CANONICAL METRICS AND HARDER-NARASIMHAN FILTRATION

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The algebraic notion of Gieseker stability is related to the existence of *balanced* metrics which are zeros of a certain moment map. We investigate some properties of balanced metrics relative to the Harder-Narasimhan filtration of a vector bundle and to blowups in the case of projective surfaces.

In Section 1 and 2, we give an overview of the relation between Gieseker stability and the existence of a sequence of canonical metrics which converge towards a (weakly) Hermite-Einstein metric if the vector bundle is Mumford stable. In Section 3, we give an approximation of the curvature of a vector bundle using natural information coming from its Harder-Narasimhan filtration. Eventually in Section 4 we look at the case of a Gieseker stable vector bundle which is not Mumford stable over a projective surface.

Let (M, ω) be a smooth projective manifold of complex dimension n with ω a Kähler form and let L be a very ample line bundle over M equipped with a smooth hermitian metric h_L.

1. Background material for stability of vector bundles

The purpose of this section is to introduce some classical notions about stability of vector bundles. Let's define for a holomorphic vector bundle E on M of rank $r(E)$,

$$\mu(E) = \mu_L(E) = \frac{\deg_L(E)}{r(E)}$$

the normalized degree of E (relative to the degree) with respect to the polarization L. Moreover, we introduce the normalized Hilbert polynomial (relative to the Euler characteristic) for E by

$$p_E(k) = p_{E,L}(k) = \frac{\chi\left(E \otimes L^k\right)}{r\,(E)}.$$

For $n \mapsto p_1(n)$ and $n \mapsto p_2(n)$ two functions with integer values, we will denote $p_1 \prec p_2$ (resp. $p_1 \preceq p_2$), if for n large enough, $p_1(n) < p_2(n)$ (resp. $p_1(n) \leq p_2(n)$).

Definition 1.1. A vector bundle E is said to be Mumford L-stable (resp. semi-stable) if for all subsheaf $\mathcal{F}$ of E with $0 < r(\mathcal{F}) < r(E)$, we have $\mu(\mathcal{F}) < \mu(E)$ (resp. $\leq$). A Mumford semi-stable vector bundle is called polystable if it is a direct sum of Mumford stable bundles (of same normalized degree).

Definition 1.2. A vector bundle E is said to be Gieseker-Maruyama L-stable (resp. L semi-stable) if for all subsheaf $\mathcal{F}$ of E with $0 < r(\mathcal{F}) < r(E)$, we have $p_{\mathcal{F}} \prec p_E$ (resp. $\preceq$). A Gieseker semi-stable vector bundle is called polystable if it is a direct sum of Gieseker stable bundles (of same normalized Hilbert polynomial).

By Riemann-Roch Theorem, Gieseker stability and Mumford stability are equivalent if M is a curve. For higher dimension, we only have the following implications:

$$E \text{ Mumford stable} \Rightarrow E \text{ Gieseker stable}$$
$$E \text{ Gieseker semi-stable} \Rightarrow E \text{ Mumford semi-stable}$$

Gieseker proved that there always exists a projective scheme $\mathcal{M}$ that parametrizes the equivalence classes of torsion free Gieseker semi-stable sheaves with fixed Chern classes. Moreover, the Gieseker stable sheaves are parametrized by the closed points of an open subscheme of $\mathcal{M}$. Gieseker and Maruyama's approach gives a natural compactification of the moduli space $\mathcal{M}$, whereas, in general, there may not exist a canonical structure for the moduli space of equivalence classes of Mumford semi-stable sheaves. However, for projective surfaces these structures of moduli spaces do exist with different compactifications related by contractions and flips and admit the same Donaldson polynomials.

2. Gieseker stability and canonical metrics

Let $\mathbf{E}$ be a hermitian holomorphic vector bundle of rank r on the projective manifold M. By Kodaira's theorem, for k large enough we get an embedding i_k given by a basis of sections (S_i) of the space $H^0\left(M, \mathbf{E}\otimes L^k\right)$:

$$\begin{aligned} i_k : M &\hookrightarrow Gr\,(r,N) \\ z_0 &\mapsto \ker\left(ev_{z_0} : H^0\left(M,\mathbf{E}\otimes L^k\right) \to \mathbf{E}\otimes L^k|_{z_0}\right)^{\vee} \end{aligned}$$

where we have set $N = N\,(k) = \dim H^0\left(M,\mathbf{E}\otimes L^k\right) = \chi(E\otimes L^k)$. Moreover, our situation is fully described by

$$\begin{array}{ccc} \mathbf{E}\otimes L^k & \to & \mathbf{U}_{r,N} \\ \downarrow & & \downarrow \\ M & \hookrightarrow & Gr\,(r,N) \end{array}$$

where $\mathbf{U}_{r,N}$ is the dual of the universal bundle over the Grassmannian of quotients of dimension r of $\mathbb{C}^N$. Over $Gr(r,N)$ equipped with the natural Fubini-Study metric, acts the group

$$SU(N) = \left\{\, \mathsf{R} \in U\,(N) : \det\,(\mathsf{R}) = 1 \,\right\}$$

and the associated moment map for the standard Fubini-Study metric is :

$$\mu_{SU(N),Gr(r,N)} : [\mathsf{Q}] \mapsto \mathsf{Q}\,{}^{\mathsf{t}}\overline{\mathsf{Q}} - \frac{r}{N} Id \in \mathfrak{su}(N) = \mathrm{Lie}\,(SU\,(N))\,.$$

where we have identified $\mathfrak{su}(N)$ with its dual. We consider an element $[\mathsf{Q}] \in Gr\,(r,N)$ as a matrix $\mathsf{Q} \in \mathbb{M}_{N\times r}\,(\mathbb{C})$ that represents r vectors of $\mathbb{C}^N$ that form an orthonormal basis, by the natural identification

$$Gr\,(r,N) = \left\{\, \mathsf{R} \in \mathbb{M}_{N\times r}\,(\mathbb{C}) : {}^{t}\overline{\mathsf{R}}\mathsf{R} = Id \,\right\}/U\,(r)$$

Moreover, one notices that the map

$$\widetilde{\mu}_{r,N} : i_k \mapsto \int_M \mu_{SU(N),Gr(r,N(k))}(i_k\,(x))dV\,(x)$$

is a moment map for the action of $SU\,(N)$ acting on $C^\infty\,(M, Gr\,(r,N))$, which is an infinite dimensional Kähler manifold by [Hi].

We also need the following definition:

Definition 2.1. Let h be a hermitian metric on a globally generated holomorphic vector bundle E. We define $B_h \in \mathrm{End}\,(E)$ as the restriction to the diagonal of the Bergman kernel associated to the L^2 metric induced by h on $H^0(M,E)$ (also called distorsion function). If we set $(s_i)_{i=1..m}$ an orthonormal basis of $H^0(M,E)$ for the metric $\int_M \langle\,.\,,\,.\,\rangle_h$ and $m = \dim(H^0(M,E))$, then for all $z \in M$,

$$B_h(z) = \sum_{i=1}^{m} s_i(z)\langle\,.\,, s_i(z)\rangle_h$$

and this definition does not depend on the choice of the basis $(s_i)_{i=1..m}$.

Note V the volume of (M,ω). Inspired by the ideas of [Do3], we have

Theorem 2.1. (Wang) *The holomorphic vector bundle* $\mathbf{E}$ *is Gieseker stable if and only if its automorphism group is finite and there exists* $k_0 \geq 0$ *such that for all* $k > k_0$*, the embedding* i_k *can be balanced, in the sense that there exists a unique* $g \in SL(N)$ *(up to action of* $SU(N)$*) such that*

$$\widetilde{\mu}_{r,N}(g \cdot i_k) = \int_M \mu_{SU(N),Gr(r,N)}(g \cdot i_k(x))dV(x) = 0$$

This is equivalent to the existence of a sequence of hermitian metrics h_k *on* $\mathbf{E}$*, called balanced metrics, such that pointwise*

$$B_{h_k \otimes h_{L^k}} = \frac{\chi\left(\mathbf{E} \otimes L^k\right)}{rV} Id_{\mathbf{E}\otimes L^k}.$$

Sketch of the proof

Gieseker stability and GIT

We refer to [H-L] for the underlying construction of *Quot* scheme of Gieseker semi-stable sheaves and [M-F-K] for notions of Geometric Invariant Theory (GIT). We shall use the following stability criterion developed by Gieseker and Maruyama in [Gi] and [Ma] that relates the condition of stability for a vector bundle with a condition of GIT-stability :

Theorem 2.2. (Gieseker-Maruyama) *Let* E *be a globally generated vector bundle of rank* r*. Let* S_i *be a basis of sections of* $H^0(M,E)$ *and let* $T(E) \in \mathrm{Hom}(\wedge^r H^0(M,E), H^0(M,\det(E)))$ *defined by*

$$T(E)(S_{i_1},\ldots,S_{i_r}) = S_{i_1} \wedge \cdots \wedge S_{i_r}.$$

We can view $T(E)$ *as a point in the space*

$$\mathbf{Z}_E := \mathbb{P}\,\mathrm{Hom}(\wedge^r H^0(M,E), H^0(M,\det(E))).$$

The vector bundle E *is Gieseker stable (resp. semi-stable) if and only if for* k *large enough,* $T(E\otimes L^k) \in \mathbf{Z}_{E\otimes L^k}$ *is GIT-stable with respect to the action of* $SL(H^0(M,E\otimes L^k))$ *and the linearisation* $\mathcal{O}_{\mathbf{Z}_{E\otimes L^k}}(1)$.

Suppose we have fixed a reference metric h on the hermitian holomorphic vector bundle $\mathbf{E}$ and that $\mathbf{E}\otimes L^k$ is globally generated. We get a L^2-metric $H = \mathrm{Hilb}(h) = \int_M \langle .,. \rangle_{h\otimes h_L{}^k} dV$ on the space $H^0(M,\mathbf{E}\otimes L^k)$. From the embedding i_k in the Grassmannian given by an H-orthonormal basis S_i, we get a metric on the bundle $\mathbf{U}_{r,N}$. Since, $i_k^*\mathbf{U}_{r,N} \simeq \mathbf{E}\otimes L^k$, we get a natural metric on $\mathbf{E}$ that we will call $FS(H)$, and therefore a metric on

$\wedge^r(\mathbf{E} \otimes L^k)$ that we will simply denote $\|\,.\,\|$. Eventually, this gives us a metric on $\mathbf{Z} := \mathbf{Z}_{\mathbf{E}\otimes L^k}$, that we can evaluate at a point $\mathbf{z} \in \mathbf{Z}$:

$$|||\mathbf{z}|||^2_{\mathbf{Z}} = \sup_{\substack{(S_i)\ H\text{-orthonormal} \\ \text{basis of } H^0(M,\mathbf{E}\otimes L^k)}} \int_M \sum_{1 \leq i_1 < \cdots < i_r \leq N} \|S_{i_1}(p) \wedge \cdots \wedge S_{i_r}(p)\|^2 \, dV$$

where the sup is *independent* of the choice of the basis. The classical Kempf-Ness's result [K-N, Theorem 0.2], gives us an analytical criterion to check the GIT-stability of a point $\mathbf{z} \in \mathbf{Z}$, with respect to the linearization $\mathcal{O}_{\mathbf{Z}}(1)$ and the $SL(N)$ action : $\mathbf{z}$ is GIT -stable if and only if the application

$$\mathcal{L}(g) : g \mapsto \log \int_M \sum_{1 \leq i_1 < \cdots < i_r \leq N} \|g \cdot S_{i_1}(p) \wedge \cdots \wedge g \cdot S_{i_r}\|^2 \, dV$$

is bounded from below by a strictly positive constant and is proper (for all $t > 0$ there exists a compact set $K \subset SL(N)$ such that $\mathcal{L}(g) > t$ if $g \notin K$).

Interlude about the notion of integral of a moment map

Consider Ξ a smooth symplectic manifold, ω its symplectic form and Γ a compact Lie group acting symplectically on Ξ. If μ is a moment map associated to this action, then one can define the functional

$$\Psi : \Xi \times \Gamma^{\mathbb{C}} \to \mathbb{R}$$

that we will call the "*integral of the moment map* μ" and that satisfies the following properties:

- for all $p \in \Xi$, the critical points of the restriction Ψ_p of Ψ to $\{p\} \times \Gamma^{\mathbb{C}}$ coincide with the points of the orbit $Orb_{\Gamma^{\mathbb{C}}}(p)$ on which the moment map vanishes;
- the restriction Ψ_p to the lines $\left\{e^{\lambda u} : u \in \mathbb{R}\right\}$ where $\lambda \in \mathrm{Lie}\left(\Gamma^{\mathbb{C}}\right)$ is convex.

Theorem 2.3. (Mundet i Riera) *There exists a unique application* $\Psi : \Xi \times \Gamma^{\mathbb{C}} \to \mathbb{R}$ *that satisfies:*

1. $\Psi(p, e) = 0$ *for all* $p \in \Xi$*;*
2. $\frac{d}{du} \Psi\left(p, e^{i\lambda u}\right)_{|u=0} = \langle \mu(p), \lambda \rangle$ *for all* $\lambda \in \mathrm{Lie}(\Gamma)$*;*

Moreover, this functional enjoys the following properties: Ψ *is* Γ*-invariant and satisfies the cocycle relation* $\Psi(p, \gamma) + \Psi(\gamma p, \gamma') = \Psi(p, \gamma'\gamma)$ *and also the relation* $\Psi(\gamma p, \gamma') = \Psi\left(p, \gamma^{-1}\gamma'\gamma\right)$ *for all* $p \in \Xi$, $\gamma, \gamma \in \Gamma^{\mathbb{C}}$*. Eventually,* $\frac{d^2}{du^2} \Psi\left(p, e^{i\lambda u}\right) \geq 0$ *for all* $\lambda \in \mathrm{Lie}(\Gamma)$ *with equality if and only if the vector field* $\overrightarrow{X}_\lambda\left(e^{i\lambda u} p\right) = 0$.

Remember that we have a diffeomorphism

$$\begin{array}{rl} \Gamma \times \mathrm{Lie}\,(\Gamma) \rightarrow & \Gamma^{\mathbb{C}} \\ (\gamma, u) \quad \mapsto & \gamma e^{iu}. \end{array} \tag{1}$$

Let $\rho : \Gamma^{\mathbb{C}} \rightarrow GL\,(W)$ be a faithful representation on a finite dimension complex vector space equipped with a hermitian metric such that $\rho\,(\Gamma) \subset U\,(W)$. We still denote ρ the representation induced on $\mathrm{Lie}\,(\Gamma)$ and on Γ. This leads to define the following metric on $\mathrm{Lie}\,(\Gamma)$ by

$$\langle a, b \rangle_{\Gamma} = Tr\left(\rho\,(a)\,\rho\,(b)^{*}\right).$$

By the diffeomorphism (1), we can associate to each element $\gamma e^{iu} \in \Gamma^{\mathbb{C}}$ its logarithm $\log_{\Gamma^{\mathbb{C}}}\left(\gamma e^{iu}\right) = u$.

Definition 2.2. We will say that Ψ is linearly log-proper respectively to the metric $\langle\, .\, ,\, .\,\rangle_{\Gamma}$ on $\Gamma^{\mathbb{C}}$ if there exists two constants $c_1 > 0$ and $c_2 > 0$ such that for all $g \in \Gamma^{\mathbb{C}}$ and for all $p \in \Xi$,

$$\left|\log_{\Gamma^{\mathbb{C}}}(g)\right|_{\Gamma} \leq c_1 \Psi_p\,(g) + c_2.$$

Balanced condition and Kempf-Ness functional

We want to measure the action of $SL(N)$ on a point $\mathbf{z} \in \mathbf{Z}$. For that reason, with the same notations as before, we introduce the following functional for $g \in SL(N)$ which depends only on the choice of H,

$$\widetilde{KN_{k,\mathbf{E}}} : g \mapsto \frac{1}{2}\int_M \log \frac{\sum_{1\leq i_1 < \cdots < i_r \leq N} \left\| g \cdot S_{i_1}\,(p) \wedge \cdots \wedge g \cdot S_{i_r}\,(p) \right\|^2}{\sum_{1\leq i_1 < \cdots < i_r \leq N} \left\| S_{i_1}\,(p) \wedge \cdots \wedge S_{i_r}\,(p) \right\|^2} dV\,(p)\,.$$

where $(S_i)_{i=1,\ldots,N}$ is an H-orthonormal basis of holomorphic sections of $\mathbf{E} \otimes L^k$. Let $[\mathsf{Q}\,(p)]$ be the point of $Gr\,(r, N)$ given by the embedding i_k at $p \in M$ and the metric H on $H^0(M, \mathbf{E} \otimes L^k)$. Our functional is related to a functional that plays a key role in Donaldson's theory [Do1,Do2,P-S]:

Definition 2.3. Let E be a hermitian holomorphic vector bundle on M and h_1, h_2 two hermitian metrics on E. We define the Kempf-Ness functional as the integral of the first Chern-Weil form:

$$KN_E\,(h_1, h_2) = \int_M \log\det\left(h_2^{-1} h_1\right) \frac{\omega^n}{n!}$$

Lemma 2.0.1. *For all* $g \in SL(N)$,

$$\widetilde{KN_{k,\mathbf{E}}}(g) = \frac{1}{2} KN_{\mathbf{E}\otimes L^k}\left(FS(H\circ g), FS(H)\right) = \frac{1}{2}\int_M \log \frac{\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}}{}^{\mathsf{t}}\overline{g}g\mathsf{Q}\right)}{\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\mathsf{Q}\right)} dV,$$

and $\widetilde{KN_{k,\mathbf{E}}}(g)$ *is the integral of the moment map* $\widetilde{\mu}_{r,N}$.

Proof. One can represent $[\mathsf{Q}(p)]$ for all $p \in M$ as a Stieffel point

$$\mathsf{Q}' = \begin{pmatrix} \mathsf{Z} \\ Id_{r\times r} \end{pmatrix}$$

with $\mathsf{Z}(p) = [z_1, \ldots, z_r] \in \mathbb{M}_{(N-r)\times r}(\mathbb{C})$. There exists an antiholomorphic application $\Phi : Gr(r, N) \to Gr(N-r, N)$ such that $\Phi([\mathsf{Q}']) = [\mathsf{Q}'^{\perp}]$, which implies $\Phi\left(\begin{pmatrix} \mathsf{Z} \\ Id_{r\times r} \end{pmatrix}\right) = \begin{pmatrix} Id_{(N-r)\times(N-r)} \\ -{}^{\mathsf{t}}\overline{\mathsf{Z}} \end{pmatrix}$.
Set $[\mathfrak{z}_1, \ldots, \mathfrak{z}_{N-r}] := -{}^{\mathsf{t}}\overline{\mathsf{Z}}$ and fix a basis $(e_i)_{i=1,\ldots,N}$ of $\mathbb{C}^N$. As it is mentioned in [Mo, Chapter 7], the potential of the Fubini-Study metric on the Grassmannian is given explicitly by

$$\begin{aligned}
&\log |(e_{N-r+1} + z_1) \wedge \cdots \wedge (e_N + z_r)|^2 \\
&= \log |(e_1 + \mathfrak{z}_1) \wedge \cdots \wedge (e_{N-r} + \mathfrak{z}_{N-r})|^2 \\
&= \log |(e_1 + \mathfrak{z}_1) \wedge \cdots \wedge (e_{N-r} + \mathfrak{z}_{N-r}) \wedge (e_{N-r+1} + z_1) \wedge \cdots \wedge (e_N + z_r)| \\
&= \log\det \begin{pmatrix} Id_{(N-r)\times(N-r)} & \mathsf{Z} \\ -{}^{\mathsf{t}}\overline{\mathsf{Z}} & Id_{r\times r} \end{pmatrix} \\
&= \log\det\left(Id_{(N-r)\times(N-r)} + \mathsf{Z}{}^{\mathsf{t}}\overline{\mathsf{Z}}\right) \\
&= \log\det\left(Id_{r\times r} + {}^{\mathsf{t}}\overline{\mathsf{Z}}\mathsf{Z}\right) \\
&= \log\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}'}\mathsf{Q}'\right) \\
&= \log\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\mathsf{Q}\right).
\end{aligned}$$

A simple computation shows that at the point $[g \cdot \mathsf{Q}]$, this potential is also given by

$$\log\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\,{}^{\mathsf{t}}\overline{g}g\mathsf{Q}\right).$$

For the second assertion, it is sufficient to consider the induced action by the 1-parameter subgroup of the form $\left\{u \to e^{\mathsf{S}u} \in SL(N)\right\}$ (here $\mathsf{S} \neq 0$ is a hermitian trace free matrix). Since we know that for all $\mathsf{A}_1, \mathsf{A}_2 \in GL(N, \mathbb{C})$, one has $D\det_{\mathsf{A}_1}(\mathsf{A}_2) = \det(\mathsf{A}_1)\,\mathrm{tr}\left(\mathsf{A}_1^{-1}\mathsf{A}_2\right)$, we obtain:

$$\begin{aligned}
&\frac{d}{du}\left(\widetilde{KN_{k,\mathbf{E}}}\left(e^{\mathsf{S}u}\right)\right) \\
&= \frac{1}{2}\int_M \frac{d}{du}\log\left(\det\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\,{}^{\mathsf{t}}\overline{e^{\mathsf{S}u}}e^{\mathsf{S}u}\mathsf{Q}\right)\right)dV, \\
&= \frac{1}{2}\int_M \mathrm{tr}\left(\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\,e^{{}^{\mathsf{t}}\overline{\mathsf{S}}u}e^{\mathsf{S}u}\mathsf{Q}\right)^{-1}\left({}^{\mathsf{t}}\overline{\mathsf{Q}}\,e^{{}^{\mathsf{t}}\overline{\mathsf{S}}u}\left(\mathsf{S}+{}^{\mathsf{t}}\overline{\mathsf{S}}\right)e^{\mathsf{S}u}\mathsf{Q}\right)\right)dV.
\end{aligned}$$

Since we have chosen an orthonormal basis, ${}^{\mathsf{t}}\overline{\mathsf{Q}}\mathsf{Q} = Id$ and consequently, for $u = 0$ we get,

$$\begin{aligned}
\frac{d}{du}\left(\widetilde{KN_{k,\mathbf{E}}}\left(e^{\mathsf{S}u}\right)\right)_{|u=0} &= \int_M \mathrm{tr}({}^{\mathsf{t}}\overline{\mathsf{Q}}\mathsf{S}\mathsf{Q}) \\
&= \int_M \mathrm{tr}({}^{\mathsf{t}}\overline{\mathsf{Q}}\mathsf{S}\mathsf{Q}) - \frac{r}{N}\int_M \mathrm{tr}\left(\mathsf{S}\right) \\
&= \left\langle \widetilde{\mu}_{r,N}\left([\mathsf{Q}]\right), \mathsf{S}\right\rangle
\end{aligned}$$

and we conclude by Theorem 2.3. □

Lemma 2.0.2. *The functional* $\widetilde{KN_{k,\mathbf{E}}}(\,.\,)$ *is linearly log-proper with respect to the action of* $SL(N)$.

Proof. Define the Kähler cone respectively to ω:

$$Ka\left(M,\omega\right) = \left\{\varphi \in C^{\infty}\left(M,\mathbb{R}\right) : \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0\right\}$$

We set

$$\varphi = \log \sum_{1\le i_1<\cdots<i_r\le N} \left\|S_{i_1}\left(p\right)\wedge\cdots\wedge S_{i_r}\right\|^2.$$

Since $\varphi \in Ka\left(M,\omega\right)$, a theorem of Kähler geometry of G. Tian [Ti] asserts that there exists two constants $\alpha = \alpha(M,\omega) > 0$ and $C = C(M,\omega) > 1$ such that

$$\int_M e^{-\alpha\left(\varphi - \sup_M \varphi\right)}\frac{\omega^n}{n!} < C,$$

which implies

$$\log\left(\int_M e^{-\alpha\left(\varphi - \sup_M \varphi\right)}\frac{\omega^n}{n!}\right) < C',$$

and by concavity of log, that

$$\begin{aligned}\int_M \varphi \frac{\omega^n}{n!} &\geq \int_M \left(\sup_M \varphi\right) \frac{\omega^n}{n!} - \frac{1}{\beta} \\ &\geq \log\left(\sup_{p\in M} \sum_{1\leq i_1<\cdots<i_r\leq N} \|S_{i_1}(p)\wedge\cdots\wedge S_{i_r}(p)\|^2\right) - \frac{1}{\beta}.\end{aligned}$$

Therefore, for all $g \in SL(N)$,

$$\widetilde{KN_{k,\mathbf{E}}}(g\cdot h) \geq \log |||g\cdot \mathbf{z}|||^2_{\mathbf{Z}} - \frac{1}{\beta},$$

where $\beta(M,\omega) > 0$ depends only on M and ω. In fact, we also get,

$$\log |||g\cdot \mathbf{z}|||^2_{\mathbf{Z}} \geq \widetilde{KN_{k,\mathbf{E}}}(g\cdot h) \geq \log |||g\cdot \mathbf{z}|||^2_{\mathbf{Z}} - \frac{1}{\beta(M)}. \qquad \square$$

By Lemma 2.0.2, the functional is $\widetilde{KN_{k,\mathbf{E}}}$ is proper and bounded from below if and only if the functional $\mathcal{L}$ is bounded from below and proper, which means that the point $T(\mathbf{E}\otimes L^k) \in \mathbf{Z}$ is GIT-stable. Moreover we obtain that the embedding i_k is balanced if and only if the Fubini-Study metric induced by i_k is a scalar multiple of the original metric on $\mathbf{E}$. This means that there exists a hermitian metric h_k on $\mathbf{E}$ which is a fixed point for $FS \circ$ Hilb (resp. Hilb(h_k) is a fixed point for Hilb $\circ\, FS$). Now, we remark that $\mathsf{Q}^t\overline{\mathsf{Q}} = \lambda Id$ if and only if ${}^t\overline{\mathsf{Q}}\mathsf{Q}$ is the matrix of the orthogonal projection to ker(Q). But ${}^t\overline{\mathsf{Q}}\mathsf{Q}$ is a bundle morphism corresponding to the Bergman kernel of $\mathbf{E}$ for the metric h_k. Therefore, we obtain the second part of Theorem 2.1.

An interesting consequence of Theorem 2.1 is an analogue of the work [Do4] of S. Donaldson in the case of vector bundles. It uses essentially two facts. First of all, one knows an asymptotic expansion in k of the Bergman kernel over a compact manifold:

$$B_{h\otimes h_{L^k}} = k^n Id + k^{n-1}\left(\frac{1}{2}\mathrm{Scal}(g_{i\bar{j}})Id + \sqrt{-1}\Lambda_\omega F_h\right) + \cdots$$

if one has assumed that $\omega = \frac{\sqrt{-1}}{2}\sum g_{i\bar{j}} dz_i \wedge d\overline{z_j}$ represents $c_1(L)$. Secondly, once one has fixed a holomorphic structure $\overline{\partial}$ on $\mathbf{E}$, the Bergman kernel can be seen as a moment map for the action of the Gauge group $\mathcal{G}$ of $\mathbf{E}$ on the infinite dimensional Kähler space

$$\mathcal{H} = \left\{ (s_1,\ldots,s_N) \in C^\infty(M, \mathbf{E}\otimes L^k)^N : \begin{array}{l} s_i \text{ are linearly independent,} \\ \overline{\partial} s_i = 0 \end{array} \right\}.$$

and the points in $\mathcal{H}//(\mathcal{G}\times SU(N))$ correspond to balanced metrics.

Theorem 2.4.

- *Suppose* $\mathbf{E}$ *is Gieseker stable. If the sequence* (h_k) *of balanced metrics on* $\mathbf{E}$ *converges, then its limit is weakly Hermite-Einstein.*
- *If* $\mathbf{E}$ *is Mumford stable then the sequence of balanced metrics* h_k *converges and the convergence is* C^m *for all* $m \geq 0$.

Theorems 2.1 and 2.4 were found independently by X. Wang [Wa1,Wa2] and J. Keller. A similar problem had already been studied in [Dr] in the case of curves. See also [Ke] for a generalization of this theorem to the case of Vortex equations and stability of pairs.

Remark. A hermitian metric h on vector bundle E is weakly Hermite-Einstein if the curvature F_h of the Chern connection relative to h satisfies the equation

$$\sqrt{-1}\Lambda_\omega F_h = \lambda_h Id_E,$$

where λ_h is a continuous function with real values. Since M is compact, there exists a unique function $f \in C^\infty(M, \mathbb{R})$ (up to a constant), such that the new metric $e^f \cdot h$ is Hermite-Einstein (i.e $\lambda_{e^f \cdot h} = \mu(E)$ is constant). A good reference on this subject is [L-T].

3. Harder-Narasimhan filtration

In this section, we give an application of Theorem 2.4. We will need to introduce the following classical notion:

Definition 3.1. If $\mathcal{F}$ is a torsion free sheaf, a Härder-Narasimhan filtration for $\mathcal{F}$ is an increasing filtration

$$0 = HN_0(\mathcal{F}) \subset \cdots \subset HN_l(\mathcal{F}) = \mathcal{F},$$

such that the factors $gr_i^{HN}(\mathcal{F}) = HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F})$ for $i = 1, \ldots, l$ are torsion free Mumford semi-stable with normalized degree μ_i satisfying

$$\mu_{\max}(\mathcal{F}) := \mu_1 > \cdots > \mu_l =: \mu_{\min}(\mathcal{F}),$$

Such a filtration exists and is unique. The graduated object

$$gr^{HN}(\mathcal{F}) = \bigoplus gr_i^{HN}(\mathcal{F})$$

is uniquely determined by the isomorphism class of $\mathcal{F}$. Moreover, there exists a unique Mumford semi-stable saturated subsheaf $\mathcal{F}_1 \subset \mathcal{F}$, called maximal destabilizing subsheaf of $\mathcal{F}$, such that:

- If $\mathcal{F}_2 \subset \mathcal{F}$ is a proper subsheaf of $\mathcal{F}$, then $\mu(\mathcal{F}_2) \leq \mu(\mathcal{F}_1)$;

• If $\mu(\mathcal{F}_2) = \mu(\mathcal{F}_1)$, then $r(\mathcal{F}_2) \leq r(\mathcal{F}_1)$.

Notice that $HN_1(\mathcal{F})$ is the maximal destabilizing subsheaf of $\mathcal{F}$.

We know that if E is a Mumford polystable vector bundle, E splits holomorphically as $E = \oplus_{i=1}^{l} E_i$, the induced metric on E_i is Hermite-Einstein and the induced filtration is given by $HN_i(E) = \oplus_{j \leq i} E_j$. Now, for an unspecified holomorphic structure, the Harder-Narasimhan filtration may not split holomorphically nor be given by vector subbundles.

Lemma 3.0.3. *Let $\mathcal{F}$ be a torsion free sheaf on M and let $\mathcal{F}' = \mathcal{F}/HN_1(\mathcal{F})$. Then*

$$HN_{i+1}(\mathcal{F}) = \ker(\mathcal{F} \to \mathcal{F}'/HN_i(\mathcal{F}'))$$

and $HN_{i+1}(\mathcal{F})/HN_1(\mathcal{F}) = HN_i(\mathcal{F}')$.

Proof. The proof is outlined in [H-L]. It uses the fact that for a sequence $0 \to A \to B \to C \to 0$, one has $\ker(HN_1(B) \to C) = HN_1(A)$. □

Proposition 3.1. *Consider the exact sequence of torsion free sheaves*

$$0 \to \mathcal{E}_1 \to E \to \mathcal{E}_2 \to 0.$$

with $l_1 := \mu_{\min}(\mathcal{E}_1) > \mu_{\max}(\mathcal{E}_2)$. Then the Harder-Narasimhan filtration of E is given by

$$\begin{aligned} 0 = HN_0(E) \subset HN_1(\mathcal{E}_1) \subset \cdots \subset HN_{l_1}(\mathcal{E}_1) \\ = \mathcal{E}_1 \subset HN_{l_1+1}(E) \subset \cdots \subset HN_l(E) = E, \end{aligned}$$

with:

$$\begin{aligned} &HN_i(E) = \ker(E \to \mathcal{E}_2/HN_{i-l_1}(\mathcal{E}_2)) \quad \textit{for} \quad i = l_1, \ldots, l, \\ &HN_i(E) = HN_i(\mathcal{E}_1) \quad \textit{for} \quad i = 0, \ldots, l_1. \end{aligned}$$

Moreover, $gr_E^{HN} = gr_{\mathcal{E}_1}^{HN} \oplus gr_{\mathcal{E}_2}^{HN}$.

Proof. Let $\mathcal{F} \subset E$ be the maximal destabilizing sheaf. We get $\mu(\mathcal{F}) \geq \mu_{\max}(\mathcal{E}_1) \geq \mu_{\min}(\mathcal{E}_1) > \mu_{\max}(\mathcal{E}_2)$ by hypothesis. The application $\phi : \mathcal{F} \to \mathcal{E}_2$ is non trivial, because otherwise we would have $\mu(\mathrm{Im}(\phi)) \geq \mu(\mathcal{F}) > \mu_{\max}(\mathcal{E}_2)$ which would contradict the semi-stability of $\mathcal{F}$. Therefore we have $\mathcal{F} \subset \mathcal{E}_1$ and $\mathcal{F} = \mathcal{E}_1$ or $\mathcal{F}$ is the maximal destabilizing subsheaf of $\mathcal{E}_1$. If $\mathcal{E}_1$ is Mumford semi-stable, then clearly $\mathcal{E}_1$ is the maximal destabilizing subsheaf of E as shows the previous lemma.

For general $\mathcal{E}_1$, we use the same kind of arguments by induction over the length of the Harder-Narasimhan filtration of $\mathcal{E}_1$. The sequence

$$0 \to \mathcal{E}_1/\mathcal{F} \to E/\mathcal{F} \to \mathcal{E}_2 \to 0$$

still satisfies the inequality $\mu_{\min}(\mathcal{E}_1/\mathcal{F}) = \mu_{\min}(\mathcal{E}_1) > \mu_{\max}(\mathcal{E}_2)$. By induction hypothesis, we get

$$\begin{gathered} 0 \subset HN_0(\mathcal{E}_1/\mathcal{F}) \subset \cdots \subset HN_{l_1-1}(\mathcal{E}_1/\mathcal{F}) = \mathcal{E}_1/\mathcal{F} \\ \mathcal{E}_1/\mathcal{F} \subset HN_l(E/\mathcal{F}) \subset \cdots \subset HN_{l-1}(E/\mathcal{F}) = E/\mathcal{F} \end{gathered}$$

with $HN_i(E/\mathcal{F}) = \ker(E/\mathcal{F} \to \mathcal{E}_2/HN_{i-l_1+1}(\mathcal{E}_2))$. From another side, by the lemma,

$$HN_i(E) = \ker(E \to (E/\mathcal{F})/HN_{i-1}(E/\mathcal{F}))$$

and therefore,

$$HN_i(E/\mathcal{F}) = \ker(E \to \mathcal{E}_2/HN_{i-l_1}(\mathcal{E}_2)).$$

By induction hypothesis, we know that for $i \leq l_1$,

$$HN_i(E)/\mathcal{F} = HN_{i-1}(E/\mathcal{F}) = HN_{i-1}(\mathcal{E}_1/\mathcal{F})$$

and we also obtain by the lemma,

$$\begin{aligned} HN_{i+1}(E) &= \ker(E \to (E/\mathcal{F})/HN_{i-1}(E/\mathcal{F})) \\ &= \ker(E \to (E/\mathcal{F})/HN_{i-1}(\mathcal{E}_1/\mathcal{F})) \\ &= \ker(E \to (E/\mathcal{F})/(HN_i(\mathcal{E}_1)/\mathcal{F})) \\ &= HN_{i+1}(\mathcal{E}_1) \end{aligned}$$

Finally, the last assertion is obvious. □

Definition 3.2. To each factor $HN_i(E)$ of a vector bundle E equipped with a hermitian metric h_E, corresponds a projection (orthogonal for h_E) π_i^{E,h_E} on $HN_i(E)$. Define the hermitian endomorphism $\Pi_{HN(E)}^{\omega,h_E}$:

$$\Pi_{HN(E)}^{\omega,h_E} = \sum \mu\left(gr_i^{HN}(E)\right)\left(\pi_i^{E,h_E} - \pi_{i-1}^{E,h_E}\right).$$

Remark. As we have seen, if E is a holomorphic vector bundle equipped with a Hermite-Einstein metric h_E, then by Uhlenbeck-Yau's Theorem E is Mumford polystable and we have the decomposition

$$(E, h_E) = (E_1, h_1) \oplus \cdots \oplus (E_k, h_k)$$

by Mumford stable vector bundles with the same normalized degree $\mu(E)$. In particular, $\sqrt{-1}\Lambda F_{h_i} = \mu(E)\, Id_{E_i}$ and

$$\Pi^{\omega,h_E}_{HN(E)} = \mu(E) \begin{pmatrix} Id_{E_1} & & \\ & \ddots & \\ & & Id_{E_l} \end{pmatrix}.$$

Definition 3.3. Let $\mathcal{F}$ be a torsion free sheaf which is Mumford semi-stable. A Jordan-Hölder filtration of $\mathcal{F}$ is a filtration

$$0 = JH_0(\mathcal{F}) \subset \cdots \subset JH_l(\mathcal{F}) = \mathcal{F}$$

such that the factors $gr_i^{JH}(\mathcal{F}) = JH_i(\mathcal{F})/JH_{i-1}(\mathcal{F})$ are all Mumford stable with same normalized Hilbert polynomial. The graduated object

$$gr^{JH}(\mathcal{F}) = \bigoplus gr_i^{JH}(\mathcal{F})$$

does not depend on the choice of the filtration.

Theorem 1. *Let E be a holomorphic vector bundle on (M,ω). If the Harder-Narasimhan filtration $HN(E)$ of E is given by subbundles, then for all $\varepsilon > 0$, for all $r \geq 0$, there exists a smooth hermitian metric h on E compatible with the holomorphic structure such that*

$$\left\| \sqrt{-1}\Lambda_\omega F_h - \Pi^{\omega,h}_{HN(E)} \right\|_{C^r} < \varepsilon.$$

Proof. We give a proof by induction on the length of the Harder-Narasimhan filtration of E.

If the rank of E is 1, this comes from the fact that we can use the Jordan-Hölder filtration since E is in particular Mumford stable, and we can apply Theorem 2.4 to get a sequence of metrics weakly Hermite-Einstein h_k which are, up to a conformal change, Hermite-Einstein metrics h'_k. Therefore, for k large enough,

$$\left\| \sqrt{-1}\Lambda F_{E,h'_k} - \mu(E)\, Id_E \right\|_{C^r} = O\left(\frac{1}{k}\right).$$

Now, if the length of the Harder-Narasimhan filtration of E is bigger than 2, then

$$0 \to E_1 \to E \to E_2 \to 0, \tag{2}$$

where E_1 is the maximal destabilizing sheaf of E which is, as E_2, a vector bundle. The filtrations $HN(E_1)$ and $HN(E_2)$ are given by vector bundles by Proposition 3.1, and for the metrics h_1 et h_2 (and respectively their curvatures F_1, F_2 of E_1 and E_2), we get

$$\left\|\sqrt{-1}\Lambda F_1 - \Pi^{\omega,h_1}_{HN(E_1)}\right\|_{C^r} < \varepsilon/3, \quad \left\|\sqrt{-1}\Lambda F_2 - \Pi^{\omega,h_2}_{HN(E_2)}\right\|_{C^r} < \varepsilon/3.$$

From (2) we have $\Pi^{\omega,h_1\oplus h_2}_{HN(E)} = \Pi^{\omega,h_1}_{HN(E_1)} \oplus \Pi^{\omega,h_2}_{HN(E_2)}$ and the holomorphic structure on E has the following form:

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_{E_1} & \alpha \\ 0 & \overline{\partial}_{E_2} \end{pmatrix},$$

with α a smooth section of $\Omega^{0,1}\left(\mathcal{H}\mathfrak{om}\left(E_1, E_2\right)\right)$ (see [Ko, §1.6]). Then,

$$\left\|\sqrt{-1}\Lambda F_E - \Pi^{\omega,h_1\oplus h_2}_{HN(E)}\right\|_{C^r} \leq \left\|\sqrt{-1}\Lambda F_1 - \Pi^{\omega,h_1}_{HN(E_1)}\right\|_{C^r} + \left\|\Lambda F_2 - \Pi^{\omega,h_2}_{HN(E_2)}\right\|_{C^r}$$
$$+ 2\sup|\alpha|^2 + 2\sup|\overline{\partial}^*\alpha|^2,$$

Up to a Gauge change of the form $g = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}$, we can assume that

$$2\left(\sup|\alpha|^2 + \sup|\overline{\partial^*}\alpha|^2\right) < \varepsilon/3.$$

This allows us to conclude, considering the new structure $g(\overline{\partial}_E)$. □

Remark. In the case of a curve, the terms of the filtration of E are locally free, and therefore subbundles of E. If we attach to each vector bundle of this filtration a Jördan-Hölder filtration, we immediately get an improvement of [Br, Theorem 5] which was our original motivation.

4. The case of surfaces

For complex surfaces, a Gieseker stable vector bundle may not be Mumford stable. By Theorem 2.4, we know that for Gieseker stable vector bundles which are not Mumford stable, the sequence of balanced metrics will not converge.

From another side, we know that in the case of surfaces, the singularities of torsion free sheaves are just points, and the reflexive sheaves are locally free [Ko, §5]:

Proposition 4.1. *Let M be a complex manifold. The singular set $S(\mathcal{F})$ of the analytic coherent sheaf $\mathcal{F}$ is defined as the closed subvariety upon which $\mathcal{F}$ is not locally free. If $\mathcal{F} \to M$ is a torsion free sheaf, $S(\mathcal{F})$ has codimension at least 2. If $\mathcal{F}$ is reflexive, then $S(\mathcal{F})$ has codimension at least* 3.

This remark is the key point of the "gluing construction" technique introduced by N. Buchdahl in [Bu2]. To a Mumford semi-stable sheaf $\mathcal{F}$ on M, one can associate a semi-stable vector bundle $\Sigma(\mathcal{F})$ that admits a Hermite-Einstein metric in the following way: $\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}^{**})$ and if $\mathcal{F}' \subset \mathcal{F}$ satisfies $\mu(\mathcal{F}') = \mu(\mathcal{F})$ then $\Sigma(\mathcal{F}) = \Sigma(\mathcal{F}') \oplus \Sigma(\mathcal{F}/\mathcal{F}')$. One checks by induction on the rank that we get a unique vector bundle $\Sigma(\mathcal{F})$. This vector bundle and $\mathcal{F}$ has same rank and determinant and $\Sigma(\mathcal{F})$ is a direct sum of Mumford stable vector bundles with the same normalized degree, i.e Mumford polystable. Moreover, we have non trivial homomorphisms $\mathcal{F} \to \Sigma(\mathcal{F})$ and $\Sigma(\mathcal{F}) \to \mathcal{F}^{**}$. For a semi-stable vector bundle E, we shall denote by $\mathbb{B}(E)$ the set of points $x \in M$ for which there exists a Mumford semi-stable vector bundle E' with $\mu(E') = \mu(E)$ and an inclusion $E' \to E$ such that $E'_x \to E_x$ is not of maximal rank. By an induction on the rank, one proves that $\mathbb{B}(E)$ is finite. In the case of a Kähler compact surface M, the following result holds [Bu2, Proposition 4.3]:

Lemma 4.0.4. *Let E be a Mumford semi-stable vector bundle such that $\Sigma(E) = \bigoplus_i \Upsilon_i \otimes E_i$ where Υ_i is vector space of dimension d_i and E_i is a Mumford stable vector bundle with $\mu(E_i) = \mu(E)$ and $E_i \ncong E_j$ for $i \neq j$. Let choose $e > r(E) \max_i (d_i/r(E_i))$. Then for all choice of e points $(x_i)_{i=1..e} \in M \backslash \mathbb{B}(E)$, there exists a vector bundle $\widetilde{E}$ on the resolution $\widetilde{M} \xrightarrow{\pi} M$ of these points such that:*

- *$\widetilde{E}$ restricts to $\mathcal{O}(1) \oplus \sum_1^{r-1} \mathcal{O}$ on each component of the exceptional divisor,*
- *$\left(\pi_* \widetilde{E}\right)^{**} = E$,*
- *$\widetilde{E}$ is Mumford stable with respect to the polarization $\omega_\varepsilon = \pi^*\omega + \varepsilon \sum_{i=1}^p s_i$ for ε small enough and where s_i is the non trivial holomorphic section that represents exactly the divisor $-\pi^{-1}(x_i)$.*

Now the fact that Mumford stability implies Gieseker stability gives us directly the following result.

Theorem 2. *Let M be a projective surface and E be a Gieseker stable vector bundle on M which is not Mumford stable. There exists a resolution $\widetilde{M} \xrightarrow{\pi} M$ consisting of a blowup of a finite number of points and a vector bundle $\widetilde{E}$ on $\widetilde{M}$ such that the balanced metrics (h_k) associated to $\widetilde{E}$ converge towards a weakly Hermite-Einstein metric and $\left(\pi_* \widetilde{E}\right)^{**} = E$.*

We now give a consequence of [Bu2, Proposition 2.4] (see also for details [Bu1]):

Definition 4.1. Let $\mathcal{F}$ be a reflexive sheaf and $\mathcal{F}_1 \subset \mathcal{F}$ with $\mathcal{F}/\mathcal{F}_1 = \mathcal{F}_2$. We note $Tor\,(\mathcal{F}_2)$ the torsion of $\mathcal{F}_2$. If $\mathcal{F}_2$ is torsion free, $\mathcal{F}_1$ is said to be saturated; otherwise its saturation is $\mathrm{Sat}\,(\mathcal{F}_1, \mathcal{F}) = \ker(\mathcal{F} \to \mathcal{F}_2/Tor\,(\mathcal{F}_2))$.

Lemma 4.0.5. *Let E be a holomorphic vector bundle on a smooth projective surface. Consider the following filtration of E*

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = E$$

by saturated sheaves. Then, there exists a resolution $\widetilde{M} \xrightarrow{\pi} M$ consisting of a blowup of a finite number of points and a filtration

$$0 = \widetilde{E_0} \subset \cdots \subset \widetilde{E_l} = \widetilde{E}$$

with $\left(\pi_* \widetilde{E}\right)^{**} = E$ and $\widetilde{E_i} = \mathrm{Sat}\left(\pi^* \mathcal{E}_i, \widetilde{E}\right)$ is a subbundle of $\widetilde{E}$.

For a blowup $\widetilde{M} \xrightarrow{\pi} M$ with $\mathbf{L}_\pi := \pi^{-1}(x_0)$ as exceptional divisor associated, the metric $\pi^*\omega$ is positive and degenerates only along the tangent directions to $\mathbf{L}_\pi$. Let $\mathbf{F}_{\mathbf{L}_\pi}$ be the curvature form of any smooth hermitian metric on the associated line bundle $\mathcal{O}(-\mathbf{L}_\pi)$. For $\delta > 0$ sufficiently small, $\omega_\delta = \pi^*\omega + \delta \mathbf{F}_{\mathbf{L}_\pi}$ is a smooth closed definite positive $(1,1)$-form. It is with respect to this polarization ω_δ that we will speak of stability on the manifold $\widetilde{M}$. We obtain under this setting a generalization of Theorem 2.

Theorem 3. *Let E be a holomorphic vector bundle on a smooth projective surface. Then, there exists a resolution $\widetilde{M} \xrightarrow{\pi} M$ consisting of a blowup of a finite number of points and a vector bundle $\widetilde{E}$ on $\widetilde{M}$ such that for $\delta > 0$ sufficiently small, and for all $\varepsilon > 0$, $r \geq 0$, there exists a smooth hermitian metric $\widetilde{h}$ on $\widetilde{E}$ with*

$$\left\| \sqrt{-1}\Lambda F_{\widetilde{h}} - \Pi^{\omega_\delta, \widetilde{h}}_{HN(\widetilde{E})} \right\|_{C^r} < \varepsilon$$

and $\left(\pi_* \widetilde{E}\right)^{**} = E$.

Proof. If the Harder-Narasimhan filtration is given by vector bundles, we apply Theorem 1. Otherwise, we prove by induction on the rank. The result is clear for rank 1. For $r(E) > 1$, we apply the previous lemma to get a filtration $\widetilde{E_i}$ of $\widetilde{E}$ and for all $\varepsilon > 0$, Theorem 1 and the hypothesis of induction allow us to find a hermitian metric $h'_i(\varepsilon)$ for $\widetilde{E_i}/\widetilde{E_{i-1}}$ such that

$$\left\| \sqrt{-1}\Lambda F_{h'_i(\varepsilon)} - \Pi^{\omega_\delta, h'_i}_{HN\left(\widetilde{E_i}/\widetilde{E_{i-1}}\right)} \right\|_{C^r} < \varepsilon.$$

By considering the smooth metric $\widetilde{h} = \oplus_{i=1}^{l} h'_i(\varepsilon)$, and by using the same kind of argument that for Theorem 1, we can conclude. □

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