CONTENT DEVELOPMENT AND/IN PROBLEM-SOLVING

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This Research Report addresses issues of mathematical content development in problem-solving contexts, using an extract from a Grade-10 classroom as an illustrative example. The analysis of the extract in relation to content development reveals how the mathematical ideas arose contingently, unfolding one after another in connection to the inquiry being undertaken. This leads to consideration of content development in problem-solving contexts as a dynamic, ongoing and non-linear process, emerging through the classroom exploration and sense making endeavors.

INTRODUCTION

The existing and ongoing literature abounds of reports illustrating the beneficial outcomes for students of a sustained practice of problem-solving in classrooms: on meaning given to mathematical concepts and their relevance for everyday life; on the development of critical, logical and autonomous thinking; on the active engagement in doing mathematics; on the development of positive relationships with mathematics, and so on (see e.g., Stein et al., 2004). Yet one main question that remains little studied, as English and Gainsburg (2015) illustrate, is about content: How is mathematical content developed through problem-solving? It is this important question that orients this Research Report, aiming to initiate reflections on how content is addressed in problem-solving environments. After grounding aspects of the study both theoretically and methodologically, a Grade-10 classroom extract is described and analysed in relation to content development, conceptualized in a dynamic fashion.

THEORIZING PROBLEM-SOLVING AS AN EMERGENT PROCESS

Grounded in the enactivist theory of cognition for conceptualizing problem-solving environments as non-linear endeavors (see e.g., Proulx & Simmt, 2016), the research is also strongly inspired by the work of Borasi (1992) and Lampert (1990) on inquiry and problem-solving, who aim to place students in authentic problem-solving situations. In their work, mathematical problem-solving is conceived of as a process that does not follow a pre-specified thread of events – analogous to the development of mathematics itself as a discipline – where numerous questions and ideas arise amid problem-solving endeavors, these often becoming central issues that can redirect the inquiry being undertaken (see also Cobb et al., 1994).

Remillard and Kaye Geist (2003) termed these “emergent” events as openings in the curriculum, where occasions offer themselves to inquiry and (can) redirect the flow of classroom events; something akin to Van Zoest et al. (2015) notion of building on ideas unfolding in the classroom or Beghetto's (2017) concept of unplanning. Borasi (1992) addressed these matters in terms of flexibility, where authentic mathematical problem-solving spaces are conceived to tackle unanticipated events:

> The open-endedness that characterizes inquiry requires extreme flexibility in terms of curriculum content and choices. A teacher will often need to deviate from the original lesson plan in order to follow a new lead, pursue valuable questions raised by the students, or let the class fully engage in a debate stimulated by difference in opinion or different solutions (p. 202)

In this context, a problem is seen as what is given to students, as well as its exploration and the questions that emerge along the way: the problem is thus a dynamic entity. Along these lines, English and Gainsburg (2015) assert that a problem *becomes* a problem in relation to what is asked of the students and not only its "text":

> Problems with high cognitive demand requires students to explain, describe, and justify, make decisions, choices, and plans; formulate questions; apply existing knowledge and create new ideas and represent their understanding in multiple formats. (p. 326)

Asserting that it needs to challenge one's thinking, this view also aligns itself with other often cited definitions, for example, like that of Lesh and Zawojewski (2007):

> A task, or goal-directed activity, becomes a problem (or problematic) when the "problem solver" (which maybe a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation (p. 782)

Thus, students' activity participates in determining the status of the problem, where one task can be a problem for one person and not for another. In this sense, tasks given can be conceived as triggers for students' mathematical activity, and this can lead to unanticipated direction (even if the task is designed with specific purposes in mind). This aligns itself with issues of events that emerge unexpectedly during the problem-solving process, where issues tackled are in relation to solvers and the unfolding events that led to them. It is through this theoretical lens that the research question is addressed, relative to content development in problem-solving contexts.

METHODOLOGICAL ISSUES

This Research Report is part of a wider research program focused on studying the teaching of mathematics through problem-solving in elementary and secondary classrooms. We collaborate with groups of teachers who regularly invite us into their classrooms to experiment various kinds of problem-solving approaches and to interact, assess and reflect with us on the teaching that goes on in these sessions. Because it inserts itself in regular classrooms, the research does not want to be disruptive and follows the teachers' teaching plans, with the tasks given in class to students being chosen by and with teachers (often coming from their teaching materials and workbooks). The problem-solving sessions usually follow the same trend, starting with a task presented to students, written on the board or handed on paper, where students are given relative amounts of time to address it. After this, in a plenary manner, students are asked to share their strategies/solutions and thoughts with the group, while ensuring that these are clearly explained and justified for other students to understand and ask additional questions if necessary. Students are also invited to interact between each other in relation to the ideas shared, to question or challenge them, add to them, etc., thus aiming to create a community of inquiry (Borasi, 1992; Lampert, 1990). These various interactions in turn often provoke new inquiries, where students can be asked to explore new issues or additional questions (Cobb et al., 1994).

Data-collection focuses on classroom discussions and interactions, as well as traces left on the board, all chronologically recorded as field notes by a research assistant (RA) or videotaped. These notes are complemented by team meetings (PI, RA, teachers) after the sessions to review/revise the events that occurred, and supplement the notes with observations and insights about issues worth reporting (here, in terms of mathematical content development). These meetings offer a first descriptive level of analysis, orienting subsequent data analysis like that reported below.

THE PROBLEM-SOLVING EXCERPT

The extract is taken from a session in a Grade-10 classroom of about 30 students, who were working on analytical geometry in relation to distances (points, midpoints, lines, etc.) and had been initiated to usual algebraic formulas. This extract was chosen for its capacity to illustrate issues of content development that were common to almost every session conducted/experimented. For this precise session, the teacher wished to experiment with tasks along a mental computation context (following our work, see e.g., Proulx, 2014), with the intention to see how students would engage in it. One task given to students was "*Find the distance between (0,0) and (4,3) in the plan*" (given orally, with points drawn on a Cartesian plan on the board), who had 15 seconds to answer without recourse to paper and pencil or any other material. Then, students were invited to share and justify their solutions to the group. The following is a synthesis of the strategies engaged in and the discussions, questions and explorations that ensued.

The first strategy referred to applying the usual distance formula ($D=\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$), leading to 5 as a distance. A second strategy suggested drawing a triangle in the plan, with sides 3 and 4, for then finding the hypotenuse by using Pythagoras (Figure 1a).

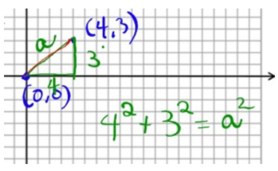


Figure 1a. Drawing the right triangle

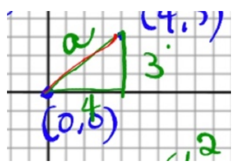


Figure 1b. Close-up on the triangle

Another student then suggested a third strategy and came to the board to trace a red segment to count on it directly from (0,0) to (4,3) as in Figure 1b. Starting from (0,0), she counted "the number of points" to arrive at (4,3), counting the number of whole-number coordinate points from (0,0) to (4,3). While doing this, she suddenly stopped and mentioned that her red segment did not go through the points that she envisaged, which made the counting difficult. The teacher then traced another segment going through square diagonals linking two separate points, which could enable counting the number of (whole-number) coordinate points from one point to the next (giving 4 as a distance, Figure 2). The student agreed that for this case, it would work.

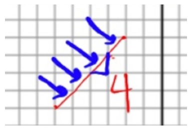


Figure 2. Line drawn through square diagonals

The teacher then asked if the measure obtained with square diagonal lengths was identical to that obtained with the side of the square (drawing ⍂ on the board).

> One student asserted that both lengths were not identical, because the diagonal of the square was not of the same length as the square's side. Another explained that both lengths were different, because the hypotenuse is always the longer side in a triangle. Finally, a student claimed that the diagonal was longer, because it faced the wider angle.

The teacher then asked if that last assertion about facing the wider angle was always true, and if so why (drawing on the board a random right triangle ◺).

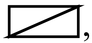
> The student who made that assertion, pointing at the triangle, stated that it was indeed the case in this drawn triangle. Another student explained that, in a triangle, the bigger the angle the longer the opposite side, mentioning that if the side-hypotenuse had been longer, the opposite angle would have been wider. And, because the sum of the (measures of the) angles in a triangle is 180º, then the 90º angle is always the wider one in a right triangle, the other 90º being shared between the remaining two angles.

Using the drawing of the triangle, the teacher simulated the variation of the right angle toward an obtuse one and traced the resulting side obtained, showing how it would become longer (drawing ◺ on the board). He then moved toward producing an acute angle, asking students if their "theory" was valid for any angle, like acute ones.

> One student asserted that it works for isosceles triangles, with equal sides facing equal angles, and another mentioned that it is the same for the equilateral triangle, since it is "everywhere the same" with same angles and same side lengths.

These explanations about the diagonal being longer than the side of the square underlined the fact that the previous strategy amounted to counting diagonals, that is, the number of diagonals of a unit square. This offered another kind of measure for the (same) distance between the two points: one in terms of units and one in terms of diagonals. A student added that if one knows the value of the diagonal, then one could find the number of unit squares for the diagonal-segment by multiplying by that factor.

One student offered a fourth strategy to find the distance, suggesting to use the sine law with angles of 45°. The teacher asked the student how he knew that the angles were 45° in the triangle. As skepticism grew in the classroom, the teacher suggested that students inquire, individually or in pairs, if the triangle's angles were 45° or not. After 5-6 minutes of exploration, students were invited to share their findings.

> One student explained that on her exam checklist there is an isosceles right triangle with 45° angles. Thus, with this triangle of side length of 4 and 3, one cannot directly assert that it is 45° because it is not an isosceles triangle as its sides are not of equal measure. Another student illustrated at the board that if one "completes" the initial triangle into a rectangle (, see Figure 3a), then the hypotenuses of both triangles are the rectangle's diagonal which cut it in two equal parts and thus cuts its angle in two equal 45° parts.

As the teacher highlighted that the two arguments were opposed, one student replied not in agreement with the last argument, drawing on the board a random rectangle with its diagonal (Figure 3b), and asserting that in this rectangle *it was not certain* that the angle was divided into two equal parts. Another student then added that because the sides of the triangle were not identical (of 3 and 4), then the diagonal would not necessarily cut the 90° angle in two equal parts of 45°.

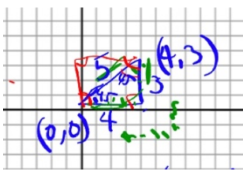


Figure 3a. The "completed" rectangle

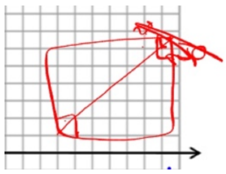


Figure 3b. The "counter" rectangle

The teacher highlighted that this last argument reused aspects of the precedent "theory" that the longer side faces the wider angle in a triangle. Hence, here, a longer side needed to face a wider angle. Then a counter-example was offered to the group.

> The student who made reference to the checklist asserted that it happens in their exams that right triangles don't have 45° angles, for example, one with angles of 32° and 58° (drawing Figure 4). She completed her drawing to create a rectangle, explaining that the diagonal cuts as well this rectangle in two parts, but that the angles obtained are not of 45°.

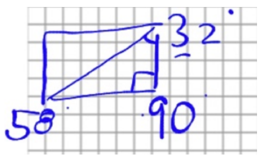


Figure 4. The triangle counter-example with angles of 32° and 58°, and its rectangle

As the teacher explained that this represented a counter-example to the 45° assertion, one student suggested calculating the angles with the sine law, for example with sin90/5=sin?/4 giving an angle of 58.1 [corrected to 53.1 and 36.9 afterwards]. The teacher then asked the group, considering all that had been said and done, where they now stood in relation to the initial claim of angles of 45º.

> One student completed the preceding argument about the different measures of the triangle sides, explaining that because all three sides of the triangle were different, then their associated angles would be different.

The teacher then highlighted the work of one student who drew a square in his notebook to assess the 45º situation. Drawing a triangle of sides 3-4-5, he extended the cathetus of 3 toward one of 4 to create a 4x4 square. Then, because in the previous unit-square the angles were of 45º, in this 4x4 they were 45º as well (Figure 5). Comparing hypotenuses of both triangles, it illustrated that in the initial 3-4-5 right triangle, the angle is smaller than the right triangle of side 4 and 4. All this led students to appear to agree with the fact that the angle was not 45º, ending the explorations (and leading to offer another task to be solved by the students).

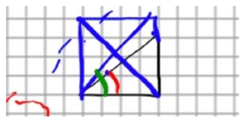


Figure 5. Comparing triangles within a square

INITIAL ANALYSES: MAPPING AND COVERING MATHEMATICS

One way of illustrating how content develops during the extract is to map the mathematical ideas that unfold one after another in it. This mapping enables one to see and highlight the mathematical ideas engaged with – how they grew/evolved – through the problem-solving session. The mapping realized here is separated in relation to the four strategies engaged with (and that led at times to additional explorations). Each mathematical idea is placed in a bubble, with links connecting one idea that follows another. Through this unfolding, a “?” shows that some mathematical ideas provoked additional questions, reorienting the exploration. Because the paper format does not allow displaying the unfolding of mathematical ideas in real time, Figure 6 offers snapshots of how the map chronologically grows along the third strategy of “counting the diagonals”. This strategy led to questions about comparing lengths of the side of the unit square and its diagonal, to issues of the largest side opposed to the widest angle, to obtuse and acute angles, to isosceles and equilateral triangles, and then back to two means of measuring the distance. [Note: even if the images are too small to be read, the intention is to give a sense of how ideas “grew” and important “terrain was covered” through the inquiry (see also Lampert, 1990). In the PME presentation, the (entire) mapping will be presented dynamically, in large scale].

Figure 6. Ongoing mapping of mathematical ideas in relation to the diagonal strategy

In total, as shown in Figure 7, the complete mapping highlights more than 50 mathematical ideas addressed in the extract, some being addressed more than once. This shows how important and numerous were the mathematical ideas shared through the inquiry. Although mostly descriptive, this mapping enables a view of the large spread of mathematical ideas addressed, but not in terms of its quality or depth of inquiry, which would require another kind of analysis. However, the mapping succeeds in illustrating the dynamic emergence of mathematical ideas engaged with, showing how much more ground is covered than only issues of distance between two points, as issues of triangles, of diagonals of squares and rectangles, of angles, of hypotenuse, measure, and so on, came continually into play. Also, these ideas were addressed not only once, but recurrently and in an interconnected way.

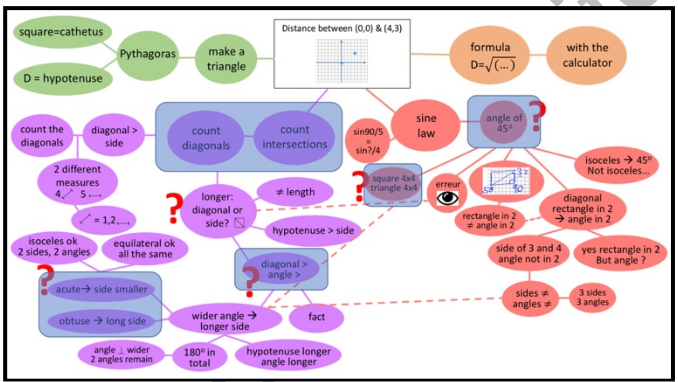


Figure 7. The final mapping of all mathematical ideas engaged with

This mapping also shows, through its links, how the exploration was not about adding isolated bits of content, but about building, unfolding and connecting mathematical ideas with one another, contingently. The interconnected unfolding of ideas points to the non-linear and emergent unfolding of events. Various openings in the curriculum (Remillard & Kaye Geist, 2003) occasioned the problem-solving process to be reoriented, one idea building on another (Van Zoest et al., 2015). As Cobb et al. (1994) explain, the mathematical strategies and ideas shared can often become new objects of inquiry: e.g., the "measuring the diagonal" strategy led to inquire into the difference between the side and the diagonal of the square; the 45° assertion (re)directed the exploration toward the need to assess the measures of the angles in the triangle. This mapping of events leads to conceptualizing content development, or one might say the covering of curriculum, as a dynamic and ongoing process, in contrast to a static one that follows a script, or a "done-once-and-for-all" or "tick-on-the-checklist" view: ideas were continually initiated, grounded in relevant inquiry, reworked, restated, argued, countered, and so forth. As Figure 8 contrasts, the development of mathematical content did not appear as a linear, pre-traced process or trajectory, but as a branching and expanding one. This invites conceptualizing curriculum content coverage in problem-solving contexts as an emergent, unfolding, and ongoing process, where mathematical ideas are interwoven and intermingled, while going forward.

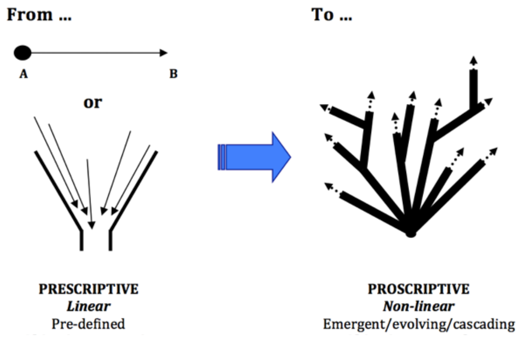


Figure 8. Toward an emergent view of content development in problem-solving

FINAL REMARKS

The analysis conducted led to addressing the issue of content development in problem-solving contexts along an emergent view, where ideas grow and unfold from one another (often) unpredictably. This disrupts a view that conceives of ideas developing one after another in a pre-planned way, and invites consideration of unpredicted events, where content is strongly grounded in relevant and contingent inquiry. Somehow, metaphorically, it is the unfolding of the inquiry that "decides" of the content covered during the inquiry (and not necessarily the teaching plans).

However inviting or compelling, these are mostly initial steps in the research, which opens up the importance of continuing studying content development in problem-solving contexts: What level of robustness or depth is attained for the ideas? In what ways are these ideas reinvested in the following classroom or inquiries? What are the long-term attainments in terms of curriculum coverage? Those are significant questions that also need to be addressed through research, in this ongoing endeavor to gain a finer understanding of the ways in which problem-solving contexts (can) achieve and contribute to the development of mathematical content, the covering of the curriculum, in mathematics classrooms.

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