one could compute $\partial\bar{\partial}\Psi$ using the standard cup product to be able to express $\int_X \nu_t \wedge \bar{\nu}_t$ as a function of t, as we show in the following example.

8.2.2 Example: the quintic in $\mathbb{P}^4$

In this section we will study different constructions on the quintic hypersurface $X = Q$ in $\mathbb{P}^4$, with $h^{1,1} = 1$ and $\dim H^1(Q, \Omega^2_Q) = h^{2,1} = 101$. Many geometrical properties of this Calabi-Yau threefold are known in the literature. For instance, one can describe its moduli very explicitly. If we define

$$W = \{\, P \,|\, P \text{ a homogeneous quintic polynomial of } Z_0, Z_1, Z_2, Z_3, Z_4\},$$

one can verify that $\dim W = 126$. Hence, for any $t \in \mathbb{P}W = \mathbb{P}^{125}$, t is represented by a hypersurface in $\mathbb{P}^4$. As two hypersurfaces that differ by an element in $\mathrm{Aut}(\mathbb{P}^4)$ are equivalent, and there exists a divisor $\mathcal{D}$ in $\mathbb{P}^{125}$ characterizing the singular hypersurfaces in $\mathbb{P}^4$, the moduli space of quintics Q is given by

$$\mathcal{M}_Q = \left(\mathbb{P}^{125}\backslash\mathcal{D}\right)/\mathrm{Aut}(\mathbb{P}^4).$$

The dimension of the moduli space is 101, as expected.

For simplicity, we study a one dimensional subspace of complex deformations defined by

$$P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4,$$

and parametrized by t. As t and $t\exp(2\sqrt{-1}\pi l/5)$, for any $l \in \mathbb{Z}$, represent the same variety, the fundamental region on the t-plane is defined as $\{\, t \,|\, 0 \leq \arg(t) < 2\pi/5 \text{ and } t \neq 1\}$. For t a fifth root of unity, i.e. $t = \exp(2\sqrt{-1}\pi l/5)$ for any $l \in \mathbb{Z}$, the quintic develops double point singularities.

Evaluating the Weil-Petersson metric on the family of quintics. Candelas, de la Ossa, Green and Parkes [Can+91] evaluated the volume $\int_X \nu_t \wedge \bar{\nu}_t$ as function of t, for the family of quintic threefolds

$$P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4, \tag{8.5}$$

by evaluating cup products. More specifically, they constructed explicitly a symplectic basis of 3-cycles (A^a, B_b) for $H_3(Q, \mathbb{Z})$, such that

$$A^a \cap B_b = \delta^a_b, \quad A^a \cap A^b = 0, \quad B_a \cap B_b = 0.$$

Also, they considered the dual basis (α_a, β^b) in cohomology so that

$$\int_{A^a} \alpha_b = \delta^a_b, \quad \int_{B_a} \beta^b = \delta^b_a,$$

with the other integrals vanishing. Then it follows that

$$\int_Q \alpha_a \wedge \beta^b = \delta^b_a, \quad \int_Q \alpha_a \wedge \alpha_b = \int_Q \beta^a \wedge \beta^b = 0.$$

Thus, the holomorphic three-form ν_t can be expanded using this basis as

$$\nu_t = z^a \alpha_a - \mathcal{G}_b \beta^b,$$

and therefore the volume can be written as

$$\int_Q \nu_t \wedge \bar{\nu}_t = \bar{z}^a \mathcal{G}_a - z^a \overline{\mathcal{G}}_a.$$

Using this, the Weil-Petersson metric is

$$g^{\text{W-P}}_{t\bar{t}} = -i \partial_t \overline{\partial}_t \log \left(\bar{z}^a \mathcal{G}_a - z^a \overline{\mathcal{G}}_a \right). \tag{8.6}$$

Hence, in order to obtain the Weil-Petersson metric, it is sufficient to evaluate the periods $z^a = \int_{A^a} \nu$, $\mathcal{G}_b = \int_{B_b} \nu$. For that, let us consider the vector space V_j generated by the vectors

$$\begin{pmatrix} \frac{\partial^k}{\partial t^k} z^a(t) \\ \frac{\partial^k}{\partial t^k} \mathcal{G}_b(t) \end{pmatrix}$$

for $0 \leq k \leq j$. For generic values t of the Kuranishi deformation, the dimension of V_j must be constant and in our case, this dimension cannot be larger than 4. Thus, expressing one element of V_5 from the others, we obtain a non-trivial ordinary differential equation relating the periods. This is the so-called Picard-Fuchs equation, and we refer to [Mor92] for a mathematical approach to this topic. Note that the form of these equations depends on the local coordinates over the space of deformations and the choice of the holomorphic form $\nu(t)$. The solution of the Picard-Fuchs equations may be singular but the types of singularities that can occur have been well studied. In [Can+91], those equations have generalized hypergeometric type and can be solved by expressing the integrands of the periods in power series of t. Each coefficient of the power series leads to an integral that can be evaluated by residue formulae. The obtained periods are extended by

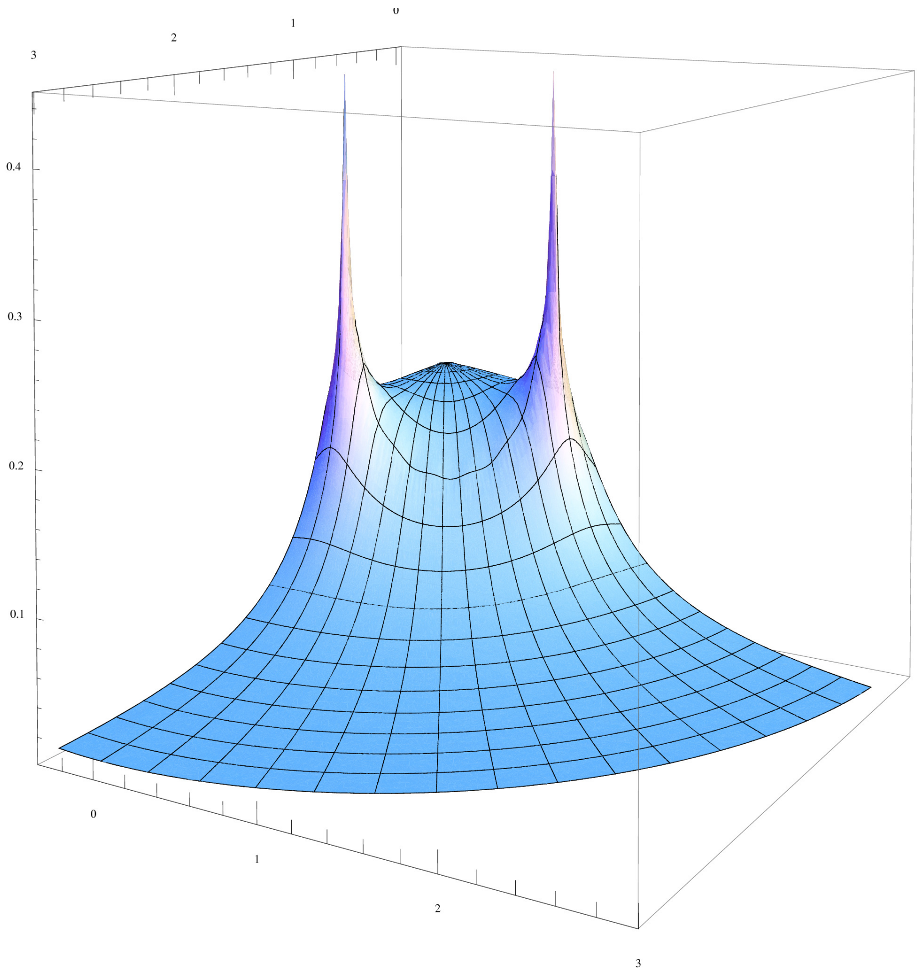


Figure 8.1: Weil-Petersson metric (vertical axis) on the t-plane (horizontal plane) of 1-dimensional moduli of Calabi-Yau quintic 3-folds, $Z_0^5+Z_1^5+Z_2^5+Z_3^5+Z_4^5-5tZ_0Z_1Z_2Z_3Z_4$.

analytic continuation over fundamental domains ($|t| < 1$ with $0 < \arg(t) < \frac{2\pi}{5}$ and $|t| > 1$ with $0 < \arg(t) < \frac{2\pi}{5}$). Although the behavior of the periods can be described at the singular points (in our case $t = 1, \infty$), it is difficult to obtain simple formulas to express exactly the periods if one is not considering hypersurfaces, as far as we know. That is why we consider this method as not satisfactory. We have written a simple program in Mathematica and Maple, for the case of the family of quintics (8.5), and computed numerically the power series that define the periods. Fig. 8.1 shows our evaluation of (8.6) for $0 < |t| \leq 3$ and $0 \leq \arg(t) < 2\pi/5$.

8.3 Numerical evaluation of the W-P metrics via deformations of the holomorphic n-form

8.3.1 Description of the method

In this section we describe how to approximate Weil-Petersson metrics by considering variations of the holomorphic n-form. First, we make the following important distinction: by X we denote a Calabi-Yau differentiable manifold with no complex structure defined on it. X_t denotes the same differentiable manifold endowed with an integrable complex structure parametrized by t. We denote by $U \subset X$ an open subset of the differentiable manifold X, such that $U \subset X$ is independent of any complex structure one defines on X.

Every element in $v \in T_{t_0}\mathcal{T}$ yields an infinitesimal deformation of the complex structure on X_{t_0}. By going to a local coordinate patch on $U \subset X$ we can relate the holomorphic coordinates on X_{t_0} with the holomorphic ones on X_{t_0+tv} by defining a proper infinitesimal diffeomorphism. Let $\{w^i\}_{i=1}^n$ be a local holomorphic coordinate system for X_{t_0} on $U \subset X$, and $\{y^i\}_{i=1}^n$ be a local holomorphic coordinate system for X_{t_0+tv} on the same subset U. Therefore, on U, we can relate the w-coordinates and the y-coordinates as:

$$y^i = w^i + v^a \vartheta_a^i(w, \bar{w}) + O(v^2), \tag{8.7}$$

with ϑ a vector field, non-holomorphic section of $T^{1,0}X_{t_0}$, and $\frac{\partial}{\partial t_a}$ is a basis for $T_{t_0}\mathcal{T}$.

Hence, we can write the holomorphic top form ν_{t_0+tv} on X_{t_0+tv}, using the w-coordinate system, as a non-holomorphic n-form in $\Omega^{n,0}(X_{t_0}) \oplus \Omega^{n-1,1}(X_{t_0})$. More precisely,

$$\nu_{t_0+tv} = \nu_{t_0} + v^a \partial_{t_a} \nu_{t_0} + O(v^2), \tag{8.8}$$

where the $O(v^2)$ terms are irrelevant for our purpose. The term $\partial_{t_a}\nu_{t_0}$ is computed as pull-back of the infinitesimal diffeomorphism defined by $\vartheta_a^i(w, \bar{w})$. Thus, given a basis of deformations $\partial_{t_a}\nu_{t_0} \in \Omega^{n,0}(X_{t_0}) \oplus \Omega^{n-1,1}(X_{t_0})$ and vectors $v_1,\, v_2 \in T_{t_0}\mathcal{T}$, we can write the Weil-Petersson inner product as

$$\langle v_1,\, v_2\rangle_{\text{W-P}} = -\frac{v_1^a \overline{v_2^b} \int_X \partial_{t_a}\nu_{t_0} \wedge \overline{\partial_{t_b}\nu_{t_0}}}{\int_X \nu_{t_0} \wedge \overline{\nu_{t_0}}} + \frac{v_1^a \overline{v_2^b} \int_X \partial_{t_a}\nu_{t_0} \wedge \overline{\nu_{t_0}} \int_X \nu_{t_0} \wedge \overline{\partial_{t_b}\nu_{t_0}}}{\left(\int_X \nu_{t_0} \wedge \overline{\nu_{t_0}}\right)^2}, \tag{8.9}$$

where we have expanded the Kähler potential (8.4) for n-forms as (8.8). Therefore, a direct calculation of the Weil-Petersson metric involves:

- A choice of $\vartheta_a^i(w,\bar{w})$, which is not unique and depends on the particular geometry of the Calabi-Yau manifold.
- To perform several integrals on X.

If X_{t_0} is a complete intersection and a Calabi Yau manifold, there is a natural choice for $\vartheta_a^i(w,\bar{w})$. Let $\{P_\alpha(Z)\}_{\alpha=1}^{m-n} = \{P_\alpha(Z,t_0)\}_{\alpha=1}^{m-n}$ be a basis of homogeneous polynomials in $\mathbb{P}^m$ whose common zero loci define X_{t_0}. Let us suppose that given two independent deformations of the complex structure, $v_1,\, v_2 \in T_{t_0}\mathcal{T} \simeq H^{0,1}_{\bar{\partial}}(TX_{t_0})$, we can find two sets of polynomials, $\{\delta_1 P_\alpha(Z)\}_{\alpha=1}^{m-n}$ and $\{\delta_2 P_\alpha(Z)\}_{\alpha=1}^{m-n}$, that parametrize isomorphic deformations of the complex structure. We set a coordinate atlas on $X \subset \mathbb{P}^m$ by choosing inhomogeneous local coordinates $\{w_i = Z_i/Z_0\}_{i=1}^m$ on $\mathbb{P}^m$, n coordinates as local coordinates on $U \subset X$, and the remaining $n-m$ coordinates as dependent of the n coordinates on $U \subset X \subset \mathbb{P}^m$. In other words, for any point $x \in X$, by making a unitary change of coordinates on $\mathbb{P}^m$ we can always set $\{w_i\}_{i=1}^n$ to be a local coordinate system on an open subset of X_{t_0} that contains x, while the remaining coordinates $\{w_i = w_i(w_1,\ldots,x_n)\}_{i=n+1}^m$ on $X_{t_0} \subset \mathbb{P}^m$ can be expressed as a function of $\{w_i\}_{i=1}^n$. We write the defining polynomials in inhomogeneous coordinates, as

$$\begin{aligned} p_\alpha(w) = p_\alpha(w,t) =& \frac{P_\alpha(Z,t)}{Z_0^{\deg P_\alpha}}, \\ \partial_{t_a} p_\alpha(w) =& \partial_{t_a}\left(\frac{P_\alpha(Z,t)}{Z_0^{\deg P_\alpha}}\right), \end{aligned}$$

where $\deg P \in \mathbb{N}$ is the degree of the homogeneous polynomial P. If $\vartheta_a^i(w,\bar{w})$ are vector fields on $X_{t_0} \subset \mathbb{P}^m$ corresponding to the deformations $\{\partial_{t_a} p_\alpha(w)\}$, and $\{y_i\}_{i=1}^m$ is a holomorphic local coordinate system on $X_{t_0+t_a v^a} \subset \mathbb{P}^m$, the following equation holds for an infinitesimal variation t_a on the moduli,

$$p_\alpha(y) + t_a \partial_{t_a} p_\alpha(y) = 0 = p_\alpha(w) + t_a \frac{\partial p_\alpha(w)}{\partial w_i}\vartheta_a^i(w,\bar{w}) + t_a \partial_{t_a} p_\alpha(w) + O(v^2), \tag{8.10}$$

where the repeated index a is not summed this time, and y^i obeys (8.7).

Proposition 8.3.1 ([KelLuk12])**.** *Let $G_{i\bar{\jmath}}$ be a Fubini-Study metric on $\mathbb{P}^m$. Let $H_{\alpha\bar{\beta}}$ be the elements*

$$H_{\alpha\bar{\beta}} = G^{i\bar{\jmath}}\frac{\partial p_\alpha(w)}{\partial w_i}\frac{\partial \bar{p}_{\bar{\beta}}(\bar{w})}{\bar{\partial}\bar{w}_{\bar{\jmath}}}.$$

Then, a natural choice for $\vartheta^i_a(w,\bar{w})$ is

$$\vartheta^i_a(w,\bar{w}) = -\left(H^{-1}\right)^{\bar{\beta}\gamma} G^{i\bar{\jmath}}\frac{\partial \bar{p}_{\bar{\beta}}(\bar{w})}{\bar{\partial}\bar{w}_{\bar{\jmath}}}\partial_{t_a}p_\gamma(w). \tag{8.11}$$

The proof is straightforward by substituting $\vartheta^i_a(w,\bar{w})$ into the equation (8.10), and observing that $p_\alpha(w) = 0$, as w lies on $X_{t_0} \subset \mathbb{P}^m$.

We can calculate the deformation of ν_t under the infinitesimal diffeomorphism defined by (8.7), by combining (8.11) and (8.8). More precisely, if

$$\nu_{t_0+t_a} = N_{i_1,\ldots,i_n}(y)dy^{i_1}\wedge\ldots\wedge dy^{i_n}, \tag{8.12}$$

is the holomorphic n-form on $X_{t_0+t_a v^a} \subset \mathbb{P}^m$, $y^i = w^i + v^a\vartheta^i_a(w,\bar{w}) + O(v^2)$, and

$$dy^i = dw^i + v^a\frac{\partial\vartheta^i_a(w,\bar{w})}{\partial w^j}dw^j + v^a\frac{\partial\vartheta^i_a(w,\bar{w})}{\bar{\partial}\bar{w}^{\bar{\jmath}}}d\bar{w}^{\bar{\jmath}} + O(v^2), \tag{8.13}$$

we can expand (8.12) as in (8.8), and determine the term $\frac{\partial\nu_t}{\partial t_a}(t_0)$ that we need to evaluate the Weil-Petersson metric (8.9).

Although one could compute (8.8) in the general case, for simplicity we just consider the case $m-n=1$, i.e. X_{t_0} is a hypersurface defined as the zero locus of a polynomial $p(w)$. By the adjunction formula we know that $\nu_{t_0+t_a}$ is pull-back of a meromorphic n-form on $\mathbb{P}^{n+1}$ that obeys the simple formula

$$\prod_{i=1}^{n+1} dy^i = d\left(p(y) + t_a\partial_{t_a}p(y)\right)\wedge\nu_{t_0+t_a}. \tag{8.14}$$

Using $\nu_{t_0+t_a} = \nu_{t_0} + t_a\partial_{t_a}\nu_{t_0} + O(v^2)$ and the transformation of dy^i (8.13), in (8.14), one can compute $\partial_{t_a}\nu_{t_0}$ as

$$\begin{aligned}\partial_{t_a}\nu_{t_0} = &-\frac{1}{\frac{\partial p}{\partial w^{n+1}}}\left(\sum_{i=1}^{n+1}\frac{\partial\vartheta^i_a(w,\bar{w})}{\partial w^i}\right)\prod_{i=1}^{n} dw^i \\ &-\sum_{i,j=1}^{n}\frac{(-1)^{n-i}}{\frac{\partial p}{\partial w^{n+1}}}\left(\frac{\partial\vartheta^i_a(w,\bar{w})}{\partial\bar{w}^{\bar{\jmath}}} + \frac{\partial\bar{w}^{\overline{n+1}}}{\partial\bar{w}^{\bar{\jmath}}}\frac{\partial\vartheta^i_a(w,\bar{w})}{\partial\bar{w}^{\overline{n+1}}}\right)dw^1..\widehat{dw^i}..dw^n d\bar{w}^{\bar{\jmath}} \\ &+\ldots\end{aligned} \tag{8.15}$$

Here, the differentials dw^i obey the Grassmann algebra of forms, $\widehat{dw^i}$ denotes the omission of dw^i, and the final ellipsis denotes further terms which do not contribute to the equation (8.9). Now, equipped with a local formula for the integrands of equation (8.9), we need a numerical way to evaluate the integrals that appear therein.

8.3.2 Monte-Carlo integration on varieties

8.3.2.1 The standard Monte-Carlo method

One of the problems that we need to solve, in order to compute the Weil-Petersson metric via local deformations of the holomorphic top-form, consists in evaluating integrals of the type

$$\int_X f\nu \wedge \overline{\nu}. \tag{8.16}$$

We can numerically approximate such integrals by introducing an auxiliary measure $d\mu$, and generating random points $\{q_l \in X\}_{1\leq l\leq N_{points}}$ on X uniformly distributed under $d\mu$. Hence, by defining the mass function

$$m(x) = \nu \wedge \overline{\nu}/d\mu(x),$$

we can estimate (8.16), *à la* Monte Carlo, as

$$\int_X f\nu \wedge \overline{\nu} \simeq \frac{\mathrm{Vol}(X)}{\sum_{l=1}^{N_{points}} m_l} \sum_{l=1}^{N_{points}} f(q_l)m(q_l) + O\left(N_{points}^{-1/2}\right),$$

where $N_{points} \in \mathbb{N}$ is the number of points used and $O(N_{points}^{-1/2})$ is the standard error for large N_{points} using the central limit theorem.

In the particular case of a polarized manifold with a very ample line bundle $\mathcal{L}$, we generate the point set and the auxiliary measure $d\mu$, by taking projective embeddings $\iota\colon X \hookrightarrow \mathbb{P}H^0(X, \mathcal{L})^*$. We endow such projective space with a Fubini-Study metric ω_{FS} and consider random sections σ in $\mathbb{P}H^0(X, \mathcal{L})$ with respect to the volume form associated to the Fubini-Study metric. The zero locus of such random sections σ are divisors with associated zero currents Z_σ. One can show [SZ99] that the expected zero current is:

$$E(Z_\sigma) = \iota^*\omega_{FS}.$$

Therefore, the expected zero loci of n independent random sections in the projective space $\mathbb{P}H^0(X, \mathcal{L})^*$ are $\int_X c_1(\mathcal{L})^n$ points on X uniformly distributed under

$$E(Z_{\sigma_1\ldots\sigma_n}) = \frac{(\iota^*\omega_{FS})^n}{n!}.$$

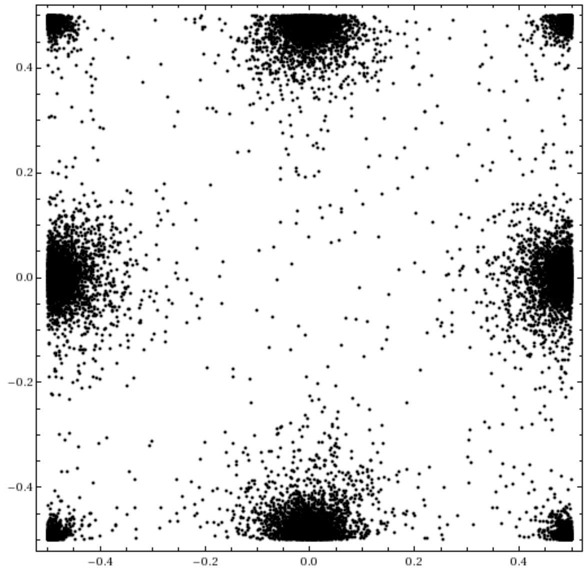


Figure 8.2: Distribution of random points on the Weierstrass cubic $Z_2^2 Z_0 = 4Z_1^3 - 60G_4(i)Z_1Z_0^2$, under the Fubini-Study metric defined by the Kähler potential $\log(1 + |Z_1/Z_0|^2 + |Z_2/Z_0|^2)$.

Thus, we take $d\mu = \iota^* \omega_{FS}^n / n!$ as the auxiliary measure and generate the points, uniformly distributed under $d\mu$, by taking the common zero loci of n independent random sections. The mass function is

$$m(x) = n! \frac{\nu \wedge \overline{\nu}}{(\iota^* \omega_{FS})^n}(x). \tag{8.17}$$

Example. Let us consider the elliptic curve E in $\mathbb{CP}^2$ defined as the zero locus of the Weierstrass cubic polynomial

$$Z_2^2 Z_0 = 4Z_1^3 - 60G_4(i)Z_1Z_0^2,$$

where $G_4(i)$, the Eisenstein series of index 4 evaluated at the complex parameter $i = \sqrt{-1}$, is $G_4(i) = -3.151212\ldots$. This elliptic curve can be seen as the square torus $\mathbb{C}/\mathbb{Z}^2$ embedded in $\mathbb{CP}^2$. The Calabi-Yau area form corresponds to the flat area form inherited from the complex plane $\mathbb{C}$, in the quotient $\mathbb{C}/\mathbb{Z}^2$. Intersections of three random sections in $\mathbb{CP}^2 = \mathbb{P}H^0(E, \mathcal{O}(3pts))$ are equivalent to intersections of the cubic $E \hookrightarrow \mathbb{CP}^2$ with random projective lines $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$. The Fubini-Study area form yields a particular distribution of points as shown in Fig. 8.2. We can still perform integrals with respect to $\nu \wedge \bar{\nu}$, because we have a precise formula for $\nu \wedge \bar{\nu}/(\iota^* \omega_{FS})^n$.

8.3.2.2 Refinements of the Monte Carlo method

When the ratio of the maximum over the minimum of the mass function (8.17), is very large, one expects a bad behavior of the Monte Carlo method just described. In this case, one can increase the number of points to try to approximate the integrals with more accuracy and precision. Also, one can use an optimal combination of several Fubini-Study metrics and subsets on X, to generate an Improved Points Set.

Remark 8.3.1. By an *Improved Points Set*, we mean a distribution of points on X whose associated mass formula $m_{IPS}(x)$ obeys

$$\frac{\max(m_{IPS}(x))}{\min(m_{IPS}(x))} \ll \frac{\max(m(x))}{\min(m(x))}.$$

For most applications, the most efficient strategy consists in working with a unique optimal Fubini-Study metric, and a point set generated by the intersection of independent random sections under the associated measure. The optimal Fubini-Study metric could be the Ω-balanced metric (see Definition 3.0.8), which is an accurate approximation to the Kähler Ricci-flat metric. The number of points should be adjusted to obtain the required accuracy and precision.

However, if we integrate functions whose evaluation maps are extremely slow from a numerical point of view, we won't be able to use a high number of points to reduce the error. In this case, we should improve the distribution of the point set, while using a constant number of points. The most obvious strategy consists in choosing several Fubini-Study metrics $\{\omega_{FS}^{[q]}\}_{q=1}^{q_{max}}$, and use the mass function

$$\begin{aligned} m_{IPS}(x) =& n!\frac{\nu\wedge\overline{\nu}}{(\iota^*\omega_{FS}^{[q]})^n}(x) \text{ for } 1\leq q\leq q_{max} \\ \text{such that } & \left| n!\frac{\nu\wedge\overline{\nu}}{(\iota^*\omega_{FS}^{[q]})^n}(x)-1\right| = \min_{1\leq j\leq q_{max}}\left| n!\frac{\nu\wedge\overline{\nu}}{(\iota^*\omega_{FS}^{[j]})^n}(x)-1\right| \end{aligned} \tag{8.18}$$

Here, we work with normalized volumes, $n!\int_X \nu\wedge\overline{\nu} = \int_X(\iota^*\omega_{FS}^{[q]})^n$. The mass function (8.18), implies that we can decompose X as a disjoint union of open subsets $X = \coprod_{q=1}^{q_{max}} U_q$ with non-zero volume. In other words, each Fubini-Study metric $\omega_{FS}^{[r]} \in \{\omega_{FS}^{[q]}\}_{q=1}^{q_{max}}$, defines a subset $U_r \subset X$:

$$x\in U_r \text{ if } m_{IPS}(x) = n!\frac{\nu\wedge\overline{\nu}}{(\iota^*\omega_{FS}^{[r]})^n}(x).$$

Therefore, in this second strategy, sets of n independent random sections with respect to the Fubini-Study volume form $\left(\omega_{FS}^{[q]}\right)^n$ yield random points

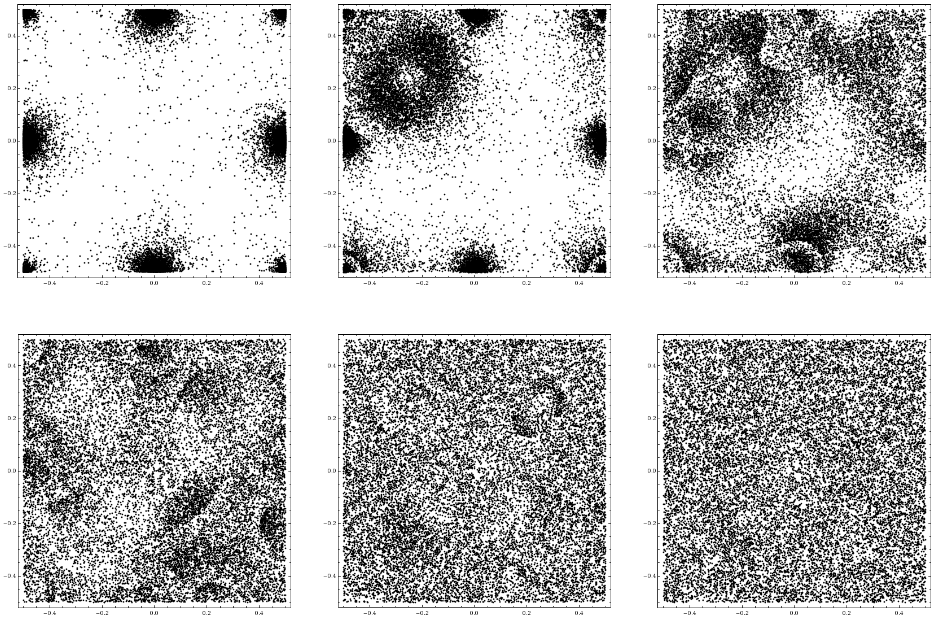


Figure 8.3: Distribution of 20,000 random points on the Weierstrass cubic, for 1, 2, 3, 5, 11, and 19 Fubini-Study metrics & subsets, optimally chosen.

on X; however, only those points that lie on $U_q \subset X$ are accepted. If a point $y \notin U_q$ is generated as common zero locus of n independent random sections under the q^{th}-measure, is rejected from the point set. This means that there exists another subset U_r and metric $\omega_{FS}^{[r]}$, with $y \in U_r$, such that n-tuples of random sections under the r^{th}-measure, generate points on U_r more closely distributed under $\nu \wedge \overline{\nu}$.

There is not a unique answer to the question of how to generate optimal sets of Fubini-Study metrics and subsets on X when $q_{max} > 1$. We propose a method that is useful in numerical applications. First we need a definition.

Definition 8.3.1 ([KelLuk12]). Given a point $x \in X$, there exists a *x-mass one* Fubini-Study metric $\omega_{FS}(\Lambda_x)$ on $\mathbb{P}H^0(X, \mathcal{L})^*$ associated to the hermitian matrix $\Lambda_x \in \mathrm{Met}(H^0(X, \mathcal{L}))$, that satisfies

$$n! \frac{\nu \wedge \overline{\nu}}{\iota^* \omega_{FS}(\Lambda_x)^n}(x) = 1.$$

Actually, the construction of $\omega_{FS}(\Lambda_x)$ goes as follows. Let us consider an orthonormal basis $\{s_\alpha\}_{\alpha=1}^{N+1}$ for $H^0(X, \mathcal{L})$ with respect to the ν-balanced Fubini-Study metric. Now, in this basis, we introduce the matrix Λ_x

$$\Lambda_x = \frac{1}{1+\epsilon} \left(\mathrm{Id} + \epsilon P_x \right),$$

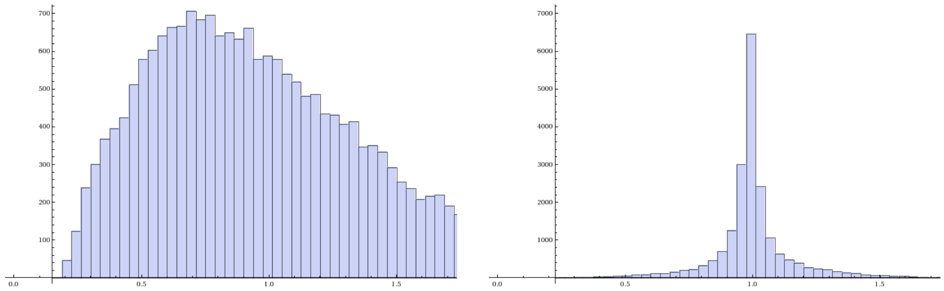


Figure 8.4: Distribution of masses for points on the Calabi-Yau quintic $Z_0^5+Z_1^5+Z_2^5+Z_3^5+Z_4^5-0.246\times Z_0Z_1Z_2Z_3Z_4$, using 1 Fubini-Study metric (left) and 19 Fubini-Study metrics & subsets, optimally chosen (right).

with Id the identity matrix, $P_x = P_x^2$ the projector on the ray generated by $x \mapsto \mathbb{P}H^0(X,\, \mathcal{L})^*$, and

$$\epsilon = \left(n!\frac{\nu \wedge \overline{\nu}}{\iota^*\omega_{FS}(\mathrm{Id})^n}(x)\right)^{\frac{1}{n}} - 1 \in \mathbb{R}.$$

It is then easy to show that if $\log\left(\sum_{\alpha\beta}\left(\Lambda_x^{-1}\right)^{\bar{\beta}\alpha} s_\alpha \bar{s}_{\bar{\beta}}\right)$ is the Kähler potential for $\omega_{FS}(\Lambda_x)$, then at the point x,

$$n!\frac{\nu \wedge \overline{\nu}}{\iota^*\omega_{FS}(\Lambda_x)^n}(x) = 1.$$

We can generate optimal sets of Fubini-Study metrics by combining iteratively the refined mass formula (8.18), and the *x-mass one* metrics. Given a set of Fubini-Study metrics $\{\omega_{FS}^{[q]}\}_{q=1}^{q_{max}}$ with $q_{max} > 0$, we can add two metrics to the set by searching for the absolute maximum x_{max} and minimum x_{min} of the mass function $m_{IPS}(x, \{\omega_{FS}^{[q]}\}_{q=1}^{q_{max}})$, and adding $\omega_{FS}(\lambda_{x_{min}})$ and $\omega_{FS}(\lambda_{x_{max}})$ to the set:

$$\{\omega_{FS}^{[q]}\}_{q=1}^{q_{max}+2} \text{ such that } \omega_{FS}^{q_{max}+1} = \omega_{FS}(\lambda_{x_{max}}),\ \omega_{FS}^{q_{max}+2} = \omega_{FS}(\lambda_{x_{min}}).$$

Figures 8.3 and 8.4 show a few examples of Improved Points Sets on the Weierstrass cubic defined above, and on a quintic threefold. Note that this refined algorithm for finding improved points set has independent interest.

8.3.3 Complexity of the algorithm

Note that to compute the Improved Point Sets over our Calabi-Yau manifold of dimension n, we need roughly to compute the determinant of a Bergman type metrics, so expressions of the form $\det(\sqrt{-1}\partial\bar{\partial}\log\sum_{i=1}^{N_r}|s_i|^2)$. If we are working let's say with sections of $\mathcal{O}_X(r)$ to embed our manifold, $N_r = h^0(\mathcal{O}_X(r))$, we need to evaluate over N_{points} points $N_r \simeq r^n$ sections of degree r, or degree $r-2$ (derivatives with respect to $\partial\bar{\partial}$ and of degree $r-1$ in n variables (derivatives with respect to ∂ and $\bar{\partial}$). Thus the complexity of this algorithm is bounded from above by a multiple of $n! q_{max} N_{points} N_r (rn + 2(r-1)n + (r-2)n)$ which is approximately

$$4n! q_{max} N_{points} r^{n+1}.$$

Since we are working in practice with small values of (n, r) it means that the complexity of this algorithm is essentially depending on the product of the number q_{max} of Improved Points Sets and the number of points N_{points} on the manifold.

8.3.4 Example: the family of quintics

Equipped with equations (8.9), (8.15) and the Monte Carlo integration technique just described, one can estimate Weil-Petersson metrics for a large class of families via deformations of n-holomorphic forms. For instance, one can compute the metric on the modulus of quintic threefolds introduced above,

$$P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4,$$

and studied independently by [Can+91]. By generating 2,000,000 points we evaluated the Weil-Petersson metric at the Fermat point $t = 0$:

$$g_{t\bar{t}}^{\text{W-P}}(0) = 0.19205 \pm 0.00104, \tag{8.19}$$

with 0.00104 the standard error. Equation (8.19) should be compared with the exact value $(0.1922\cdots)$, obtained by computing the volume of the quintic, as function of t, via integration of its 3-cycles, (8.6). In Fig. 8.5 we computed the Weil-Petersson metric in the same region of the t-plane studied in Fig. 8.1.

Another algorithm that allows to estimate the Weil-Petersson metric consists in evaluating the logarithm of numerical volumes for several threefolds near the manifold that we want to study, fitting a quadratic function for the values therein, and computing its Hessian. This method is highly inefficient despite is much simpler to implement. As an example, if we evaluate the function

$$\Psi_Q(t, \bar{t}) = -\log\left(\int_{Q_t} \nu_t \wedge \overline{\nu_t}\right), \tag{8.20}$$

for 300 random values of t near $t = 0$ with 100,000 points on each quintic threefold Q_t, and fit a quadratic function around the Fermat point, we find

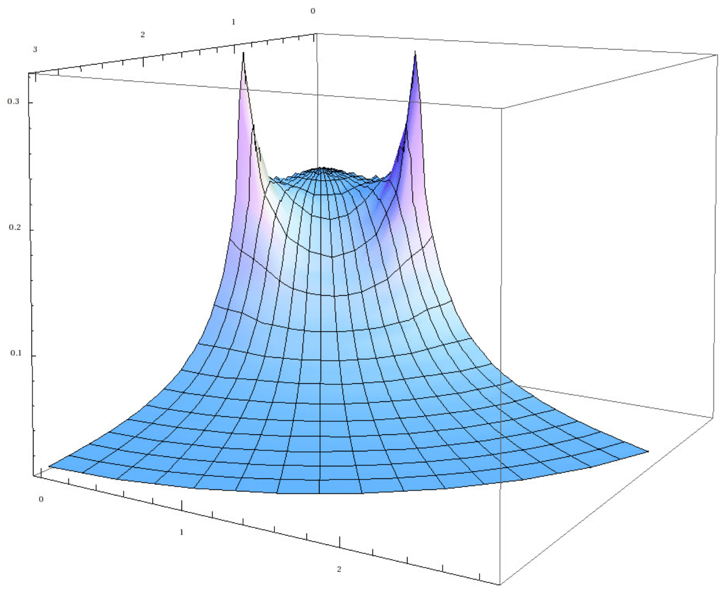


Figure 8.5: Approximation of the Weil-Petersson metric (vertical axis) on the t-plane (horizontal plane) of 1-dimensional moduli space of Calabi-Yau quintic threefolds, $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$, using local deformations of the holomorphic form and Monte Carlo integration.

that the Hessian at $t = 0$ is

$$g_{t\bar{t}}^{\text{W-P}}(0) = 0.209693 \pm 0.03.$$

In other words, by using 15 times more points than in (8.19), we evaluate $g_{t\bar{t}}^{\text{W-P}}(0)$ with an error 30 times bigger. In Fig. 8.6 we represent the graph of the fitted function (8.20), for 300 points on the t-plane.

8.4 Numerical evaluation of the W-P metrics via Donaldson's quantization approach

8.4.1 Quantized Weil-Petersson metric for constant scalar curvature Kähler metrics

As we explained in details in Section 2.4, there is a quasi-projective scheme $Hilb(N, \chi)$, the Hilbert scheme of subschemes of $\mathbb{P}^N$ with fixed Hilbert polynomial χ. From the natural action of $SL(N+1, \mathbb{C})$ over $\mathbb{P}^N$, the group $SL(N+1, \mathbb{C})$ will act equivariantly on $Hilb(N, \chi)$ and the universal family $Univ_{N,\chi}$. In this section, we restrict our attention to the smooth orbits of the moduli space $\mathcal{M}_{N,\chi} = Hilb(N, \chi)^{ps}//SL(N+1)$. In the case of

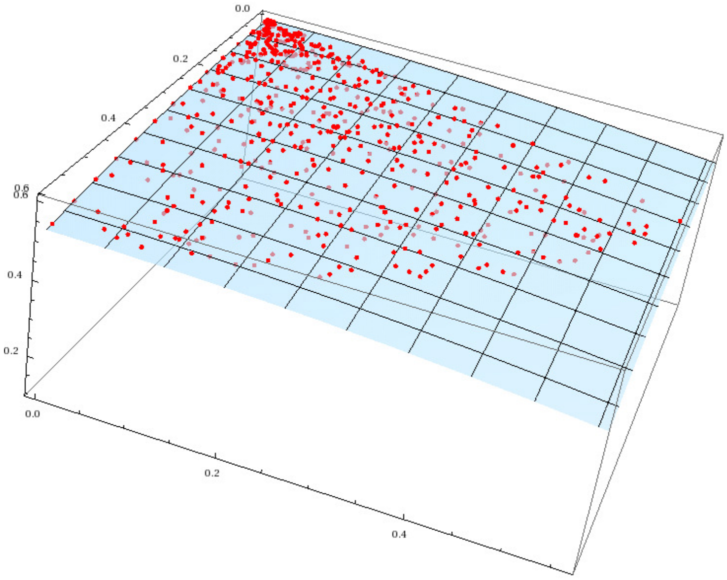


Figure 8.6: Quadratic fit of the function Ψ_Q for 300 random Calabi-Yau quintic threefolds on the t plane near $t = 0$.

polarized Calabi-Yau manifolds, the dimension of this quotient is given by $\dim H^1(X, TX)_L$, i.e elements of $H^1(X, TX)_L$ that keeps the polarization L invariant. Thus, $\mathcal{M}_{N,\chi}$ is strictly included in the Kuranishi space of deformations of the manifold, leaving some non-algebraic deformations. But, on the other hand, let us consider $\mathcal{M}_H$ the moduli space of isomorphisms classes of polarized Hodge manifolds $(X, \mathcal{L}_X)$ with $(X, c_1(\mathcal{L}_X))$ diffeomorphic to (X_0, α) with $\alpha \in H^2(X_0, \mathbb{R})$. Then, in the particular case of Calabi-Yau's, the moduli space $\mathcal{M}_H$ carries a structure of orbifold complex space and $\mathcal{M}_{N,\chi} \subset \mathcal{M}_H$ is open and closed in $\mathcal{M}_H$ [FS90, Section 5 and 11].

On $\mathcal{M}_{N,\chi}$, we shall see that there exists a quantized Weil-Petersson metric obtained from (2.1) by restriction. If X carries a cscK metric in the class of L, then we know that there exists a convergent sequence of balanced metrics in $c_1(L)$, where each element of the sequence corresponds to a point in $\mathcal{M}_{N,\chi}$. For our purpose, we consider local algebraic deformations of the complex structure on X, which correspond to tangent vectors in $\mathcal{M}_{N,\chi}$. An integrable complex structure $J \in \mathcal{J}_{int}$ on X can be deformed by any element $v \in \Omega^{0,1}T^{(1,0)}X_J$. We can be more precise by fixing the differentiable structure on X and the complex line bundle L^k, and considering integrable complex structures on X with corresponding Dolbeault operators on L^k. If L^k is a fixed complex line bundle on X and J is a complex structure on X, we say that $\mathcal{L}^k$ is the associated holomorphic line bundle on X_J endowed with a Dolbeault operator $\bar{\partial} = \bar{\partial}_J$. The deformation of the complex structure $J + v$

on X induces a deformation of the Dolbeault operator $\bar\partial = \bar\partial_J \colon \Omega^{p,q}(\mathsf{L}^k) \to \Omega^{p,q+1}(\mathsf{L}^k)$ as

$$\bar\partial_{J+v} = \bar\partial + v\partial + O(v^2).$$

With h a Hermitian metric on L, we obtain a L^2 metric on $\Omega^0(X, \mathsf{L}^k)$; we denote by $L^2(X, \mathsf{L}^k)$ its L^2 completion. If the dimension $\dim\ker\overline{\partial}_J = N+1$, is constant as the integrable complex structure J varies on $\mathcal{J}_{int}$, there is a natural embedding of $\mathcal{J}_{int}$:

$$\begin{aligned} \tau \colon \mathcal{J}_{int} &\to \mathrm{Gr}(N+1, L^2(X, \mathsf{L}^k)), \\ J &\mapsto \ker\overline{\partial}_J \subset L^2(X, \mathsf{L}^k) \end{aligned} \tag{8.21}$$

with $\mathrm{Gr}(N+1, L^2(X, \mathsf{L}^k))$ the Grassmannian of $N+1$ planes in $L^2(X, \mathsf{L}^k)$. If (s_α) is an orthonormal basis of sections for the finite dimensional vector space $\ker\overline{\partial}_J$, the infinitesimal deformation v is pushforwarded to a tangent vector on the Grassmannian $\mathrm{Gr}(N+1, L^2(X, \mathsf{L}^k))$ under (8.21). The induced vector can be computed as the infinitesimal deformation of the basis of sections $\{s_\alpha\}$. Thus, if $\{s_\alpha + \delta s_\alpha\}$ is a basis for $\ker\overline{\partial}_{J+v}$, then

$$\left(\overline{\partial}_J + v\partial + O(v^2)\right)(s_\alpha + \delta s_\alpha) = 0$$

and $\delta s_\alpha = -\overline{\partial}_J^{-1}(v \lrcorner \partial s_\alpha)$, neglecting $O(v^2)$ corrections. One has to define the inverse $\overline{\partial}_J^{-1}$ properly on the orthogonal complement $\ker\overline{\partial}_J^{\perp} \subset L^2(X, \mathsf{L}^k)$. As δs_α also denotes a tangent vector to $\mathrm{Gr}(N+1, L^2(X, \mathsf{L}^k))$ in homogeneous coordinates, the $(N+1)$-plane spanned by the $\overline{\partial}_J^{-1}(v \lrcorner \partial s_\alpha)$ in $L^2(X, \mathsf{L}^k)$, is orthogonal to $\ker\overline{\partial}_J$. Therefore, one can alternatively define (2.2) as follows.

Definition 8.4.1 (Quantized Weil-Petersson metric [KelLuk12])**.** For $H_t \in \mathrm{Met}(H^0(X_t, L^k))$ the balanced metric at level k in the sense of Definition 2.4.1, and for $v_1, v_2 \in \Omega^{0,1}T^{(1,0)}X$ representatives of $v_i \in T_{J_0}\mathcal{J}_{int} \simeq H^{0,1}_{\bar\partial}(TX_t)$ under the Kodaira-Spencer map, we define the quantized Kähler form $\Omega_k^{\text{W-P}}$ in $\Omega^{1,1}(\mathcal{T})$ as[1],

$$\Omega_k^{\text{W-P}}(v_1, v_2) = k^n H_t^{\alpha,\beta} \int_{X_t} \langle \bar\partial^{-1}(v_1 \lrcorner \partial s_\alpha), \bar\partial^{-1}(v_2 \lrcorner \partial s_\beta)\rangle_{FS(H_t)} \frac{(\frac{1}{k}c_1(FS(H_t))^n}{n!} \tag{8.22}$$

for $\{s_\alpha\}$ an orthonormal basis with respect to H_t. The definition is independent of the choice of the basis $\{s_\alpha\}$ and of the scaling of the balanced metric.

Inspired from Donaldson's results, we expect that as k tends to infinity, the sequence of Kähler forms $\Omega_k^{\text{W-P}}$ converges to the Weil-Petersson metric

[1] Remark that in the formula below the ∂ operator corresponds to the $(1,0)$ part of the connection on L^k so depends on the balanced fibrewise metric. Thus, the terms $v_1 \lrcorner \partial s_\alpha, v_2 \lrcorner \partial s_\beta$ can be also written as $v_1 \lrcorner \nabla_{FS(H_t)} s_\alpha, v_2 \lrcorner \nabla_{FS(H_t)} s_\beta$.

defined in Equation (8.3), locally uniformly, over the smooth points of the moduli space $\mathcal{M}_{N,\chi}$.

8.4.2 Quantized Weil-Petersson metric for projective Calabi-Yau manifolds

Of course Kähler Ricci-flat metrics can be seen as constant scalar curvature Kähler metrics. Nevertheless, if we consider a holomorphic family of projective Calabi-Yau manifolds, we can slightly modify the notion of quantized Weil-Petersson metric of Definition 8.4.1, by substituting the Fubini-Study volume form by $\Omega = \nu \wedge \overline{\nu}$. This modification has important advantages in numerical applications. We refer to Definition 3.0.8 for the definition of Ω-balanced metric.

Definition 8.4.2 ([KelLuk12])**.** With the notations of Section 8.2.1, let us consider $H_t \in \mathrm{Met}(H^0(X_t, L^k))$ to be the Ω_t-balanced metric at level k for k sufficiently large with

$$\Omega_t = \nu_t \wedge \overline{\nu_t}.$$

For $v_1, v_2 \in \Omega^{0,1}T^{(1,0)}X_t$, one has a Kähler form on $\mathcal{T}$ defined as

$$\Omega_k^{\text{W-P}}(v_1, v_2) = k^n H_t^{\alpha,\beta} \int_{X_t} \langle \bar{\partial}^{-1}(v_1 \lrcorner \partial s_\alpha), \bar{\partial}^{-1}(v_2 \lrcorner \partial s_\beta)\rangle_{FS(H_t)} \nu_t \wedge \bar{\nu}_t \quad (8.23)$$

for $\{s_\alpha\}$ an orthonormal basis with respect to H_t. Making an abuse of notation, we also call $\Omega_k^{\text{W-P}}$ the quantized Weil-Petersson metric at level k, as in Equations (8.22).

As $\omega_{\infty,t}^n = \nu_t \wedge \bar{\nu}_t$, with $\omega_{\infty,t} \in c_1(L)$, is the Calabi-Yau metric, we know from the construction of the balanced metrics (more precisely from the asymptotic of the Bergman kernel [MM07, Remark 5.1.5]) that

$$\|\omega_{FS(H_t)} - \omega_{\infty,t}\|_{C^\infty} = O\left(\frac{1}{k^2}\right).$$

In particular, $\frac{n!\omega_{FS(H_t)}^n}{\nu_t \wedge \bar{\nu}_t} = 1 + O\left(\frac{1}{k^2}\right)$, and therefore, we expect using [Don09; Kel09] that $\Omega_k^{\text{W-P}}$ converges to the Weil-Petersson metric of Definition 8.2.1.

8.4.3 Algorithm for numerical computation of the W-P metrics using Quantization

In [Don09], Donaldson showed how to numerically compute *balanced metrics* and Ω-balanced metrics, in order to approximate cscK metrics on varieties and Kähler Ricci-flat metrics on Calabi-Yau manifolds. Such metrics can be constructed explicitly if one has analytic control on the projective embeddings and knows how to evaluate integrals. The same technical difficulties arise if we want to evaluate quantized Weil-Petersson metrics on moduli spaces (8.23).

One can find the balanced metric on $X_{t_0} \hookrightarrow \mathbb{P}^N$ by introducing any initial definite positive Hermitian matrix $H(0)$ on $H^0(X_{t_0}, \mathcal{L}^k)$, and iterating the map, $T_{\Omega_{t_0},k}\colon \mathrm{Met}(H^0(\mathcal{L}^k)) \to \mathrm{Met}(H^0(\mathcal{L}^k))$

$$H(q+1)_{\alpha\bar{\beta}} = T_{\Omega_{t_0},k}(H(q))_{\alpha\bar{\beta}} \tag{8.24}$$

$$= \frac{N+1}{\mathrm{Vol}_L(X_{t_0})}\int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{(H(q)^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}}\Omega_{t_0}, \tag{8.25}$$

up to reaching convergence with the limit point $H_{t_0} = H(+\infty)$.
Let us denote as before t_a an infinitesimal deformation on the moduli space and $v \in T_{t_0}\mathcal{T}$. In order to find the infinitesimal deformation of the balanced embedding for the manifold $X_{t_0+t_a v^a} = X_{t_0+tv}$ into $\mathbb{P}^N$ and be able to evaluate (8.23), it is convenient to work with a family of line bundles on X. If $\pi\colon \mathcal{X} \to \mathcal{T}$ denotes a family of complex structures on X, with $\pi^{-1}(t) = X_t$, there exists a holomorphic line bundle $\mathcal{L}^k \to \mathcal{X}$ such that $\mathcal{L}^k|_t = \mathcal{L}_t^k \to X_t$. In other words, the restriction of $\mathcal{L}^k$ to the fibers of $\pi\colon \mathcal{X} \to \mathcal{T}$ is identical to the holomorphic polarization $\mathcal{L}_t^k$ on X_t. The natural Hermitian structure on $\mathcal{L}_t^k \to X_t$ whose curvature is the corresponding Kähler Ricci-flat metric, lifts to a hermitian structure on $\mathcal{L}^k \to \mathcal{X}$. When we approximate the Kähler Ricci-flat metrics on X_t by balanced metrics, $\mathcal{L}^k \to \mathcal{X}$ also admits a compatible hermitian structure. More precisely, if $\{s_\alpha(t,\bar{t}) = \eta_\alpha(t,\bar{t})\hat{e}_t\}_{\alpha=1}^{N+1}$ is a basis of holomorphic sections for $H^0(X_t, \mathcal{L}^k)$, $\hat{e}_t$ is the holomorphic frame in a local trivialization, the parameters t denote the moduli dependence, and $H_t = H_{(t,\bar{t})}$ is the associated balanced matrix, we endow $\mathcal{L}^k \to \mathcal{X}$ with the hermitian metric

$$h_t = \frac{\hat{e}_t \otimes \hat{e}_t^*}{(H_t^{-1})^{\bar{\gamma}\delta} s(t)_\delta \bar{s}(\bar{t})_{\bar{\gamma}}}. \tag{8.26}$$

Therefore, given the diffeomorphism between X_{t_0} and X_{t_0+tv}, defined in local holomorphic coordinate charts (8.7), as

$$y^i = w^i + v^a \vartheta_a^i(w,\bar{w}) + O(v^2),$$

one can compute the infinitesimal deformation of the embedding $X_{t_0+tv} \hookrightarrow \mathbb{P}^N$, as the covariant derivative of $s_\alpha(t)$

$$\nabla_v \eta_\alpha \hat{e}_t = v^a \frac{\partial \eta_\alpha}{\partial t_a}\hat{e}_t + v^a h_t^{-1}\frac{\partial h_t}{\partial t_a}\eta_\alpha \hat{e}_t,$$

where $\frac{\partial \eta_\alpha}{\partial t_a} = \frac{\partial \eta_\alpha}{\partial w^i}\vartheta_a^i(w,\bar{w})$. In other words, if $\hat{e}(y)$ is a holomorphic frame for $\mathcal{L}_{t_0+tv}^k \to X_{t_0+tv}$, we can write the basis of holomorphic sections as

$$s_\alpha(y) = \eta_\alpha(y)\hat{e}(y) = \eta_\alpha(w)\hat{e}(w) + \nabla_v \eta_\alpha(w,\bar{w})\hat{e}(w) + O(v^2).$$

The proof is straightforward. Thus, $\nabla_v \eta_\alpha \hat{e}$ are smooth sections in $L^2(X_{t_0}, \mathcal{L}^k)$ that represent components of vector fields along $T^{1,0}|_{X_{t_0}}\mathbb{P}^N$. The sections $\nabla_v \eta_\alpha \hat{e} \in L^2(X_{t_0}, \mathcal{L}^k)$, can be expressed as the sum of a holomorphic section plus a non-holomorphic section, because of the decomposition

$$L^2(X_{t_0}, \mathcal{L}^k) = H^0(X_{t_0}, \mathcal{L}^k) \oplus H^0(X_{t_0}, \mathcal{L}^k)^{\perp}$$

under the L^2-metric induced by the $Hilb_{\Omega_{t_0}}$ map. As we need the normal components of $\nabla_v \eta_\alpha \hat{e}$ to $H^0(X_{t_0}, \mathcal{L}^k)$, we have to project out the holomorphic part, $P_{t_0} \nabla_v \eta_\alpha \hat{e} \in H^0(X_{t_0}, \mathcal{L}^k)$. The holomorphic part of $\nabla_v \eta_\alpha \hat{e}$ is computed using the Bergman kernel projector $P_{t_0} \colon L^2(X_{t_0}, \mathcal{L}^k) \to H^0(X_{t_0}, \mathcal{L}^k)$. In an orthonormal basis $\{s'_\alpha\}_{\alpha=1}^{N+1}$ one can express P_{t_0} as

$$P_{t_0}(\sigma) = \frac{N+1}{\mathrm{Vol}_L(X_{t_0})} \sum_\alpha s'_\alpha \int_X \frac{\sigma \overline{s}'^{\alpha}}{(H_{t_0}^{-1})^{\bar{\gamma}\delta} s'_\delta \overline{s}'_{\gamma}} \nu_{t_0} \wedge \overline{\nu}_{t_0}. \tag{8.27}$$

Therefore, the term $(\mathrm{Id} - P_{t_0})\nabla_v \eta_\alpha \hat{e}$ denotes the projection of $\nabla_v \eta_\alpha \hat{e}$ onto the orthogonal complement in $\Gamma\left(T^{1,0}|_X \mathbb{P}^N\right)$ of the subspace defined by the infinitesimal action of $GL(N+1, \mathbb{C})$, which is isomorphic to $H^0(X_{t_0}, \mathcal{L}^k)$. Hence, the quantized Weil-Petersson metric (8.23) can be written as:

$$\Omega_k^{\text{W-P}}(v_1,\, v_2) = \left(H_{t_0}^{-1}\right)^{\bar{\beta}\alpha} \int_X \frac{((\mathrm{Id} - P_{t_0})\nabla_{v_1}\eta)_\alpha \overline{((\mathrm{Id} - P_{t_0})\nabla_{v_2}\eta)_{\bar{\beta}}}}{(H_{t_0}^{-1})^{\bar{\gamma}\delta} \eta_\delta \bar{\eta}_{\bar{\gamma}}} \nu_{t_0} \wedge \overline{\nu}_{t_0}. \tag{8.28}$$

Here, the basis of sections $\{s_\alpha = \eta_\alpha \hat{e}\}_{\alpha=1}^{N+1}$ is not necessarily orthonormal, although due to the simplicity of P_{t_0} when expressed in an orthonormal basis (8.27), it is convenient to work with $\{s_\alpha\}_{\alpha=1}^{N+1}$ orthonormal (where $H_{t_0} = \mathrm{Id}$).

Thus, evaluating (8.28) involves the following algorithm [KelLuk12]:

1. Building an explicit basis of holomorphic sections $\{s_\alpha = \eta_\alpha \hat{e}\}_{\alpha=1}^{N+1}$ for $H^0(X_{t_0},\, \mathcal{L}^k)$, with $k = 1, \ldots, k_{max}$, and k_{max} some maximum value of k that one can handle numerically.

2. Developing a numerical algorithm to evaluate integrals on X_{t_0} under the measure $\nu_{t_0} \wedge \overline{\nu}_{t_0}$, as we did in Section 8.3.2.

3. Computing the balanced metric H by iterating the $T_{\Omega_{t_0},k}$ map.

4. Choosing a basis of infinitesimal diffeomorphisms $\vartheta_a^i(w, \bar{w})$ on X_{t_0}, isomorphic to the basis $\frac{\partial}{\partial t_a}$ for $T_{t_0}\mathcal{T} \simeq H^1(X_{t_0}, \Omega^{n-1})$.

5. Solve the linearized balanced equations for $\partial_{t_a} H_{\alpha\bar{\beta}}$ at $t = t_0$:

$$\begin{aligned}\frac{\partial H_{\alpha\bar{\beta}}}{\partial t_a} =& c_0 \int_X \frac{\nabla_a \eta_\alpha \hat{e} \bar{s}_{\bar{\beta}}}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \nu_{t_0} \wedge \overline{\nu}_{t_0} + c_0 \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \frac{\partial \nu_t}{\partial t_a} \wedge \overline{\nu}_{t_0} \\ & - \frac{c_0}{\mathrm{Vol}_L(X_{t_0})} \int_X \frac{\partial \nu_t}{\partial t_a} \wedge \overline{\nu}_{t_0} \times \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{(H^{-1})^{\bar{\gamma}\delta} s_\delta \bar{s}_{\bar{\gamma}}} \nu_{t_0} \wedge \overline{\nu}_{t_0},\end{aligned} \tag{8.29}$$

where $c_0 = \frac{N+1}{\mathrm{Vol}_L(X_{t_0})}$. We obtain these relations by differentiating the condition on H to be a fixed point of the $T_{\Omega_{t_0},k}$ map and the fact that the Bergman function associated to the balanced metric is constant. The system (8.29) is a non-trivial system of linear equations, as $\partial_{t_a} H_{\alpha\bar{\beta}}$ is contained in the $\nabla_a \eta_\alpha$ term. One can solve (8.29) by using Gauss' elimination method, or one can solve it iteratively by setting $\partial_{t_a} H_{\alpha\bar{\beta}}(q = 0) = 0$ as initial value and interpreting (8.29) as a linearized $T_{\Omega_{t_0},k}$ map.

6. Computing $\nabla_a \eta_\alpha$, given $\frac{\partial H_{\alpha\bar{\beta}}}{\partial t_a}$, and its projection $(\mathrm{Id} - P_{t_0})\nabla_a \eta_\alpha$ using (8.27).

7. Finally: Evaluating the inner products (8.28).

8.4.4 Complexity of the algorithm

Let us denote $N \simeq k^n$ as above the number of sections in $H^0(X_{t_0}, L^k)$ where n is the dimension of the manifold.
Firstly, we have already computed the complexity of Step 2 in Subsection 8.3.3.
Moreover, we notice that we need to fix N_{points} points on the manifold with $N_{points} > (N+1)(N+2)/2$ since we are need to compute balanced metrics and thus we need to solve $(N+1)(N+2)/2$ equations at least. In practice, we believe that

$$N_{points} \simeq k^{2n}$$

is a reasonable choice. To compute the balanced metrics (Step 3), we need to inverse a matrix of size $N \times N$ in order to get a basis of orthonormal sections. This requires approximately $N^2 \log(N) \simeq nk^{2n} \log(k)$ operations. We also need to compute the Bergman function $\sum_{i=1}^N |s_i|^2$ for the N_{points} points, and this can be seen as to evaluate N polynomials of degree k in n variables.
Thus each iteration of the $T_{\Omega_{t_0},k}$ map has complexity

$$C(T_{\Omega_{t_0},k}) = nk^{2n} \log(k) + k^{2n} k^n (kn) \simeq nk^{3n+1},$$

since we need to do k products to evaluate the different powers for each of the n variables. From [Don09], we know that we have exponential speed of convergence of the iterates of the $T_{\Omega_{t_0},k}$ map. This means that if we expect an error of size ϵ for the convergence of these iterates, one needs approximately $k\log\left(\frac{1}{\epsilon}\right)$ iterations of the $T_{\Omega_{t_0},k}$ map. Since, the balanced metric is close to the Ricci-flat metric with an error of size $O(1/k^2)$ (this can be proved from the Bergman asymptotic expansion), it is relevant to assume that $\epsilon \simeq 1/k^2$. In that case, the complexity of Step 3 is approximately equivalent to

$$C(\text{Step 3}) \propto \log(k)k^{3n+1}n.$$

For Step 5, we need to compute the term

$$c_0\int_X \frac{1}{(H^{-1})^{\bar\gamma\delta}s_\delta\bar s_{\bar\gamma}}\left(\nabla_a\eta_\alpha\hat e\bar s_{\bar\beta} + \left(\frac{\frac{\partial\nu_t}{\partial t_a}\wedge\overline{\nu}_{t_0}}{\nu_{t_0}\wedge\overline{\nu}_{t_0}} - \frac{\int_X\frac{\partial\nu_t}{\partial t_a}\wedge\overline{\nu}_{t_0}}{\mathrm{Vol}_L(X_{t_0})}\right)s_\alpha\bar s_{\bar\beta}\right)\nu_{t_0}\wedge\overline{\nu}_{t_0}$$

we remark that we need to evaluate the Bergman function over the N_{points} points (so as above, $k^{2n}k^n n$ operations are required) and few extra terms of the same order of complexity like $\nabla_a\eta_\alpha\hat e$. The terms involving the volume form have complexity $n!N_{points}$ so are negligible. To take into account that this step requires more operations than for the integrand in the $T_{\Omega_{t_0},k}$ map (this extra depends on the dimension of the manifold and not k), we estimate that its complexity is similar to n^2k^{3n}. Furthermore the resolution of this system of $N\times N$ equations will require approximately $nk^{2n}\log(k)$ operations at least. In practice, we used in the examples (see Section 8.4.5) an iterative process that has exponential speed of convergence (note that we have not studied formally the rate of convergence). Thus, Step 5 has approximately for complexity

$$C(\text{Step 5}) \propto \log(k)k^{3n+1}n^2.$$

For similar reasons, the complexity of Step 6 and 7 are bounded from above by $k^{3n+1}n$ and don't contribute much in the total complexity of the algorithm.
For instance, since the quantized Weil-Petersson metric approximates the Weil-Petersson metric with an error of size $O(1/k)$, for a relative precision of order 10% on a manifold of dimension 3 we need $\sim 3\times 10^{11}$ operations. It is reasonable for a recent computer. In dimension 2, it is also reasonable to ask a precision of 1%. The size of memory required to run the algorithm is not an issue in dimension 2 for $k\leq 100$. In complex dimension 3, it is necessary to use the symmetries to reduce the number of parameters.

8.4.5 Example: the family of quintics

The results of this section have to be compared with Section 8.3.4. We implemented the algorithm that we have just described, for the family of quintic threefolds Q_t in $\mathbb{P}^4$ defined by the polynomial family

$$P(Z) = Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4.$$

We studied the region of the t-plane given by $0 < |t| \leq 3$ and $0 \leq \arg(t) < 2\pi/5$, as we did in the examples of sections 2 and 3. We divided the region in a lattice of more than 300 points, and computed the corresponding balanced metrics for embeddings in linear spaces of sections, up to degree $k = 6$. We chose monomials of degree k defined on $\mathbb{P}^4$, modulo the ideal generated by $P(Z)$, as the basis of sections. We evaluated the integrals that appear in the $T_{\Omega_{t_0},k}$ map (8.25) by using the Monte Carlo method described in Section 3. In order to compute the variation of the sections $\frac{\partial s_\alpha}{\partial t}$, we used the infinitesimal diffeomorphism defined by (8.11), with

$$\frac{\partial \eta_\alpha}{\partial t} = \sum_{i=1}^{4} \frac{\partial \eta_\alpha}{\partial w_i}\frac{\partial w_i}{\partial t} = \sum_{i=1}^{4} \frac{\partial \eta_\alpha}{\partial w_i}\vartheta^i(w,\bar{w}). \tag{8.30}$$

For this family of quintics, Proposition 8.3.1 that defines a choice of vector field $\vartheta^i(w,\bar{w})$ provides

$$\vartheta^i(w,\bar{w}) = -\frac{G^{i\bar{j}}\frac{\partial \bar{p}(\bar{w})}{\partial \bar{w}_{\bar{j}}}}{G^{m\bar{n}}\frac{\partial \bar{p}(\bar{w})}{\partial \bar{w}_{\bar{n}}}\frac{\partial p(w)}{\partial w_m}}(-5w_1w_2w_3w_4),$$

with $G^{i\bar{j}}$ the inverse of the Fubini-Study metric in $\mathbb{P}^4$, and $w_i = Z_i/Z_0$ local coordinates on $Q_t \subset \mathbb{P}^4$. Given the Hermitian metric h_t from Equation (8.26) we can compute the covariant derivative $\nabla_t \eta_\alpha \hat{e}$ as

$$\begin{aligned}\nabla_t \eta_\alpha \hat{e} =& \sum_{i=1}^{4} \frac{\partial \eta_\alpha}{\partial w_i}\vartheta^i(w,\bar{w})\hat{e} \\ &- \frac{\partial}{\partial t}\left((H_t^{-1})^{\bar{\gamma}\delta}\eta(t)_\delta\bar{\eta}(\bar{t})_{\bar{\gamma}}\right)\frac{\eta_\alpha\hat{e}}{(H_t^{-1})^{\bar{\gamma}\delta}\eta(t)_\delta\bar{\eta}(\bar{t})_{\bar{\gamma}}},\end{aligned}$$

which we computed by using (8.30), and solving the linearized balanced equations (8.29) for $\frac{\partial H_{\alpha\bar{\beta}}}{\partial t}$. The method that we implemented to solve the linearized balanced equations, consisted in iterating the linearized $T_{\Omega_{t_0},k}$ map; such iterating scheme reached good estimates of the solutions within 5 or 6 iterations.

In Fig. 8.7 we plot the sequence of metrics $\Omega_k^{\text{W-P}}$, defined in (8.28), for $k = 1, 2, \ldots, 6$. We present all the plots computed at the previous steps in

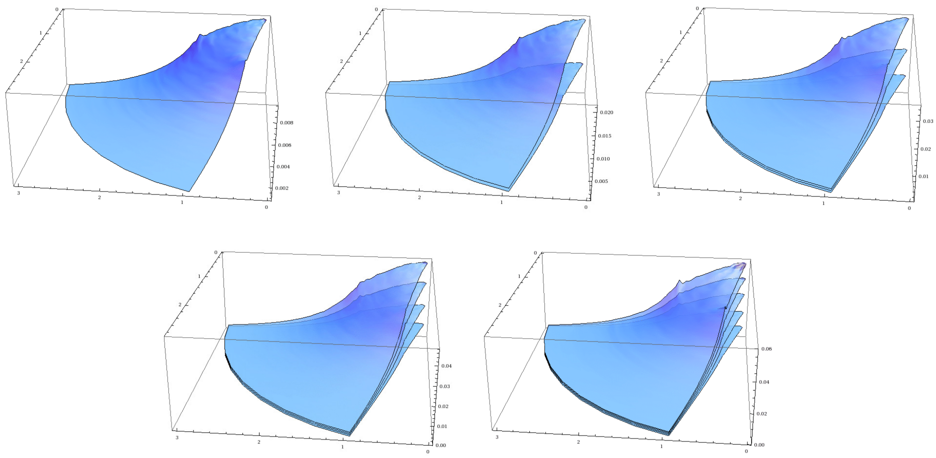
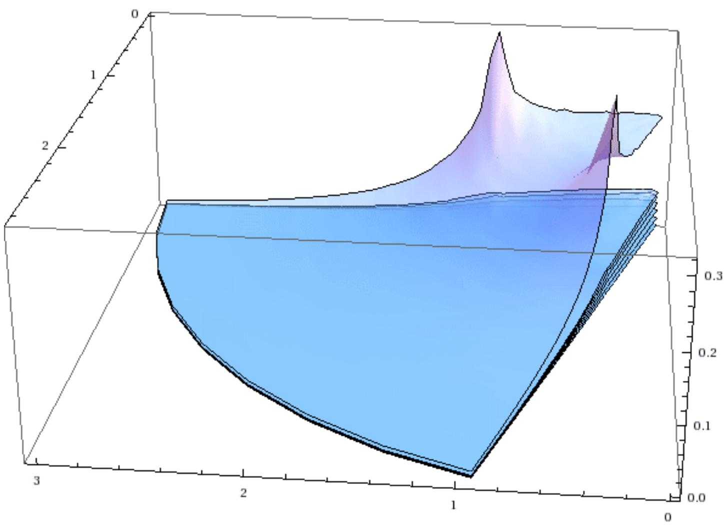


Figure 8.7: Quantized Weil-Petersson metrics (vertical axis) on the t-plane (horizontal plane), of Calabi-Yau quintic 3-folds $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$, for $k = 1, 2, 3, 4, 5$. Below, a zoom on the result for $k = 6$.



order to show the convergence and the presence of the singularities that appear clearly for $k = 6$. The time that took to compute each value $\Omega_k^{\text{W-P}}(t, \bar{t})$ per point in the t-plane, was approximately equal to 4 times the time needed to compute the balanced metric. One can observe that for $|t|$ large, the rate of convergence of the sequence is higher than in other regions of the t-plane. In points near the Fermat quintic, $t = 0$, and for $k = 6$, the quantized Kähler metric is approximately $(0.07\cdots)$ vs the exact value $(0.19\cdots)$. One expects deviations smaller than 0.01 in this region of the t-plane, when $k > 12$. The worst rate of convergence is located near the points $t = \exp(2\sqrt{-1}\pi\mathbb{Z}/5)$, where the quintic develops double point singularities. In such region of the family, the approximation of the corresponding Ricci-flat metric by Ω-balanced metrics, is also much less accurate. One should develop further techniques using weighted projective spaces to approximate accurately the metric near singular points of the moduli space.

One can explain intuitively why this scheme cannot be accurate near singular points. In Kähler geometric quantization, the limit $k \to +\infty$ of the quantum systems, associated to the Planck's constant $\hbar = \text{Vol}_\omega(X)^{1/n}/k$, corresponds to the semiclassical system (X, ω). Due to quantum uncertainty in regions of volume smaller than $\hbar^n$, one expects that accurate approximations of geometric features in X occur when the size of such features are located is actually bigger than $\frac{\text{Vol}_\omega(X)}{k^n}$. Therefore, as a singularity is a geometric object of zero volume, these numerical constructions should fail near singularities.

8.4.6 Extra remarks

Another – more difficult and slower – way to approximate Weil-Petersson metrics involves to evaluate the Weil-Petersson formula itself on a family of balanced metrics; instead of a family of Kähler-Einstein metrics on varieties or Hermite-Einstein metrics on bundles. In other words, instead of using (8.28) to approximate Weil-Petersson metrics one could evaluate

$$\Upsilon_k(v_1, v_2) = \frac{1}{\text{Vol}(X_t)} \int_X v_1^a \bar{v}_2^{\bar{b}} g_t^{i\bar{j}} \frac{\bar{\partial}}{\partial \overline{w}_{\bar{j}}} \left(h_t^{-1} \frac{\partial h_t}{\partial t_a} \right) \left(\frac{\bar{\partial}}{\bar{\partial} \overline{w}_{\bar{\imath}}} \left(h_t^{-1} \frac{\partial h_t}{\partial t_b} \right) \right)^* \nu_t \wedge \bar{\nu}_t, \tag{8.31}$$

with h_t the family of balanced metrics defined in (8.26). As the formula (8.31) would become the Weil-Petersson metric if h_t was Hermite-Einstein, as in [ST92], one expects that if h_t is balanced, (8.31) should converge to the Weil-Petersson metric in the $k \to \infty$ limit.
We have implemented an algorithm to compute this metric on the family of quintics that we have studied in this chapter. As one can see in Equation (8.31), it is slightly more difficult to implement this formula numerically due to the higher number of derivatives. Also, the numerical calculation itself is much slower in comparison with a numerical evaluation of (8.28).

For instance, to compute the metric (8.31) for $k = 3$ took as much time as computing (8.28) for $k = 6$. Due to problems with the speed of the numerical calculation, we decided not to resume a detailed analysis of a numerical method based on Equation (8.31).

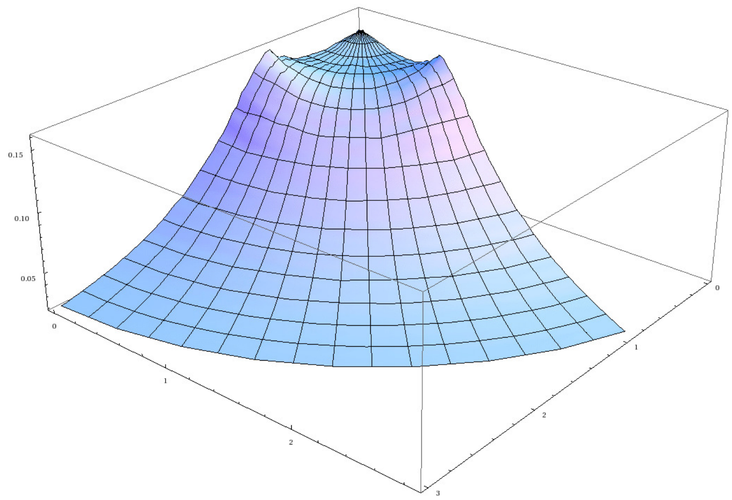


Figure 8.8: Approximation of the Weil-Petersson metric of quintic 3-folds, $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 - 5tZ_0Z_1Z_2Z_3Z_4$, using (8.31) and a family of $k = 1$ balanced metrics.

Part V

Perspectives

Chapter 9

Perspectives

In the future, we would like to address the following issues.

9.1 About balancing flows

1. There is certainly a natural interpretation of the existence of J-balanced metrics in terms of G.I.T. We give now an attempt of a possible definition in terms of test configurations. For a test configuration $\mathcal{T} = (\mathcal{M}, \mathcal{L}_1)$ of (M, L_1) as in Section 2.5, we have the weight $w_1(k)$ of the induced $\mathbb{C}^\times$ action. Set $P_1(k) = \dim H^0(M, L_1^k)$ and $P_2(k) = \dim H^0(M, L_2^k)$. Take now D a sufficiently general element in the linear system $|L_2^{k'}|$. We have also a weight $w_2(k')$ since the test configuration $\mathcal{T}$ induces a test configuration for D. We form the normalized weight
$$\tilde{w}_{L_1,L_2} = w_2(s)kP_1(k) - w_1(k)sP_2(s) = \sum_{i=0}^{n+1} e_i^{L_1,L_2}(s)k^i.$$
We expect for s large that the positivity of $e_{n+1}^{L_1,L_2}(s)$ is equivalent to the existence of J-balanced metrics on L_1^s thanks to Kempf-Ness theory.
The leading term of the polynomial expression $e_{n+1}^{L_1,L_2}(s)$ has been interpreted in [LS13] as the analog of the Donaldson-Futaki invariant in this context.
Another possible ingredient in this frame is coming from the characterization of J-balanced metrics in terms of Deligne pairings and the action of a 1-parameter subgroup in $SL(N+1)$. Using the notations of [PS10, Section 6], the deformation of the pairing
$$\langle L_2, L_1, ..., L_1\rangle$$

under the metric change $h \mapsto he^{-\phi} \in \mathrm{Met}(L_1)$ is given by the functional

$$\mathfrak{F}(\omega,\phi) = \sum_{i=0}^{n-1} \int_M \phi\,\chi \wedge (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^i \wedge \omega^{n-1-i}$$

where ω is the curvature of h. This functional is related to the J-flow. Actually the derivative $\frac{d\,\mathfrak{F}(\omega,\phi_t)}{dt}$ along a path in the space of Kähler potentials is given, up to a constant factor, by the functional $\frac{d\,J_\chi}{dt}(e^{-\phi_t})$, see (4.8), which provides another expression for the functional J_χ. Therefore, with respect to 1-parameter subgroups in $SL(N+1)$ (which correspond also to test configurations, see [PS10, Section 6.1.7]), the deformation of the considered pairing is controlled by $I_{\mu^0_{k,\chi}}$. It remains to identify this Deligne pairing as a positive line bundle over a certain projective variety in view of a complete G.I.T construction, cf. [Zha96, Theorem 3.6].

2. In the case of a manifold with negative first Chern class, we expect that the work of Y. Odaka and X. Wang can be useful to derive K-stability for the polarizations in a neighborhood of the canonical bundle. Let us give some hints of the techniques in the case of a surface X, with $K_X > 0$. The cone $\mathcal{C}_0$ studied by Donaldson, Chen, Weinkove and Song is formed of the polarizations L such that

$$(L.K_X)L > \frac{1}{2}(L^2)K_X.$$

Define $\mathcal{C}_0' \subset \mathcal{C}_0$ the cone formed by the polarizations L such that

$$(L.K_X)L > \frac{3}{4}(L^2)K_X.$$

We aim to show the positivity of $DF(\mathcal{B}, \mathcal{L} - E)$ for $L \in \mathcal{C}_0'$ in view of Theorem 2.5.6. Using the formalism of Section 2.5.2, one can do the following decomposition

$$DF(\mathcal{B}, \mathcal{L} - E) = DF_1 + DF_2 + DF_3,$$

where

$$\begin{aligned} DF_1 &= (\mathcal{L} - E)^2(-2(L.K_X)\mathcal{L} + (L^2)\mathcal{K}_X), \\ DF_2 &= (\mathcal{L} - E)^2(2(L.K_X)E + 2(L^2)\mathcal{K}_X), \\ DF_3 &= (\mathcal{L} - E)^2(3(L^2)\mathcal{K}_{\mathcal{B}/X\times\mathbb{P}^1}). \end{aligned}$$

We refer to [OS12, Lemma 4.2],[Der13, Lemma 3.7] for next lemma.

Lemma 9.1.1. *Let $\pi : \mathcal{B} \to X \times \mathbb{P}^1$ be the blow-up map. Then*

$(\mathcal{L}-E)^2.\mathcal{R} \leq 0$ *for any* $\mathcal{R} = \pi^*(p^*R)$, *where* R *is nef.*

A direct consequence of the previous lemma is that $DF_1 \geq 0$ for any $L \in \mathcal{C}_0$ since the pull-back of ample is nef. Now, let's show that DF_2 is positive. Define

$$s = \dim(Supp(\mathcal{O}/\mathcal{I})),$$

in the blow up process.

Lemma 9.1.2 ([Oda12, Lemma 2.8], [OS12, Lemma 4.7])**.** *The following inequalities hold:*

$$\begin{aligned}(-E)^3 < 0,\ &\text{if } s = 0\\ E^2.\mathcal{L} < 0,\ &\text{if } s = 1 \text{ or } s = 2,\\ E^2.(\mathcal{L}-E) \leq 0.&\end{aligned}$$

We will also need the next lemma, see [Oda13b, Lemma 3.5 (i)] that shows that $E.\mathcal{L}^2 = 0$ and $E.\mathcal{L}.\mathcal{K}_X = 0$ since geometrically, there is no intersection of the supports of the divisors.

Lemma 9.1.3. *For any Cartier divisors* $D_1, ..., D_q$ *on* $X \times \mathbb{P}^1$, *and any Cartier divisors* $E'_1, .., E'_{n+1-q}$ *with* $\dim(\pi(\cap_{i=1}^{n+1-q}\mathrm{Supp}(E'_i))) < q$, *the intersection* $(\pi^*D_1..\pi^*D_q.E'_1..E'_{n+1-q})$ *vanishes.*

Case $s = 0$**.**

Then $E^3 > 0$ by Lemma 9.1.2. But then

$$(\mathcal{L}-E)^2.(1/2E+\mathcal{L}) = -(3/2)\mathcal{L}^2.E + (1/2)E^3 = (1/2)E^3$$

thanks to Lemma 9.1.3. So

$$(\mathcal{L}-E)^2((L.K_X)E + (L^2)\mathcal{K}_X) > (\mathcal{L}-E)^2(-2(L.K_X)\mathcal{L} + (L^2)\mathcal{K}_X)$$

But the right hand side of the previous inequality is non-negative by Lemma 9.1.1. Eventually, $DF_2 > 0$ for any $L \in \mathcal{C}_0$.

Case $s \neq 0$**.**

We remark that $\mathcal{K}_X.\mathcal{L}^2 = 0 = \mathcal{L}^3$ because $\mathcal{L}$ and $\mathcal{K}_X$ are pull-back of ample divisors from X that has dimension 2. Thus, we have with Lemma 9.1.3,

$$\begin{aligned}DF_1 + DF_2 =& - E^2.(6(L.K_X)\mathcal{L} - 3(L^2)\mathcal{K}_X - 2(L.K_X)E)\\ =& - 2(L.K_X)E^2.(\mathcal{L}-E) - E^2.(4(L.K_X)\mathcal{L} - 3(L^2)\mathcal{K}_X)\end{aligned}$$

Thanks to Lemma 9.1.2, we get that the first term is non-negative and that the second term is positive if L belongs to $\mathcal{C}'_0$. Thus, we get $DF_1 + DF_2 > 0$ as expected.

To conclude, we remark that we only need to control the term DF_3. But this term is non negative as soon as X has only log-canonical singularities (this is the discrepancy term, see [Oda12, Proof of Theorem 2.6]). Here we use the fact that the restriction of $\mathcal{L} - E$ to the central fibre is semi ample and so $(\mathcal{L} - E)^2.E_i \geq 0$ where $E = \sum_i c_i E_i$ and also $(L - E)^2.E > 0$, see [OS12, Lemma 4.7].

Eventually, we have proved the following proposition.

Proposition 9.1.1. *Assume X is a complex projective surface with K_X ample. Let L an ample line bundle on X such that*

$$(L.K_X)L > \frac{3}{4}(L^2)K_X.$$

Then (X, L) is K-polystable.

This has to be compared with the result of Panov-Ross [PR09, Example 5.8] (and [RT06, Theorem 5.5]) where it is proved the slope stability of the polarization in the cone $\mathcal{C}_0$. At the moment we don't know if Proposition 9.1.1 can be extended from the set $\mathcal{C}_0'$ to the set $\mathcal{C}_0$.

Moreover, we expect that a generalization to higher dimension holds involving the condition introduced by M. Lejmi and G. Székelyhidi [LS13], namely that the polarization L belongs to the cone

$$\mathcal{C}_1 = \{L \text{ s.t. } (n\gamma L^p - pL^{p-1}.K_X) \cap [V] > 0, \forall V \subset M, \dim(V) = p\}.$$

In the case of dimension 2, $\mathcal{C}_1 = \mathcal{C}_0$. Of course, it is natural to wonder if the classes in $\mathcal{C}_0$ admit a cscK metric (it is known that Mabuchi's K-energy is proper on $\mathcal{C}_0$).

3. Another direction of investigation is to consider the analogue of the Ω-Kähler flow for Fano manifolds. In that case, we would like to develop the tools to derive the equivalent of Perelman's estimates for the evolving potentials. Since the flow we will consider will have a geometric interpretation in terms of moment maps, we believe these estimates could actually turn out to be simpler. In a similar vein, we have been studing the equivalent of the Ω-Kähler flow for higher rank bundles using the notion of balanced metrics in the sense of Wang [Wan02; Wan05]. In that case, we proved with R. Seyyedali that at the quantization limit, it converges towards the flow

$$\begin{aligned} h(t)^{-1}\frac{dh(t)}{dt} = &-\Big(\sqrt{-1}\Lambda F_{(E,h(t))} + \frac{1}{2}\mathrm{scal}(\omega) Id_E \\ &-\frac{1}{r}\int_X \mathrm{tr}\Big(\sqrt{-1}\Lambda F_{(E,h(t))} + \frac{1}{2}\mathrm{scal}(\omega) Id_E\Big)\frac{\omega^n}{n!}\Big), \end{aligned}$$

where $h(t)$ is a family of metrics over the bundle E. Up to a conformal change, this flow is Donaldson's heat flow for hermitian endomorphisms (which is equivalent by a change of the holomorphic structure to the classical Yang-Mills flow for connections). Eventually, it remains to study the flows investigated in [FLM11] using a finite dimensional approach.

4. We explained in [CaoKel12] that most of the techniques of Chapter 3 can be extended to singular positive forms. With a slight modification of Proposition 3.1.1, one can show that the asymptotic expansion of the Bergman function holds when one considers a positive volume form Ω that can be written as
$$\Omega = f_\Omega \omega^n$$
with $f_\Omega > 0$ on M and $f_\Omega \in L^1_\omega(M, \mathbb{R})$ and ω a smooth Kähler form. The asymptotic result for the operator Q_k (Theorem 3.1.3) is valid when applied to the space of functions $f \in L^p_\omega(M, \mathbb{R})$ with $p > 1$. This comes from the techniques of [LM07] that can be extended from L^2 to L^p topology, $p > 1$. To be more precise, the regularity of the function f is only needed in [LM07, Equation (27)] and the Cauchy-Schwartz inequality in [LM07, Equation (28)] can still be applied in the L^p spaces. This implies that we get a more general version of Theorem 3.1.1 for $\Omega \in L^p(M)$ positive volume form ($p > 1$) but with a weaker underlying convergence (the error terms are only controlled in L^p norms instead of C^∞ norm for the sequence $\omega_k(t)$ and $\frac{\partial \omega_k(t)}{\partial t}$). Finally, when one considers non smooth forms $\Omega = f_\Omega \omega^n$ with $f_\Omega \in L^p_\omega(M), f_\Omega > 0$ with $p > 1$, the limit of the balancing flows is still the Ω-Kähler flow (3.3). Note that we don't expect the potential of the involved metric in (3.3) to be smooth and we shall speak instead of weak Ω-Kähler flow. We also remark that the balancing flow will converge towards a balanced metric again. Actually a notion of balanced metric for L^p volume forms (and even more general) has been studied in details in the recent work [Ber+13, Section 7]. Furthermore the technical results of Section 3.3 still hold. Eventually, all these remarks show that the study of balancing flows in finite dimension leads naturally to define flows in Kähler geometry but among non smooth potentials. This is very natural in view of the recent results for the Kähler-Ricci flow involving degenerate metrics, see for instance [EGZ09].

9.2 About projective bundles

1. It was proved by Y. Rollin-M. Singer [RS09] that if E is a holomorphic vector bundle of rank 2 endowed with a parabolic structure over a curve B, such a structure encodes an iterated blow-up Y of $X = \mathbb{P}(E)$.

Furthermore, if E is parabolically polystable, then it was shown recently by Rollin that Y carries an extremal Kähler metric, see [RS13]. A natural objective is to obtain a higher-dimensional analog of Rollin-Singer's result when the bundle has higher rank and the base is still a curve. The existence part is not clear anymore in this case. Moreover, given a parabolic bundle over a base which is a curve or a surface, suppose that Y carries an extremal Kähler metric (in a suitable Kähler class). Does this imply that the bundle is parabolically polystable for a certain polarization? Does it imply a certain stability for a Kähler class on the underlying base? A first step is to extend the result of Section 5.4 to define a notion of "relative slope stability" (a weaker version of the relative K-stability introduced by G. Székelyhidi) and compute the relative Donaldson-Futaki invariants DF_1^{rel} of the test configurations induced by deformations to the normal cone of a subbundle of E. As a by-product, this should also provide new criteria on the structure of E to detect non-existence of extremal Kähler metrics on the projectivization X. We have done some computations of DF_1^{rel} in that direction (using equivariant Riemann-Roch formula on $H^*(\mathbb{P}(E))_{S^1\times S^1}$ and the computation of $\mathrm{ch}_3(S^kE)$ in full generality) that recover the results of [Szé07, Section 4]. We expect that it will help to prove the main conjecture of [Apo+11].

2. We wish to study stable parabolic structures over higher-dimensional bases and their natural relationship with extremal Kähler metrics with conical singularities, using the formalism developed by O. Biquard, C. Simpson and T. Mochizuki. To have good analytic estimates about the curvature tensors, we would need to require the base to be endowed with a conical Kähler-Einstein metric with small angle. We expect to generalize the results of Chapter 7 with Kai Zheng.

3. We would like to describe a general test configuration of a projective bundle in terms of a test configuration induced by a subsheaf (cf Section 5.4.1) and a test configuration for the base manifold. We expect that it is possible in the case the base manifold has dimension 1 or 2 thanks to the work of B. Crauder about degenerations of ruled manifolds. In practice, this would help us to compute the Donaldson-Futaki invariant for any test configuration of such manifolds. We hope to deduce from this result a classification of Fano threefolds in terms of K-stability and a study of the Kähler cone for certain examples (see Sections 6.1.2, 6.1.3). This is also related to the study of Donaldson's J-flow, see our discussion at the beginning of Section 4.2.

4. We would also like to investigate a stabilization process for ruled manifolds. The general topic here is to construct extremal metrics (or balanced type metrics, or to check K-stability) on the blow-up at a very

large number of points of the projectivization of an unstable bundle. In the Chapter 7, we have seen an elementary method to obtain stable parabolic structures from unstable Mumford bundle is introduced. Another direction is given by some results concerning stabilization of bundles over algebraic surfaces due to R. Brussee and R. Friedman-J. Morgan (cf Section 6.1.1), which were applied by N.P. Buchdahl in order to construct moduli space of bundles. Here we have in mind the following general problem. It is natural to ask whether a manifold X blown up sufficiently many times can be equipped with an extremal Kähler metric. For instance, one could imagine an analog of a famous result of C.H. Taubes in the case of self-dual metrics: given a 4-dimensional oriented manifold M, then $M\#n\mathbb{P}^2$ admits a self-dual metric for n sufficiently large. Intuitively, almost every point of X is to be blown up and the metric is almost everywhere replaced by a model metric which is extremal. A rather difficult perturbation theory has to be developed. We believe that the particular case of ruled manifolds should be investigated first.

9.3 About numerical approximations of canonical metrics

1. Let us consider as in Chapter 8 a family of Kähler-Einstein manifolds of general type or a family of polarized Calabi-Yau manifolds. In [Sun12], X. Sun used the Kuranishi-divergence gauge and the expansion of the Kähler forms of Kähler-Einstein metrics to give a new proof of the full curvature formula of the L^2 metric on the direct image sheaves of the relative pluricanonical bundles (previous proofs were established by G. Schumacher and B. Berndtsson). His quite simple formula is true for Kähler-Einstein metrics on general type manifolds and Calabi-Yau manifolds. We would like to investigate if similar techniques could hold for constant scalar curvature Kähler metrics and extremal metrics.

   Furthermore, the normalized Ricci curvatures of the L^2 metrics on the direct image $R^0K^m_{\mathcal{X}/\mathcal{T}}$ converge to the Weil-Petersson metric when $m \to +\infty$. Its expression involves the Laplacian operator. Thus, it is actually possible to approximate these Ricci curvature if one is able to quantize the Laplacian operator. This means that we should be able to provide another sequence of metrics obtained by algebraic methods that converges towards the Weil-Petersson metric. We intend to compare this approach with the one in Chapter 8. As we said, it is then crucial to be able to quantize the Laplacian operator for a Kähler metric within an integral class. But this is related to the techniques developed in Chapter 3 as shall see in details now.

Given A a hermitian endomorphism of $\mathbb{C}^{N+1}$ and ζ_A the associated vector field on $\mathbb{P}^N$ we define $P_k : Lie(U(N+1)) \to \Gamma(T\mathbb{P}^N{}_{|\iota(M)})$ such that $P_k(A) = \zeta_A$, where ι is an holomorphic embedding of the manifold M in $\mathbb{P}^N$. Using the Fubini-Study on $T\mathbb{P}^N{}_{|\iota(M)}$ and the volume form Ω on M, one gets a L^2 inner product on $\Gamma(T\mathbb{P}^N{}_{|\iota(M)})$. This allows to define the adjoint map $P_k^* : \Gamma(T\mathbb{P}^N{}_{|\iota(M)}) \to Lie(U(N+1))$ using the Killing form. We are interested in the asymptotic behavior of the operator $P_k^* P_k$ that satisfies

$$\mathrm{tr}(AP_k^* P_k B) = \int_M (\zeta_A, \zeta_B)\Omega$$

for any $A, B \in Lie(U(N+1))$. For $f \in C^\infty(M, \mathbb{R})$, let us consider the hermitian operator

$$Q_{\Omega,k}(f)_{\alpha,\beta} = k \int_M f(s_\alpha, s_\beta)\Omega$$

where $\{s_i\}$ is an orthonormal basis of $H^0(L^k)$ with respect to L^2 inner product $Hilb_\Omega(h^k)$. It is the derivative at $t = 0$ of $Hilb_\Omega(h_t^k)$ along the path of metrics $h_t = e^{tf}h$. Assume the curvature ω of h satisfies $\omega^n/n! = \Omega$. For $d = \partial + \bar\partial$ the exterior differentiation, we denote $\Delta_\omega = 2\Delta_{\bar\partial} = 2\bar\partial^*\bar\partial$ the usual Laplacian on $(0,0)$ forms on M. Then we claim that the following result holds.

Theorem 9.3.1. *Let $f \in C^\infty(M, \mathbb{R})$. The gradient of the moment map μ_Ω satisfies as $k \to +\infty$*

$$\mathrm{tr}\left(Q_{\Omega,k}(f)P_k^* P_k Q_{\Omega,k}(f)\right) = \frac{1}{4\pi k}\int_M f\Delta_\omega(f)\,\frac{\omega^n}{n!} + O(k^{-2}),$$

and this estimate is uniform in f and the metric $h \in \mathrm{Met}(L)$ if they vary in a compact set in C^∞ topology. This implies that for any $f, g \in C^\infty(M, \mathbb{R})$,

$$\mathrm{tr}\left(Q_{\Omega,k}(f)P_k^* P_k Q_{\Omega,k}(g)\right) = \frac{1}{4\pi k}\int_M f\Delta_\omega(g)\,\frac{\omega^n}{n!} + O(k^{-2}).$$

Building on these techniques and the results of [Sun12], we would like to estimate numerically the Weil-Petersson volume of the moduli space over the family of quintics, which is a rational number. A major issue is here to understand how we can approximate the Weil-Petersson metric close to a singular point. Another possible direction is to investigate the case of moduli space of stable vector bundles in some special cases.

List of Main Symbols

Symbol	Pages
$\mathcal{J}_{int}$	16
$Univ_{N,\chi}$	18
Ω_k	18
$Hilb_k$	19
FS_k	19
T	20
ρ_k	21
$Chow_s$, $e_{n+1}(s)$	27, 122
$\tilde{w}(s,k), \mathrm{Hilb}_{s,k}$	26, 125
DF_1	27
$Hilb_\Omega$	33
$T_{\Omega,k}$	33
μ_{FS}	34
μ_Ω	34
ϖ_Ω	34
μ^0_Ω	35
Q_k	38
d_k, dist_k	51
$\hat{h}_k(t)$	51, 75
$\tilde{h}_k$	52, 77
$\overline{h}_k(t)$	53, 77
$\overline{\mu}$	64

$\Omega_{\chi,\omega}$	69
$Hilb_{k,\chi}$	72
$T_{k,\chi}$	72
$\mu_{k,\chi}$	72
$\mu^0_{k,\chi}$	73
J_χ	81
$I_{\mu^0_{k,\chi}}$	81
I_{μ_J}	82
$\hat{I}_k$	85
$\mathcal{L}_k$	94
B_k	97
$\hat{h}_E$	100
$\tilde{B}_k$	100
$\tilde{R}$	102
$\Delta, \delta_L, \delta_{K_B^*}$	116
$\mathcal{L}_{r,m}$	115
Ψ, Ψ_Q	150, 162
W-P, $g^{\text{W-P}}$	150
$\Omega_k^{\text{W-P}}$	165

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