On Iteration of the Ricci Operator on the Space of Kähler Metrics

Abstract. In this note we consider iteration on the c_1 class of a Fano manifold using the Ricci operator. Motivated by a problem posed by Nadel we define negative powers of the Ricci operator and demonstrate that while orbits of positive powers have no limit points, orbits of negative powers converge in C^∞ precisely when the manifold is Kähler-Einstein and in that case the set of limits coincides with the set of Kähler-Einstein metrics. Our work is closely related to the study of energy functionals on the space of Kähler metrics and we show how it can be used to simplify the proof of some of the results of Song and Weinkove.

1 Introduction

Let M be a compact closed Kähler manifold of complex dimension n with positive first Chern class $c_1(M)$. For any Kähler metric g we let $\omega := \omega_g = \sqrt{-1}g_{i\bar{j}}(z)dz^i \wedge d\bar{z}^j$ denote its corresponding Kähler form, a closed positive (1,1)-form on M. The motivation for this note comes from a problem posed by Nadel, later investigated by the first author [K]. In a short note [N] Nadel considers iteration on the space $\mathcal{H}_{c_1(M)}$ of Kähler metrics cohomologous to $c_1(M)$, defined inductively using the Ricci operator as follows. Let $\mathcal{H}^{(0)}_{c_1(M)}$ denote the set of all smooth forms in the $c_1(M)$ class and let $\mathrm{Ric}^{(0)} := \mathbb{I}$ be the identity operator. For any Kähler metric we let $\mathrm{Ric}\,(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$ denote the Ricci form of ω. It is well-defined globally and lies again in the $c_1(M)$ class. If it is positive we let $\mathrm{Ric}^{(2)}(w)$ denote its Ricci form, and in a similar fashion we define higher powers of the operator as long as the positivity is preserved. If we let $\mathcal{H}^{(k)}_{c_1(M)}$ denote the maximal domain of definition of $\mathrm{Ric}^{(k)}$, we obtain a filtration of $\mathcal{H}^{(0)}_{c_1(M)}$. We also let $\overline{\mathcal{H}^{(k)}_{c_1(M)}}$ denote the set of all metrics in $\mathcal{H}^{(k-1)}_{c_1(M)}$ whose Ricci curvature is nonnegative. The motivation for this construction comes from the simple fact that, when they exist, Kähler-Einstein metrics are by definition fixed points of the iteration process. Nadel asks whether these are all periodic points and proves in that direction the absence of periodic points of order two and three. Furthermore he raised the question whether Kähler-Einstein metrics could be related to higher order periodic points of this iteration. Using an inequality on Monge-Ampère masses (also proved independently by Blocki [Bl1]), the first author obtained the

Theorem 1.1. *[K] Let (M,ω) be as above and assume that $\mathrm{Ric}^{(k)}(\omega) = \omega$ for some $k \in \mathbb{N}$. Then ω is Kähler-Einstein.*

The nonexistence of periodic points motivates a natural reinterpretation of Nadel's original second question to include limit points as periodic points of infinite order. The purpose of this paper is to address this question.

To that end we generalize Nadel's iteration scheme to include negative powers (defined in Section 2 below). We then prove that the iteration process of Nadel has no non-trivial limit points, while, in contrast, the orbits of the modified dynamical system converge if and only if a Kähler-Einstein metric exists, and that the limit (when it exists) is a Kähler-Einstein metric (Theorem 2.1). This shows that indeed the behavior of the dynamical system is closely tied with the existence of Kähler-Einstein metrics and hence with the complex structure of M

and answers Nadel's second question. Our proof also demonstrates the nonexistence of fixed points. The proof makes use of Mabuchi's K-energy functional and the work of Bando and Mabuchi [BM],[B]. We then remark that the result of this note can be used to simplify the proofs of two of the main theorems in the work of Song and Weinkove [SW] on the Chen-Tian energy functionals, a collection of functionals which include the K-energy [CT]. The requisite background material and results on the various energy functionals is collected in Section 3 and the proof of Theorem 2.1 and some of its immediate applications are given in Section 4. We conclude with some remarks and further questions for future study.

2 Ricci iteration

The key observation of this note lies in the fact that Nadel's construction can be reversed. Let α be any form representing the $c_1(M)$ class. By the Calabi-Yau (C-Y) Theorem [Y] there exists a unique Kähler form in $\mathcal{H}_{c_1(M)}$, which we denote by ω_1, whose Ricci form equals α. We define the *inverse Ricci operator* by $\mathrm{Ric}^{(-1)}(\alpha) := \omega_1$. Similarly we define higher powers $\mathrm{Ric}^{(-k)} := \mathrm{Ric}^{(-1)} \circ \cdots \circ \mathrm{Ric}^{(-1)}$, making repeated use of the powerful C-Y Theorem. The advantage of this construction lies in the fact that iterations *improve* the positivity of α. Indeed, if we consider the filtration $\{\mathcal{H}^{(j)}_{c_1(M)}\}_{j\geq 0}$ to measure positivity in some sense then the image of $\mathrm{Ric}^{(-k)}$ lies in $\mathcal{H}^{(k)}_{c_1(M)}$ and we may iterate the inverse Ricci operator to any desired power. Note that this yields homeomorphisms $\mathrm{Ric}^{(k)} : (\mathcal{H}^{(l)}_{c_1(M)}, ||\cdot||_{\mathcal{C}^{m,\beta}}) \to (\mathcal{H}^{(l+k)}_{c_1(M)}, ||\cdot||_{\mathcal{C}^{m-2k,\beta}})$ for all $\beta \in (0,1)$, $m, l \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}$ such that $l + k \geq 0$ and $m \geq 2k$. We let $\mathcal{H}^{(\infty)}_{c_1(M)}$ denote the set of all $L^\infty(M)$ limits $\lim_{k\to\infty} \mathrm{Ric}^{(-k)}\alpha$, for $\alpha \in \mathcal{H}^{(0)}_{c_1(M)}$ (when they exist). Let $\mathcal{E} \subseteq \mathcal{H}_{c_1(M)}$ denote the set of Kähler-Einstein metrics (possibly empty). The result we wish to present in this note is the following

Theorem 2.1. *Let (M,ω) be a compact closed Kähler manifold with $[\omega] = c_1(M)$.*
(i) Suppose that $\mathrm{Ric}^{(k)}(\omega) = \omega$ for some $k \in \mathbb{Z}$. Then ω is a Kähler-Einstein metric.
(ii) In general, $\mathcal{H}^{(\infty)}_{c_1(M)} = \mathcal{E}$.
(iii) In contrast, there are no limit points for iterations of positive powers of Ric.

Therefore one may view the Kähler-Einstein condition as equivalent to the existence of a contraction on the space of smooth Kähler potentials defined up to a constant. In order to prove such a statement it is natural to seek for a sort of norm which is decreased along the orbits in this function space. Indeed the proof makes use of such a functional, which we introduce in the next section.

3 Energy functionals on the space of Kähler metrics

We call a function $A : \mathcal{H}_{c_1(M)} \times \mathcal{H}_{c_1(M)} \to \mathbb{R}$ an *energy functional* if it is zero on the diagonal. By an *exact energy functional* we will mean one which satisfies in addition $A(\omega_1,\omega_2) + A(\omega_2,\omega_3) = A(\omega_1,\omega_3)$ (the *cocycle condition* after Mabuchi [M]).

Let $V := \int_M \omega^n = [\omega]^n([M])$ and let $||\varphi||^2_{W^{1,2}(M,w)} := |\varphi|^2_{W^{1,2}(M,w)} + |\varphi|^2_{L^2(M,w)}$ where $|\varphi|^2_{W^{1,2}(M,w)} := V^{-1}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \omega^{n-1}$ and $|\varphi|^2_{L^2(M,w)} := V^{-1}\int_M \varphi^2\omega^n$. The energy functionals I, J, introduced by Aubin in [A2], are defined for each pair $(\omega, \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi)$

by

$$I(\omega,\omega_\varphi)=V^{-1}\int_M\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{i=0}^{n-1}\omega^i\wedge\omega_\varphi^{n-1-i},$$

$$J(\omega,\omega_\varphi)=\frac{V^{-1}}{n+1}\int_M\sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\sum_{i=0}^{n-1}(n-i)\omega^i\wedge\omega_\varphi^{n-1-i}.$$

Note that I,J and $I-J$ are all nonnegative and equivalent. In particular, $(I-J)(\omega,\omega_\varphi)\geq C|\varphi|^2_{W^{1,2}(M,\omega)}$ for some $C=C(n)>0$.

The Chen-Tian energy functionals $E_k,\ k=0,\ldots,n$, are defined as follows. Connect each pair $(\omega,\omega_{\varphi_1}:=\omega+\sqrt{-1}\partial\bar\partial\varphi_1)$ with a piecewise smooth path $\{\omega_{\varphi_t}\}$ and put

$$\begin{aligned}E_k(\omega,\omega_{\varphi_1}) \ &= \ (k+1)V^{-1}\int_{M\times[0,1]}\Delta_{\varphi_t}\dot\varphi_t\mathrm{Ric}\,(\omega_{\varphi_t})^k\wedge\omega_{\varphi_t}^{n-k}\wedge dt\\ &\quad-(n-k)V^{-1}\int_{M\times[0,1]}\dot\varphi_t(\mathrm{Ric}\,(\omega_{\varphi_t})^{k+1}-\mu_k\omega_{\varphi_t}^{k+1})\wedge\omega_{\varphi_t}^{n-1-k}\wedge dt,\end{aligned}\tag{1}$$

where $\mu_k:=\frac{c_1(M)^{k+1}\cup[\omega]^{n-k-1}([M])}{[\omega]^n([M])}$. This gives rise to well-defined *exact* energy functionals independent of the choice of path [CT]. The *K-energy*, E_0, was introduced by Mabuchi [M]. The importance of those functionals lies in the fact that they integrate the *generalized Futaki invariants* $\mathcal{F}_k:h(M)\to\mathbb{R}$ defined on the Lie algebra of holomorphic vector fields $h(M)$ and that the non-vanishing of those invariants is an obstruction to the existence of Kähler-Einstein metrics on Fano manifolds, as first proved by Mabuchi for $k=0$ [M] and by Chen and Tian for all k [CT]. This implies that E_k vanishes on pairs joined by a one parameter subgroup of automorphisms through the identity [CT, Corollary 5.5].

Bando and Mabuchi proved the following

Theorem 3.1. *[BM, Theorem A],[B, Theorem 1] Given a Kähler-Einstein metric $\omega_{KE}\in\mathcal{H}_{c_1(M)}$, one has $E_0(\omega_{KE},\omega)\geq 0$ for all $\omega\in\mathcal{H}_{c_1(M)}$ with equality if and only if ω is Kähler-Einstein and in that case there exists a holomorphic automorphism homotopic to the identity h such that $h^\star\omega_{\mathrm{KE}}=\omega$.*

The proof relies on showing that E_0 is (strictly) monotonically decreasing along a certain well-chosen deformation path ending at ω_{KE} . The deformation $\{\omega_{\varphi_t}\}\subseteq\mathcal{H}_{c_1(M)}$ chosen is constructed from two paths, solutions of the following Monge-Ampère equations

$$\omega_{\varphi_t}^n=\begin{cases}e^{tf+c_t}\omega^n, & t\in[0,1]\\ e^{f-(t-1)\varphi_t}\omega^n, & t\in[1,2]\end{cases}\tag{2}$$

where $\mathrm{Ric}\,\omega-\omega=\sqrt{-1}\partial\bar\partial f$ with the normalizations $\int_M e^{tf+c_t}\omega^n=\int_M e^{f-(t-1)\varphi_t}\omega^n=V$.

The theorem then follows from

Proposition 3.2. *One has*

$$\frac{d}{dt}E_0(\omega_{\mathrm{KE}},\omega_{\varphi_t})=\begin{cases}-(1-t)V^{-1}\int_M(\Delta_{\omega_{\varphi_t}}\dot\varphi_t)^2\omega_{\varphi_t}^n-\frac{d}{dt}(I-J)(\omega,\omega_{\varphi_t}), & t\in[0,1]\\ -(2-t)\frac{d}{dt}(I-J)(\omega_{\mathrm{KE}},\omega_{\varphi_t}), & t\in[1,2]\end{cases}\tag{3}$$

from which the Theorem follows. Note that the first path is the one used in Yau's continuity method proof [Y]. It connects any point ω in $\mathcal{H}_{c_1(M)}$ to $\mathrm{Ric}^{(-1)}\omega$ in $\mathcal{H}^{(2)}_{c_1(M)}$. The second path, introduced by Aubin in [A2], is used to connect any point in $\mathcal{H}^{(2)}_{c_1(M)}$ to a Kähler-Einstein metric.

We now state two of the main results in [SW] on the boundedness of the E_k functionals on the c_1 class of Fano manifolds. The first establishes their partial boundedness.

Theorem 3.3. *[SW, Theorem 1.1] Let M be as above. If M admits a Kähler-Einstein metric ω_{KE} then for any $\omega \in \overline{\mathcal{H}^{(2)}_{c_1(M)}}$ and for each $k = 0, \ldots, n$ one has $E_k(\omega_{\mathrm{KE}}, \omega) \geq 0$, with equality if and only if ω is Kähler-Einstein.*

Their proof analyzes the behavior of E_k along the second path. A delicate point arises here since while the solution for the Monge-Ampère equation always exists at $t = 2$ it may fail to extend to a path even for nearby t when M admits non-trivial holomorphic vector fields. This lies at the heart of the work of Bando and Mabuchi [BM]. Therefore in order to complete their proof of Theorem 3.2, Song and Weinknove rely on a careful approximation result from [BM] needed in order to circumvent this difficulty. Then they carry out a detailed computation showing that while E_k may not necessarily be monotone (when the path exists), one still has $E_k(\omega_{\mathrm{KE}}, \omega_{\varphi_1}) \geq E_k(\omega_{\mathrm{KE}}, \omega_{\varphi_2}) = 0$.

Their second theorem is a strengthening of the above for the case $k = 1$ providing the analogue of Theorem 3.1 for E_1.

Theorem 3.4. *[SW, Theorem 1.2] Let M be as above. If M admits a Kähler-Einstein metric ω_{KE} then for any $\omega \in \mathcal{H}_{c_1(M)}$ one has $E_1(\omega_{\mathrm{KE}}, \omega) \geq 0$, with equality if and only if ω is Kähler-Einstein.*

Their proof makes use of Theorem 3.2 together with an analogous calculation for the first path showing that similarly to the second path one has $E_1(\omega_{\varphi_1}, \omega) \geq E_1(\omega_0, \omega) = 0$. Explicitly, their computation shows that

$$\begin{aligned} E_k(\omega_{\varphi_1}, \omega) =& V^{-1} \int_M \sqrt{-1}\partial\varphi_1 \wedge \bar{\partial}\varphi_1 \wedge \sum_{i=0}^{n-1} a_i \omega^i \wedge \omega_{\varphi_1}^{n-1-i} \\ &+ (k+1)V^{-1} \int_{M\times[0,1]} (1-t)(\Delta_{\omega_{\varphi_t}} \dot{\varphi}_t)^2 \omega_{\varphi_t}^n \wedge dt \\ &- V^{-1} \int_M \sum_{i=1}^{k} \binom{i+1}{k+1} f(\sqrt{-1}\partial\bar{\partial} f)^i \wedge \omega^{n-i}, \end{aligned} \tag{4}$$

with $a_i = \begin{cases} \frac{(n-k)(i+1)}{n+1}, & 0 \leq i \leq k-1 \\ \frac{(k+1)(n-i)}{n+1}, & k \leq i \leq n \end{cases}$. Since the last term is positive on $\mathcal{H}_{c_1(M)}$ for $k = 1$ they conclude their proof.

Later we will make use of the following observation.

Lemma 3.5. *For all $\omega \in \overline{\mathcal{H}^{(2)}_{c_1(M)}}$ and for all $k = 0, \ldots, n$, $E_k(\omega_{\varphi_1}, \omega) \geq 0$ with equality if and only if ω is Kähler-Einstein.*

Proof. From the definition of f it follows that

$$\sum_{i=1}^{k} \binom{k+1}{i+1} (\sqrt{-1}\partial\bar{\partial} f)^{i-1} \wedge \omega^{n-i} = \sum_{i=1}^{k} \binom{k+1}{i+1} (\mathrm{Ric}\ \omega - \omega)^{i-1} \wedge \omega^{n-i}$$

$$
\begin{aligned}
&= \sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^{i-1-j}\binom{k+1}{i+1}\binom{i-1}{j}\mathrm{Ric}\,(\omega)^j \wedge \omega^{n-1-j}\\
&= \sum_{j=0}^{k-1}\mathrm{Ric}\,(\omega)^j \wedge \omega^{n-1-j}\sum_{i=1}^{k-j}(-1)^{i-1-j}\binom{k+1}{i+1}\binom{i-1}{j}\\
&= \sum_{j=0}^{k-1}(k-j)\mathrm{Ric}\,(\omega)^j \wedge \omega^{n-1-j}
\end{aligned}
$$

where we made use of the combinatorial identity

$$
\sum_{i=1}^{k-j}(-1)^{i-1-j}\binom{k+1}{i+1}\binom{i-1}{j} = k-j.
$$

Therefore the third term in (4) may be rewritten in the form

$$
\int_M \sqrt{-1}\partial f \wedge \bar{\partial} f \wedge \sum_{i=1}^{k} i\omega^{n-1-(k-i)} \wedge \mathrm{Ric}\,(\omega)^{k-i}. \tag{5}
$$

Note that it is nonnegative on $\overline{\mathcal{H}^{(2)}_{c_1(M)}}$ with equality if and only if f is constant (thanks to the $i=k$ term). Since the other two terms in (4) are nonnegative the Lemma follows. □

In the following section we show how Theorem 2.1 may be applied to provide a simplified proof of Theorems 3.2 and 3.3 of Song and Weinkove, in particular avoiding both the lengthy computations involving the second path (in (2)) and the difficulties arising from the existence of holomorphic vector fields (cf. [SW, Section 3]).

4 Negative first Chern class

5 Proof of Theorem 2.1 and Applications

Proof of Theorem 2.1. The proof is a simple consequence of the work of Bando and Mabuchi. Note that the nonexistence of fixed points of negative order implies that of positive order, and vice versa. Therefore assume that for some $\omega \in \mathcal{H}_{c_1(M)}$ and some $l \in \mathbb{N}$ one has $\mathrm{Ric}^{(-l)}(\omega) = \omega$. By the cocycle condition we therefore have

$$
0 = E_0(\omega, \mathrm{Ric}^{(-l)}\omega) = \sum_{i=0}^{l-1} E_0(\mathrm{Ric}^{(-i)}\omega, \mathrm{Ric}^{(-i-1)}\omega). \tag{6}
$$

On the other hand, from the first part of (3)

$$
E_0(\omega, \mathrm{Ric}^{(-1)}\omega) = -V^{-1}\int_{M\times[0,1]}(1-t)(\Delta_{\omega_{\varphi_t}}\dot{\varphi}_t)^2\omega^n_{\varphi_t}\wedge dt - (I-J)(\omega, \mathrm{Ric}^{(-1)}\omega) \le 0, \tag{7}
$$

with equality if and only if $\mathrm{Ric}^{(-1)}\omega = \omega$. Therefore each of the terms in (6) must vanish identically and we conclude that (M,ω) is Kähler-Einstein. This proves (i).

We now turn to the limiting behavior of the iteration and assume that $\omega \in \mathcal{H}_{c_1(M)} \setminus \mathcal{E}$. Again from the cocycle condition one has that

$$C_\omega := \lim_{l\to\infty} E_0(\omega, \mathrm{Ric}^{(-l)}\omega) = \sum_{i=0}^{\infty} E_0(\mathrm{Ric}^{(-i)}\omega, \mathrm{Ric}^{(-i-1)}\omega). \tag{8}$$

From (7) we know that each of the summands is negative and hence the limit exists in $\mathbb{R}^- \cup \{-\infty\}$. For each $i \geq 0$ write $\mathrm{Ric}^{(-i-1)}\omega = \mathrm{Ric}^{(-i)}\omega + \sqrt{-1}\partial\bar{\partial}\varphi^{(i)}$. Set $\varphi := \lim_{l\to\infty} \sum_{i=0}^{l} \varphi^{(i)}$ when the limit exists.

Now suppose that ω is such that $\lim_{l\to\infty} \mathrm{Ric}^{(-l)}\omega \in \mathcal{H}^{(\infty)}_{c_1(M)}$. Rewriting (8)

$$\lim_{l\to\infty} E_0(\omega, \omega + \sqrt{-1}\partial\bar{\partial}\sum_{i=0}^{l}\varphi^{(i)}) = \sum_{i=0}^{\infty} E_0(\mathrm{Ric}^{(-l)}\omega, \mathrm{Ric}^{(-l-1)}\omega) \leq -C_\omega < \infty.$$

This implies that the limit satisfies the Kähler-Einstein equation. Since by assumption the limit is in $L^\infty(M)$

the usual theory of non-degenerate Monge-Ampère equations implies that φ has a $\mathcal{C}^{2,\alpha}(M)$ bound and consequently, from elliptic regularity theory, it follows that $\omega_\varphi \in \mathcal{E}$. Hence $\mathcal{H}^{(\infty)}_{c_1(M)} \subseteq \mathcal{E}$.

Next, we show that if $\mathcal{E} \neq \emptyset$ then $\mathcal{H}^{(\infty)}_{c_1(M)} \neq \emptyset$. Indeed, for any $\omega_{\mathrm{KE}} \in \mathcal{E}$ and $\omega \in \mathcal{H}_{c_1(M)}$

$$E_0(\omega_{\mathrm{KE}}, \mathrm{Ric}^{(-l)}\omega) = E_0(\omega_{\mathrm{KE}}, \omega) + \sum_{i=0}^{l-1} E_0(\mathrm{Ric}^{(-i)}\omega, \mathrm{Ric}^{(-i-1)}\omega). \tag{9}$$

If we had a uniform a priori $\mathcal{C}^0$ bound on $\sum_{i=0}^{l}\varphi^{(i)}$, then (9) would imply that φ exists, lies in $W^{1,2}(M, w)$ and satisfies the Kähler-Einstein equation $\mathrm{Ric}\,(\omega_\varphi) = \omega_\varphi$. Since then φ would be a weak solution of a non-degenerate Monge-Ampère equation elliptic regularity would imply that $\varphi \in C^\infty(M)$. We now prove

Lemma 5.1. *For each $\omega \in \mathcal{H}_{c_1(M)}$ there exists $C > 0$ depending only on (M, ω) such that $||\sum_{i=0}^{l}\varphi^{(i)}||_{L^\infty(M)} < C$ for all $l \geq 0$.*

Hence to complete the proof of (ii) we only need to show that $\mathcal{E} \subseteq \mathcal{H}^{(\infty)}_{c_1(M)}$ when $\mathcal{H}^{(\infty)}_{c_1(M)} \neq \emptyset$. But indeed take a single point $\omega \in \mathcal{H}^{(0)}_{c_1(M)}$ whose orbit converges (necessarily) to a Kähler-Einstein metric ω_{KE}. The fact that $\mathrm{Ric}^{(-1)}$ commutes with pull-back by automorphisms together with the second part of Theorem 3.1 then imply that

$$\left\{ \lim_{l\to\infty} \mathrm{Ric}^{(-l)} h^\star\omega : h \in \mathrm{Aut}(M)_0 \right\} = \mathcal{E},$$

where $\mathrm{Aut}(M)_0$ denotes the identity component of the holomorphic automorphisms group.

Next, to prove (iii) note that positive powers of Ric can never converge on $\mathcal{H}_{c_1(M)} \setminus \mathcal{E}$. Indeed by our previous argument the limit must be Kähler-Einstein when it is regular, yet since E_0 strictly increases along iterations this would imply the existence of metrics whose K-energy with respect to this Kähler-Einstein metric is negative, contradicting Theorem 3.1. □

Proof of Theorem 3.2. Let ω_{KE} be a Kähler-Einstein metric in $\mathcal{H}_{c_1(M)}$ and let ω be any Kähler metric with nonnegative Ricci curvature. By Theorem 2.1 $\omega_1 := \lim_{l\to\infty} \mathrm{Ric}^{(-l)}\omega$ exists and is Kähler-Einstein. Then Lemma 3.4 gives

$$E_k(\omega_{\mathrm{KE}}, \omega) = E_k(\omega_{\mathrm{KE}}, \omega_1) + \sum_{i=0}^{l} E_k(\mathrm{Ric}^{(-i-1)}\omega, \mathrm{Ric}^{(-i)}\omega) \geq E_k(\omega_{\mathrm{KE}}, \omega_1).$$

Since $\omega_1 = h^\star\omega_{\mathrm{KE}}$ for some $h \in \mathrm{Aut}(M)_0$, the right hand side vanishes by the invariance of the Chen-Tian functionals (cf. Section 2) and we are done. □

Proof of Theorem 3.3. The proof is identical except that for $k = 1$ (4) is non-positive on the whole of $\mathcal{H}_{c_1(M)}$ and so we may start our iteration at any point $\omega \in \mathcal{H}_{c_1(M)}$. □

6 Remarks

Iteration on other classes. Let $\omega \in \mathcal{H}_\Omega$ be a Kähler representative of an arbitrary class Ω in the Kähler cone of a Fano manifold M. Let $\nu \in \mathcal{H}^{(0)}_{c_1(M)}$ be a representative of $c_1(M)$. By the C-Y Theorem there exists a unique Kähler representative ω_1 of Ω whose Ricci form equals ν. Define a map $\mathrm{Ric}^{(-1)}_\Omega : \mathcal{H}^{(0)}_{c_1(M)} \to \mathcal{H}_\Omega$ by $\mathrm{Ric}^{(-1)}_\Omega \nu := \omega_1$ (when appropriate norms are chosen, it defines a homeomorphism of Banach spaces, as in Section 2). We may therefore define an operator Ric_Ω on $\mathcal{H}_\Omega$ by $\mathrm{Ric}_\Omega := \mathrm{Ric}^{(-1)}_\Omega \circ \mathrm{Ric} \circ \mathrm{Ric}$. This gives rise to an iteration on Ω by setting $\mathrm{Ric}^{(k)}_\Omega := \mathrm{Ric}^{(-1)}_\Omega \circ \mathrm{Ric}^{(k)} \circ \mathrm{Ric}$ for each $k \in \mathbb{Z}$. Note that this induces a filtration on $\mathcal{H}_\Omega$ defined by $\mathcal{H}^{(k)}_\Omega := \{\omega \in \mathcal{H}_\Omega : \mathrm{Ric}\,\omega \in \mathcal{H}^{(k)}_{c_1(M)}\}$.

From Theorem 2.1 we see that that orbits of the new dynamical systems converge, if and only M is Kähler-Einstein, to a Kähler representative which is characterized by the property that its Ricci form is Kähler-Einstein. For $\Omega \neq c_1(M)$, how is this dynamical system related to the study of the space $\mathcal{H}_\Omega$? In particular, it would be interesting to relate this to the existence problem of extremal metrics for Kähler classes near c_1.

Upper bounds for E_k and escape rates. For a Kähler-Einstein manifold, E_0 and E_1 are known to be bounded from below on the c_1 class (cf. Section 3). In fact, a stronger statement is true: E_0 and E_1 are *proper* in the sense of Tian if and only there exists a Kähler-Einstein metric in the class [T], [SW, Theorem 1.4]. In general though, one knows that they are not bounded from above. For example along a one-parameter subgroup of automorphisms for which the corresponding generalized Futaki invariant does not vanish. Or (for E_0 at least) simply take a sequence of potentials for which $I - J$ blows up. It is natural to ask whether the filtration $\{\mathcal{H}^{(k)}_{c_1(M)}\}_{k\geq 0}$ corresponds to different energy sub-level sets of the E_k. In particular we ask whether there exists a constant depending only on $(M.\omega_0)$, $l \in \mathbb{N}$ and $\epsilon > 0$ such that E_k is bounded from above on $\mathcal{H}^{(l)}_{c_1(M)}(\epsilon) := \{\omega \in \mathcal{H}^{(l)}_{c_1(M)} : \mathrm{Ric}^{(l)}\omega \geq \epsilon\omega_0\}$.

In light of Theorem 2.1 one readily sees that such a result would relate to the question of determining the limiting behavior of positive iterates of Ric. At the moment it is only obvious that many orbits escape to $\mathcal{H}_{c_1(M)} \setminus \mathcal{H}^{(2)}_{c_1(M)}$ after finitely many iterations, but it is not clear if there are Kähler metrics which do not lie in $\mathrm{Ric}^{(-l)}(\mathcal{H}_{c_1(M)} \setminus \mathcal{H}^{(2)}_{c_1(M)})$ for some l. It would be interesting to determine whether one may find orbits of infinite length and whether asymptotically they could correspond to points in $\overline{\mathcal{H}^{(2)}_{c_1(M)}} \setminus \mathcal{H}^{(2)}_{c_1(M)}$ or otherwise blow up (special degenerations).

Questions of stability. When M is not Kähler-Einstein one may still apply the iteration process, though it must blow-up. It would be interesting to know whether such a sequence corresponds to a special degeneration and hence explore the relation of the dynamics presented here to questions of G.I.T stability related to the existence problem for Kähler-Einstein metrics.

Let $\omega = \omega_0$ denote the initial metric and let φ_1 be a Kähler potential with

$$\operatorname{Ric}(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_1) = \omega_0.$$

Put $\sqrt{-1}\partial\bar{\partial}h = \operatorname{Ric}\omega_0 - \omega_0$. The function h thus given is for the moment determined only up to a an additive constant. The equation then becomes

$$-\sqrt{-1}\partial\bar{\partial}\log\det g_{\varphi_1} = \omega_0 = \omega_0 - \operatorname{Ric}\omega_0 + \operatorname{Ric}\omega_0 = -\sqrt{-1}\partial\bar{\partial}\log\det g - \sqrt{-1}\partial\bar{\partial}h$$

or

$$\sqrt{-1}\partial\bar{\partial}\log\frac{\omega_{\varphi_1}^n}{\omega^n} = \sqrt{-1}\partial\bar{\partial}h$$

that is

$$\omega_{\varphi_1}^n = \omega^n e^h$$

together with the volume normalization

$$\frac{1}{V}\int_M e^h\omega^n = 1.$$

This determines φ_1 only up to a constant, which will be fixed in the the next step.

Put $\omega_1 = \omega_{\varphi_1}$. In the second step we solve

$$\operatorname{Ric}(\omega_1 + \sqrt{-1}\partial\bar{\partial}\varphi_2) = \omega_1$$

and $\omega_1 - \operatorname{Ric}\omega_1 = \omega_1 - \omega_0 = \sqrt{-1}\partial\bar{\partial}\varphi_1$. The Monge-Ampère equation is now

$$\omega_{\varphi_1+\varphi_2}^n = \omega_{\varphi_1}^n e^{-\varphi_1} = \omega^n e^{h-\varphi_1},$$

with φ_1 determined uniquely by

$$\frac{1}{V}\int_M \omega^n e^{h-\varphi_1} = 1.$$

Iterating this procedure we have $\operatorname{Ric}^{(-l)}\omega = \omega_{\Sigma_{j=1}^l\varphi_j}$ for each $l \in \mathbb{N}$ where

$$\omega_{\Sigma_{j=1}^l\varphi_j}^n = \omega^n e^{h-\Sigma_{j=1}^{l-1}\varphi_j}, \tag{10}$$

and each of the φ_j is uniquely determined by

$$\frac{1}{V}\int_M \omega^n e^{h-\Sigma_{j=1}^{l-1}\varphi_j} = 1. \tag{11}$$

From now on we set $\Phi_l = \Sigma_{j=1}^l\varphi_j$ and $\omega_l = \omega_{\Phi_l}$.

The following formula is taken from [T,§7.2]. For the derivation (for any Kähler class) we refer to [C] where an equivalent expression is derived.

Proposition 6.1. *Let h be a function satisfying* $\operatorname{Ric}\omega - \mu\omega = \sqrt{-1}\partial\bar{\partial}h$. *One has*

$$E_0(\omega, \omega_\varphi) = \frac{1}{V}\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n - \mu(I-J)(\omega,\omega_\varphi) + \frac{1}{V}\int_M h(\omega^n - \omega_\varphi^n). \tag{12}$$

As an immediate corollary we have

Proposition 6.2. *Let $\mu = 1$. Then*

$$E_0(\omega, \omega_l) = -(I-J)(\omega,\omega_l) - \frac{1}{V}\int_M \sum_{k=1}^{l-1}\varphi_k\omega_l^n + \frac{1}{V}\int_M h\omega^n \leq 0, \tag{13}$$

with equality if and only if ω is Kähler-Einstein.

Proof. The formula follows directly from the Proposition. To show the inequality we note that

$$\begin{aligned} E_0(\omega_{k-1}, \omega_k) =& \frac{1}{V}\int_M -\varphi_{k-1}\omega_k^n - (I-J)(\omega_{k-1},\omega_k) + \frac{1}{V}\int_M -\varphi_{k-1}(\omega_{k-1}^n - \omega_k^n). \\ =& -(I-J)(\omega_{k-1},\omega_k) - \frac{1}{V}\int_M \varphi_{k-1}\omega_{k-1}^n. \end{aligned}$$

The first term is nonpositive with equality iff $\omega_k = \omega_{k-1} = \operatorname{Ric}\omega_k$, while the second term is nonpositive since

$$1 = \frac{1}{V}\int_M \omega_k^n = \frac{1}{V}\int_M e^{-\varphi_{k-1}}\omega_{k-1}^n \geq \frac{1}{V}\int_M (1-\varphi_{k-1})\omega_{k-1}^n.$$

Since

$$E_0(\omega, \omega_l) = \sum_{k=1}^{l} E_0(\omega_{k-1},\omega_k)$$

the conclusion follows. □

Let G denote the Green function for $\Delta = \Delta_{\bar\partial}$ WRT (M,ω) with $\int_M G(x,y)\omega^n(y) = 0$ and $A(M,\omega) = -\inf G$ such that

$$f(x) - \frac{1}{V}\int_M f\omega^n = -\frac{1}{V}\int_M G(x,y)\Delta f(y)\omega^n(y), \quad \forall\, f \in C^\infty(M).$$

Theorem 6.3. *[BM] One has*

$$A(M,\omega) \le \frac{c_n}{2}\mathrm{diam}(M,\omega)^2.$$

If $\mathrm{Ric}\,\omega \ge \epsilon\omega$ *then* $\mathrm{diam}(M,\omega)^2 \le \pi^2(2n-1)/\epsilon$ *by Myers' Theorem [P, p.245].*

Proposition 6.4. *Assume that* $\mathrm{Aut}(M,J)$ *is finite. Assume that there exists an integer* $l_0(\omega)$ *and an* $\epsilon > 0$ *such that* $\mathrm{Ric}\,\omega_l \ge \epsilon\omega_l, \quad \forall\, l \ge l_0$. *Then there exists a constant* C_1 *depending only on* (M,w) *such that*

$$||\Phi_l||_{L^\infty(M,\omega)} \le C_1, \quad \forall\, l \in \mathbb{N}.$$

Proof. Let G_l be the Green function for $\Delta_l = \Delta_{\bar\partial,\omega_l}$ (i.e., the Laplacian WRT (M,ω_l)) satisfying $\int_M G_l(x,y)\omega_l^n(y) = 0$. Set $A_l = -\inf_{M\times M} G_l$.

Since $-n < \Delta_0\Phi_l$ and $n > \Delta_l\Phi_l$ the Green formula gives

$$\begin{aligned}
\Phi_l(x) - \frac{1}{V}\int_M \Phi_l\omega^n &= -\frac{1}{V}\int_M G(x,y)\Delta\Phi_l(y)\omega^n(y) \le nA_0,\\
\Phi_l(x) - \frac{1}{V}\int_M \Phi_l\omega_l^n &= -\frac{1}{V}\int_M G(x,y)\Delta\Phi_l(y)\omega_l^n(y) \ge -nA_l.
\end{aligned}$$

Hence

$$\mathrm{osc}\,\Phi_l \le n(A_0 + A_l) + I(\omega,\omega_l). \tag{14}$$

The existence of a Kähler-Einstein metric implies the properness of E_0 in the sense of Tian [T1], [T2, Ch.6]. In other words, if $E_0(\omega,\cdot)$ is bounded from above on a subset of $\mathcal{H}$ so is $I(\omega,\cdot)$. Since for each l one has $E_0(\omega,\omega_l) \le 0$ we conclude that $I(\omega,\omega_l)$ is uniformly bounded independently of l.

Finally, the Proposition follows from Theorem 6.3 which provides for a uniform bound for A_l. □

Note: even to show the curvature bound for a subsequence would be good at this point.

Assume that (M,J) admits a Kähler-Einstein metric ω_{KE}. Consider the family of Monge-Ampère equations (2). Define the *Aubin operators* Aub_ϵ by setting $\mathrm{Aub}_\epsilon(\omega) = \omega_{\varphi_{1+\epsilon}}$ for each $\epsilon \in [0,1]$. Note that $\mathrm{Aub}_0(\omega) = \mathrm{Ric}^{(-1)}\omega$ and $\mathrm{Aub}_1(\omega) = \omega_{\mathrm{KE}}$. Formally, $\mathrm{Aub}_\epsilon = (\frac{1}{1-\epsilon}(\mathrm{Ric} - \epsilon\mathbb{I}))^{-1}$.

Corollary 6.5. *Assume that* $\mathrm{Aut}(M,J)$ *is finite. Let* $\epsilon \in (0,1]$. *The limit points of the operator* Aub_ϵ *are* $\{\omega_{\mathrm{KE}}\}$.

Proof. As before we obtain (14) with ω_l replaced by $\mathrm{Aub}_\epsilon^{(l)}(\omega)$. Since the K-energy decreases along iterates (cf. (3)) we still have a uniform bound on I along the orbits. By Theorem 6.3 we also have a uniform bound (depending on ϵ) on $A(M,\mathrm{Aub}_\epsilon^{(l)}(\omega))$. □

Notons que dans le cas où $\alpha(M) = 1$ l'on peut définir une autre itération (qui n'est pas tout à fait $\mathrm{Ric}^{(-s)}$ pour $s < 1$ (Cf. mon email précédent) mais qui y ressemble et qui converge. Il s'agit de

$$\omega^n e^h = \omega_{\varphi_1}^n, \quad \omega_{\varphi_1}^n e^{-s\varphi_1} = \omega_{\varphi_1+\varphi_{1+s}}^n, \quad \omega_{\varphi_1+\varphi_{1+s}}^n e^{-s\varphi_{1+s}} = \omega_{\varphi_1+\varphi_{1+s}+\varphi_{2+s}}^n, \cdots$$

avec les normalisations nécessaires.

7 Negative first Chern class

Let $\mathrm{Ric}\,\omega + \omega = \sqrt{-1}\partial\bar{\partial}h$ with $\frac{1}{V}\int_M e^h\omega^n = 1$. We define the iteration now by putting $-\mathrm{Ric}\,\omega_l = \omega_{l-1}$. This can be rewritten in terms of the following Monge-Ampère equations

$$\omega^n_{\Sigma^l_{j=1}\varphi_j} = e^{h+\Sigma^{l-1}_{j=1}\varphi_j}\omega^n, \qquad \frac{1}{V}\int_M e^{h+\Sigma^{l-1}_{j=1}\varphi_j}\omega^n = 1, \quad \forall\, l\in\mathbb{N}. \tag{15}$$

Proposition 7.1. *Let $\mu=-1$. Then*

$$E_0(\omega,\omega_l) = (I-J)(\omega,\omega_l) + \frac{1}{V}\int_M\sum_{k=1}^{l-1}\varphi_k\omega_l^n + \frac{1}{V}\int_M h\omega^n. \tag{16}$$

$\star$ *Proof.* Once again the formula follows directly from the Proposition above. Nous devrions pouvoir montrer que l'énergie décroit.

Proposition 7.2. *Let $k\in\mathbb{Z}$. If $\mathrm{Ric}^{(k)}\omega=\omega$ then ω is Kähler-Einstein.*

Proof. Recall that (cf. §3) that $I(\omega,\omega_\varphi)\geq|\varphi|^2_{W^{1,2}(M,\omega)}$. Assume that $\{\mathrm{Ric}^{(-j)}\omega\}_{j=0}^k$ is the orbit of ω under $\mathrm{Ric}^{(-1)}$. Note that $\varphi_j, 1\leq j\leq k$ are all uniquely determined. Moreover, $\Phi_k=-h$. Then, with the above notations, we have

$$\int_M\sqrt{-1}\partial\varphi_k\wedge\bar{\partial}\varphi_k\wedge\omega^{n-1}_{k-1}\leq\int_M\varphi_k(\omega^n_{k-1}-\omega^n_k) = \int_M(-\Phi_{k-1}-h)(1-e^{\varphi_{k-1}})\omega^n_{k-1}.$$

Proposition 7.3. *Let $p\in(1,\infty)$. There exists a constant C_1 depending only on p, M and ω such that*

$$||\Phi_l||_{L^\infty(M,\omega)}\leq C_1, \quad \forall\, l\in\mathbb{N}.$$

Proof. Let $p\in(1,\infty)$. According to the work of Kolodziej [Ko] (or Blocki [Bl2] if $p\in(2,\infty)$ which will suffice for the proof) and in view of (15) it suffices to prove that

$$||e^{h+\Phi_l}||_{L^p(M,\omega)}\leq C_2, \quad \forall\, l\in\mathbb{N},$$

for $C_2=C(M,\omega,p)$. From the normalization in (15) it follows that $\frac{1}{V}\int_M(h+\Phi_l)\omega^n\leq 0$. In particular $\sup(h+\Phi_l)\leq-\frac{1}{V}\int_M(h+\Phi_l-\sup(h+\Phi_l))\omega^n$. Therefore

$$\frac{1}{V}\int_M e^{p(h+\Phi_l)}\omega^n\leq e^{p\sup(h+\Phi_l)}\leq e^{-p\frac{1}{V}\int_M(h+\Phi_l-\sup(h+\Phi_l))\omega^n}\leq e^{p\,\mathrm{osc}\,h}e^{-p\frac{1}{V}\int_M(\Phi_l-\sup\Phi_l)\omega^n}.$$

Let G be the Green function for $\Delta=\Delta_{\bar{\partial}}$ satisfying $\int_M G(x,y)\omega^n(y)=0$. Since $-n<\Delta\Phi_l$ the Green formula gives

$$\frac{1}{V}\int_M(\sup\Phi_l-\Phi_l)\omega^n = -\frac{1}{V}\int_M G(x_0,y)\Delta\Phi_l(y)\omega^n(y)\leq nA$$

where $A = -\inf_{M\times M} G(x,y)$ and $x_0 \in M$ satisfies $\Phi_l(x_0) = \sup\Phi_l$. Therefore $C_2 = e^{p(\operatorname{osc} h + nA)}$. $\square$

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