

**How Do We Know?**

**An Epistemological Journey in the Day-to-Day, Moment-to-Moment of Researching,  
Teaching and Learning in Mathematics Education**

by

Jean-François Maheux  
B .Sc. Université Laval, 2004  
M. Sc. Université du Québec à Montréal, 2007

A Dissertation Submitted in Partial Fulfillment of the  
requirement for the Degree of

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## **ABSTRACT**

In this dissertation, I offer an epistemological journey in the day-to-day, moment-to-moment of mathematics education. Drawing on enaction and cultural historical activity theory, I examine various episodes from research involving children in second and third grade doing geometry with their regular teacher and a research team using tools from the tradition of interaction and conversation analysis.

My interest is to go beyond interpreting teachers' and students' mathematical activity to explore the question of "How do we know" in mathematics education, including a reflection upon the researcher's own actions. I want to better understand how the actions of researchers, teachers and students intertwine to co-produce mathematics education in its actual form and, from that angle, articulate some of the aspects by which mathematics education becomes a (more) meaningful undertaking for all of us.

In total, I present five studies from a travel journal (first written as book chapters or journal articles) that came to fruition from this journey. The first one looks at how geometrical knowings came into being in a second grade classroom, and articulates the interdependence of abstract, concrete, cultural and bodily mathematical knowings. The second takes a more critical look at

the analysis of classroom episodes from video data to produce such “knowledge” about students’ knowing. The third study examines student-teacher communication. It articulates the irreducible, dynamical nature of mathematical knowing through communicative activity that is always knowing-with another and therefore constitutes an ethical relation. The fourth study takes yet another look at the role of researchers and that of knowledge production to appreciate how, in collecting data, research can create learning opportunities for both teachers and their students. The final study returns to the first one, and presents a more elaborated understanding of what it means to know geometrically from the students’ perspective. Rethinking knowing through relationality with oneself, others, and the material world, it concludes with a reflection on the ethical responsibility that comes with knowing mathematically.

As a whole, the dissertation presents itself like a single (textual) utterance, a “turn taking” in our ongoing conversations about researching, teaching and learning in the field of mathematics education. Running through the studies themselves and the reflections surrounding them, metaphors (such as that of a journey across a landscape of theories, methods, and concrete observations in the day-to-day, moment-to-moment of mathematics education) invite the reader to “walk the walk” of thinking differently about “how we know.” In the last chapter, I call upon the reader to join the conversation by questioning, taking up and accepting, or even rejecting what has been done, hence acknowledging it in/as present, so that the dialogue is furthered.

## TABLE OF CONTENTS

SUPERVISORY COMMITTEE .....	ii
ABSTRACT .....	iii
TABLE OF CONTENTS .....	v
LIST OF FIGURES .....	viii
ACKNOWLEDGEMENTS .....	ix
EPIGRAPH .....	x
CHAPTER 1 .....	1
Where it All Began .....	1
Going to School: Mathematics Education from the Inside .....	2
The Passage: Learning on Teaching .....	3
Entering Academia .....	4
The Big Move: Beginning a PhD .....	6
I Keep Arriving .....	8
CHAPTER 2 .....	10
Walking a Landscape as a Metaphor .....	10
Two Theoretical Landscape .....	13
The Biological Theory of Cognition .....	13
Cultural Historical Activity Theory .....	19
An Uneven Topography (for an Uneven Topology) .....	23
Walking the Walk .....	28
(Method) Against Method .....	28
The Participants, the Schools, the Project .....	31
An Orientation Toward what Emerges .....	33
Mathematics Education In and Through the Day-to-day, Moment-to-moment .....	34
Understanding the Senseable (with Others) .....	36
Like Roads in the Country .....	39
A Path Laid Down in Walking .....	41
At Walker's-eye View .....	41
Maps, Not the Territory .....	46
CHAPTER 3 .....	51

Preface .....	51
This is Not a Cube .....	54
On Forms of Mathematical Knowing.....	56
The Double Ascension of the Abstract and the Concrete .....	56
Body, Culture and the Nature of Mathematics .....	59
What Makes a Cube a Cube?.....	64
A Classroom Episode .....	64
Concrete and Abstract from Everyday Knowings in the Mathematics Classroom .....	66
Concrete and Abstract in Transactions with the Material World and with Others .....	69
The Translation of Sensual, Cultural, Concrete Abstract Experiences .....	70
Communication With and For Oneself and Others .....	74
The Role of Cultural Artifacts .....	79
The Co-existence, Co-emergence and Co-evolution of Abstract, Concrete, Bodily and Cultural Knowing in Mathematics .....	83
CHAPTER 4 .....	86
Preface .....	86
Two Perspectives on the Observer .....	91
The Observer from a Psychological Perspective.....	92
The Observer from a Sociocultural Perspective.....	97
An Historical Take on Observation.....	101
How the Observer and the Observed Co-emerge .....	103
Students' Actions Are Made With and For the Other(s).....	108
Research and the Ethical Ground .....	114
CHAPTER 5 .....	121
Preface .....	121
Mathematics and Communication.....	124
Knowing Always Already is Knowing-With .....	126
From Knowing-With to Ethics .....	138
Opening: The Relational Dimension of Communicating Research .....	151
CHAPTER 6 .....	153
Preface .....	153
Why Research?.....	156

Researching in the Middle of Teaching and Learning .....	159
Researching in the Classroom: Co-Introducing a Lesson? .....	160
The Inseparability of Researching, Teaching, and Learning.....	162
Researching-in-the-Middle and Learning Opportunities.....	166
Engaging with Teachers: Researching Creates Teaching Opportunities .....	168
Discussing with Students: Researching In or By the Way? .....	171
Engaging with Students: Researching Learning Possibilities .....	173
Toward a New Ethics of Researching .....	176
Final Remarks: The End of Educational Knowledge as We Know it? .....	179
CHAPTER 7 .....	182
Preface .....	182
On Relationality .....	185
First Fold: Relationality with the Material World, or Nadia Sees a Circle .....	186
Second Fold: Relationality with Others, or Nadia and Nate Find another Circle .....	191
Third Fold: Relationality with Oneself, or Nadia (as Nate) Creates a Geometrical Model .....	196
Knowing-With-In: An Enfolding Relationality.....	200
Answerability in Knowing Mathematically .....	202
CHAPTER 8 .....	206
How (Not) to (Not) Conclude a Dissertation .....	206
Back Cover .....	213
Coda.....	218
REFERENCES .....	219

## LIST OF FIGURES

Figure 2.1: The relation between the branches and the tree is <i>similar</i> to that of any mathematics classroom session to mathematics education as a whole .....	35
Figure 2.2: A map of the main routes traveled during my epistemological journey .....	50
Figure 3.1: A map of my walk along “what makes a cube a cube” .....	53
Figure 3.2: This is not a cube.....	54
Figure 3.3: Four ways of touching the yellow block in gesture [g], [j], [m] and [o].....	71
Figure 3.4: Some students’ orientation while Eugene explains that what makes a cube a cube is “all the sides that it has” .....	75
Figure 4.1: A map of my walk when “looking at the observer” .....	88
Figure 4.2: Sonia, Jade and Maeve experiencing with the cone.....	103
Figure 4.3: Maeve’s gestures in turn 04.....	109
Figure 5.1: A map of my walk in “teacher-student’s knowing-with”.....	123
Figure 5.2: Jennifer slides her finger back and forth along an edge of the block. Next to her, there is a poster on the wall.....	130
Figure 5.3: Tobin and Jennifer coordinating with one another as she hands him the block.....	137
Figure 5.4: Tara holding the cylinder and the cube for Tobin, and the classroom .....	140
Figure 6.1: A map of my walk in “researching in the middle of teaching and learning” .....	155
Figure 6.2: Two views from Miki’s camera. ....	161
Figure 6.3: Chi-Chi gesturing for herself (a), but also for Miki and the camera (b) .....	172
Figure 7.1: A map of my walk in “rethinking knowing” .....	184
Figure 7.2: Nadia squats, pointing to the “circle” she found .....	187
Figure 7.3: (a) The students find shapes and record their observation; (b) a flower similar to the ones they observed.....	193
Figure 7.4: The students’ model of a flower.....	197

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*Hope is like a road in the country;  
there was never a road, but when many people walk on it,  
the road comes into existence*

Lin Yutang, Chinese American writer

## CHAPTER 1

### Introduction: Where am I Coming From?

In this first chapter, I introduce myself to the reader, and in so doing, also present this dissertation as an *Epistemological Journey in the Day-to-day, Moment-to-moment of Mathematics Education*. Going back to where I came from, I discuss how and why I have, over the past three years, explored forms of scholarship which permits me to not only look into teachers' and students' knowings, but also reflect upon my own actions as a researcher. Hence, I explain my intention to better understand how researchers', teachers' and students' actions intertwine to co-produce mathematics education in its actual form, and from that perspective, articulate some of the aspects by which mathematics education becomes a (more) meaningful undertaking for us all.

#### Where it All Began

I am coming from somewhere  
And this somewhere I am taking within me  
And this coming is also my doing, my being  
It is a becoming, a coming to be  
In new places in which, it seems to me  
I keep arriving  
I keep arriving

Some years ago, at around midnight on December 31<sup>st</sup>, 1999, I was sitting in front of a collection of computer screens. This was in a security office of an international information technology company in Montreal. I was in charge of monitoring the system's passage into the third millennium. Chatting the night away with the (Y2K!) "ghost in the machine," I realized I was ready to try something new with my life, and decided to become a mathematics teacher.

### **Going to School: Mathematics Education from the Inside**

Why mathematics education? Perhaps naïvely, education was, to me, one of the most important aspects of human existence. Through education, one has access to "knowledge", what generations of people patiently worked out, but also to ways of understanding that one does not necessarily experience in his or her everyday life. In my opinion, education was a springboard to appreciate and serve the best of human nature, and schooling was is part of the collective endeavor to give all of us the means to access to it. For me, mathematics was a powerful and playful tool to think beyond surface level.

At the same time, I could not help but remember my days as a high school student. Back then, doing mathematics was not much fun, and it gave me the durable impression that its power consisted in to affirming the authority of the teacher, not in promoting heartfelt involvement in the thick of thinking. Little mistakes would detract from the value of most reasoning, and attempts to think outside the box, the exercise or the problem were not encouraged whatsoever. In my memory, I left high school mathematics with the feeling that there was "something" interesting there, although I could never really get at it. That night in December, with the reflection of myself in the computer screens, I remember meditating what a waste this was. I decided to try to make things a little bit better in mathematics education, at least for a few

students at the time.

### **The Passage: Learning on Teaching**

Those noble, quixotic intentions took me all the way to university, where I encountered something quite unexpected: research. Very quickly, I now realize, it appeared to me that the world of education was divided into three domains. On the one hand, there were the actual practices, what teachers and students do and experience every day in school. Then, within the university and among mathematics educators and student-teachers, there were educational discourses about those realities and what they *should* be. Finally, we had research: people attempting to break from current practices and discourse to invent new ways of understanding life in school, and change the face of education. I was immediately drawn toward research, especially because I found it hard to believe in the ability of educational discourse to really make a change in dominant practices. As my (much younger) student colleagues were attempting to resolve the tensions between what teacher-educators told us in university classes and what was asked from us in our practicums, I could see my (much younger) student colleagues becoming the same type of teachers I had in the past. I observed differences of course: mathematics (and science, for I was also trained as a chemistry teacher) was made more “empirical” using contexts, hands-on experiments, or illustrations involving all sorts of manipulatives. But it seemed to me that even us, pre-service teachers, were still basically thinking about mathematics education in a similar way.

Like my colleagues, I could easily argue against teaching as *transmission* of knowledge and make a case for *construction* of knowledge by the students. Certainly, there was an important shift in approach between those two perspectives. What was it then? It all became

clear the day I began to read about a theory called “situated learning” (Lave & Wenger, 1991), in which learning is considered in terms of *identity*. From that perspective, students and teachers in a classroom are not merely dealing with “knowledge” to be transmitted or constructed: they are *being* and *becoming*, and knowledge as reified in curricula, textbooks, or teachers preparation notes is but a dimension (albeit an important one) of what is happening for them. That is, I was realizing that by focusing on the construction of knowledge, we (teachers) were still placing this knowledge *before* the students themselves, and that this attitude was perhaps one of the main reasons classroom mathematics hardly reached me as a student, and why I felt the approach used by constructivists would not do much better. If knowledge comes first, am I not, as an individual, but a tool to ensure its preservation? Should not mathematics be in the service of being and becoming rather than the other way around?

### **Entering Academia**

It is such personal reflections that took me into research with the hope (again, one could say naïve) to “make off” from dominant discourses and practices and envision new ways of thinking and going about mathematics education. I then began a Master’s degree to explore the relation between knowing/learning mathematics and students construction of identity following the model developed by Etienne Wenger in his seminal book *Communities of Practices: Meaning, Learning and Identity* (Wenger, 1998). In the year 2007, I concluded a Master’s thesis in mathematics education with a reflection about the relationship between knowledge and “action in common.” I studied my own work, as a researcher, in order to develop a framework and build teaching and learning situations. I then analyze my collaboration with a teacher in re-inventing one those situations, to finally examine how its realization with students also transform that same

situation (Maheux, 2007a). Along the way, I realized that these “lesson design” undertakings were quite different from one another, but were also strongly related. Together with what happens with the students in teaching and learning, my work as a researcher and that of the teacher outside the classroom were also a part of what we call mathematics education. I noticed the importance of the envisioned collaboration with teachers and students alike in what I was doing, observing that I had, in all of this, a consistent attitude toward “knowledge production.” Knowledge, as a research outcome, in terms of teachers’ professional development, as well as in students’ activity, had to be considered contextually, and had to be based on the actual and previous experiences of the person (Maheux, 2007b). Moreover, it occurred to me that, at least in my presence, teachers and students gave signs of concern not only for teaching and learning, but for my researching endeavor as well. In other words, I found myself left with the impression that mathematics education was something researchers, teachers and students concretely do with and for one another, and that there is some epistemological consistency in how all of them come to know.

Central to this impression was the idea that students, teachers or researchers can never simply be “wrong,” although what they do is open to question. On the one hand, my initial formation as a teacher in Québec was permeated with constructivist ideas. Students had to be placed in situations that they personally and willingly engaged in meaning making, and that what they “construct” as their own understanding matches specific curricular expectations, or at least valid mathematical knowledge. However, I had often observed that despite all of our efforts, students not only interpret situations in various, frequently “incorrect” mathematical ways, but that their understandings also “make sense” and need to be valued in that respect. This is important because mathematics education is not simply a game in which abstract and more or

less interested players try to gain knowledge. It is an activity involving real people, children, made of flesh and blood, of feelings and dreams, and who put their own selves at stake, for the present and for the future. Needless to say, constructivism had little to offer. Yet, my training as a researcher introduced me to approach the improvement of mathematics education from the perspective of the teachers. Whereas research often presents “findings” supposed to aid teachers and students to teach and/or learn better, top-down models are largely criticized, particularly because they cannot take into account the very contextual nature of teaching and learning. Hence, as we derive recommendations, research (often through the voice of teacher educators) seems to aim at telling teachers what or what not to do, devaluating practices that do not match the direction of the researchers. Although I frequently observed a teacher (even myself, for that matter) acting in dubious ways, I had to admit that those actions also had a validity of their own, responding in the best possible way to the situation *as it appeared* to that teacher. I can say the same about the work of my colleagues: although I often disagreed with the way researchers conduct research in general or with their analyses, it seems important to value those efforts as consequent and legitimate. However, it seems necessary to question those different ways of doing in how they contribute to the production of mathematics education. Students, teachers and researchers are human beings reacting to sociomaterial conditions, and to their own reading of them, but it is important to question those reactions because they also produce and reproduce these conditions and interpretations for themselves and for one another.

### **The Big Move: Beginning a PhD**

I was in the middle of all these thoughts when I arrived in Victoria (B.C.) to begin my doctoral studies, only a few days after I finished writing my Master’s thesis. I was arriving (in

this new destination) to work on a project pertaining to elementary school children's mathematical understandings. Designed by my two supervisors, this ongoing project problematizes the articulation of knowing in its embodied, situated nature (especially in and through the physical body) and knowledge in its abstract, socio-cultural and historical dimension. A theoretical rationale for the research was roughly laid out, comprising elements of cultural historical activity theory and the biological theory of cognition. Some data involving a second grade teacher and her students doing geometry that had already been collected. Besides that, the field was wide open for research opportunities. Indeed, the project aimed to better understand teaching and learning in its concrete, moment-to-moment realization and for that reason, was not organized around very specific and closed-ended questions. Hence, from day one I was invited to look at the data and identify points or moments of interest to me, while developing my own theoretical articulation of those issues, drawing at my discretion on literature addressing embodied, cultural and historical aspect of knowing. I then wholeheartedly engaged with the data and the literature, coming to realize that I had, in that project, a unique opportunity to explore some of the questions and observations I had coincidentally brought with me.

To truly understand where I am coming from with this dissertation, I need to explain a little bit more how exceptional my situation was. First of all, I had the freedom to research according to my own interest, my own sensitivity, and thus to formulate my own questions. Although this is what generally happens for graduate students creating their own research projects "from scratch," it is less likely to be the case when they join an ongoing project. I was placed in a situation where I actually had access to a project, which in and of itself, constituted a fabulous case study of mathematics education. That is, I had the opportunity to look at mathematics education research as developed and understood by people other than myself, while

also contributing (from that moment) to its unfolding. Hence, I would not have to develop a project for the specific purpose of examining how the actions of researchers, teachers and students intertwine to co-produce mathematics education, with all the methodological complication one can imagine. As a consequence, I adopted a format for my dissertation, the collection of five studies, which allows me to start from whatever I can actually observe around me and in the data. I could then follow the sequence of my emerging understandings in the thick of researching, rather than having to lay out a research “project” to frame my exploration from the start or without having to give it such a shape after the fact (which may even have been worse). All in all, we know that research “findings” are not simply derived from the questions that we formulate at first, and that an important part of researching is precisely in that work by which we form a fit between relevant interrogations, concrete observations, and more general conclusions. Finally, and of greatest importance to me, my research was taking place under remarkable circumstances because I was unexpectedly offered a rich theoretical landscape that allowed me to articulate mathematics education as something researchers, teachers and students concretely do with and for one another. Because it was developed to articulate embodied, situated knowing with knowledge at the socio-cultural level, this theoretical ground (that I outline in the following chapter) had features that permitted me to examine and formulate epistemological consistency in how researchers, teachers and students come to know in the day-to-do, moment-to-moment of mathematics education.

### **I Keep Arriving**

Where am I coming from? As I write these lines today, setting up the reader to appreciate my various papers as a whole, I realize that there is no single starting point, but instead a never-

ending process of *coming* by which I bring my own history to the here and now of researching. More specifically, it is a process of coming that is also a process of *becoming*: of coming to be. Getting to the point when I started to work on the studies I present here, I was not only coming, but making history: My own history.

Such is the account I create, in this chapter, of my own coming to the studies therein contained. What I articulate here is not, cannot be, an “innocent” historical account of my becoming. I am not objective or impartial, and I did not write those paragraphs as a high school student, an undergraduate, nor even a researcher before or during the preparation of the studies. This opening chapter in which I introduce myself (or, better, introduce the reader to me by leading him/her inside my work) is again, always, already, a moment in the constitution of my narrative self. As an agent of action living his obligation to act (Ricœur, 1992), I am writing myself inasmuch as I keep arriving to those studies, which also tells the story of my epistemological journey.

Hence, as I now reread those pieces, appreciating how they knit into one another, I realize that there is not one place I am arriving at, and no truly final product of my epistemological exploration. We do not arrive somewhere and start investigating whatever is out there: we co-evolve with the world around us in and through our observations. This is exactly what this collection of articles makes evident. It is of course my own reading (here and now) which researches patterns and link the chapters together, just as my reader will. There is no single place I entered and researched; no *one* place I finally reached and portray to the reader.

There is a coming and becoming in which I keep arriving, and in which I invite the reader to join in, coming and becoming with me.

## CHAPTER 2

### **Theoretical and Methodological Landscape: Like a Road in the Country**

In this second chapter, I lay out the theoretical and methodological landscape of the five studies constituting this dissertation. I composed the studies as standalone pieces, and for that reason, the reader does not actually need to be introduced to a general “framework” in order to understand how each chapter works, what questions are investigated, and how the data is analyzed. However, I briefly present elements of a theoretical and methodological *landscape* to develop a certain sensitivity in the reader, and so he/she can better appreciate the whole to which each chapter contributes. I do so only under the condition of making explicit my theoretical/methodological posture *against* theory and method as commonly practiced, and in favour of a heartfelt, unsubmitive, and publicly open engagement with people, ideas, and data. I then briefly present the five studies composing this dissertation, and the themes that emerged from this path I laid down upon walking it.

#### **Walking a Landscape as a Metaphor**

In this section, I describe the “theoretical landscape” in which my studies take root. I use the term “landscape” as oppose to “framework” to make it clear that this section should not be read as a “supporting structure” in the way beams and joists serve to build a house. The landscape metaphor also illustrates how I guarantee to avoid the dangers of abstractions that ignore themselves as such (Bourdieu & Wacquant, 1992). In other words, how I keep away from taking theoretical concepts as the thing itself, or as its essence, and use methods as objective means of observation.

To me, theories and method are *resources* (Barry, 2009) for my epistemological journey; they are a special type of element in my environment that I use to situate and orient myself. I lever with them to raise questions and doubts as I travel through the day-to-day, moment-to-moment of researching, teaching and learning in mathematics education. Consistent with the emergent, almost organic nature of my work, those theories and methods are more like the surrounding environment, the ecosystem, which permits a tree to grow from a seed. What I attempt to do is prepare the reader for some of the constituents of that ecosystem, and the relationships therein.

I do this not in a manner to explain the different parts of the tree or how it grew: this is the type of work I do in the chapters themselves. More artistically (as the word landscape suggests), I want to sketch the theoretical scenery in which my studies germinated. My intention, in doing so, is to develop a certain sensitivity in the reader. For example, the reader should feel this growing sensitivity in what I describe as the uneven topography of the theoretical landscape I expose. That is, my intention is not to create and present a smooth, softly articulated network of ideas and techniques that would lead the reader right to my observations and some naturally following conclusions. On the contrary, I want to set up, or better: to *upset* the reader so that, in reading the following studies, he or she can better appreciate the struggles of working the ideas, which constitute an important dimension of the *whole* to which each chapter contributes.

I do this because my intention with this dissertation is not to discuss, prove or develop theory, or to explain and illustrate some research method. I am concerned here with the day-to-day, moment-to-moment of mathematics education, and the various senses we can give to “knowledge” as it realizes in and through researching, teaching and learning. My research is *a*

*journey* in that territory, what Bourdieu calls an “epistemological experiment,” and what truly makes this journey a whole is not so much the initial question(s) I might have asked, but the actual “walk” itself. I am not writing these lines before engaging with ideas, people and data, but after the fact. If I had a clear map and a determined itinerary when I began my journey three years ago, it is of little use today in comparison to what the excursion actually was, which is precisely what the reader has access to in the studies. The writing of those studies *is* the journey itself in the best manner that I can communicate for those who did not walk the walk by my side throughout the entire journey. They are themselves like maps that are not the territory out of which they rose, but *as such* the landmark of my traveling. At this point, and despite what is usually done in academia, presenting and defending a protocol that the following studies “implement” would simply be fallacious. More importantly, it would only serve what Feyerabend (1976) describes as the fairy-tale of science and scientific method we use to mask our ideological dimensions and arrogantly impose science and its method to the detriment of all other ways of knowing (e.g., arts, myths, etc.) and, in the long run, of science itself.

The most vital task of social science is to break with both ordinary AND scholarly common sense to provoke *conversions of the gaze*, a transformation of one’s vision of the social world (Bourdieu & Wacquant, 1992, p. 251). In the next section, I begin by articulating some tenets of the biological theory of cognition, and then introduce elements from cultural-historical activity theory. These two theoretical streams form the original background of the large research project (designed by my two supervisors around embodied knowing and learning in elementary mathematics) whereto my dissertation also contributes. From chapter to chapter, the reader will find similar and different literature and concepts from these two theories. I use them not to provide explanations or to prove anything, but as contextually relevant resources to raise

awareness to unseen aspects of the production of mathematics education. In that sense, each study naturally calls upon its own theoretical and methodological constructs in a self-sufficient manner. However, when I take this dissertation as a whole, as a unitary utterance of our ongoing academic conversations (Bakhtin, 1986), I want to try and do a little bit more than what the chapters “themselves” are doing. Hence, I do propose some theory in the next section, but the two streams I present are to work together and *not* to present one or two lenses that I want the reader to use to read the five following chapters. Rather, I want them to produce a textured, contrastive, complex and (explicitly assumed to be) incomplete (back)ground that disturbs. Not fabricates, not order. I vigorously assume this polemical position precisely because I want to *trigger* conversions of the gaze. I do not engineer preconceive visions, but introduce the reader into the whirlwind of methods and theory that was mine, and in and through which I journeyed. Walking this landscape with me in such a way, he or she will perhaps better appreciate what I have seen, and transform his or her vision of the world of researching, teaching and learning in mathematics education.

## **Two Theoretical Landscape**

### *The Biological Theory of Cognition*

The biological theory of cognition occupies a growing importance in the field of mathematics education (Proulx, Simmt & Tower, 2009). Tracing to biologists Gregory Bateson and, later, to Chilean biologists Humberto Maturana and Francisco Varela, themselves followed by many others. The theory develops from an investigation into the biological roots of cognition, and starting from the very question of what “cognition” means. Overtly assuming a non-anthropocentric perspective, epistemology, As Bateson (1979) puts it, includes here:

the starfish and the redwood forest, the segmenting egg, and the Senate of the United States. ... And in the anything which these creatures variously know, I included "how to grow into five-way symmetry," how to survive a forest fire," "how to grow and still stay the same shape," "how to learn," "how to write a constitution," "how to invent and drive a car," "how to count to seven," and so on. (p.4)

Simply put, cognition then relates to the observable behaviors of living “organisms,” and more precisely to the way they maintain themselves (i.e., exist, survive, resist disintegration) in doing whatever they do, should it be a cell digesting nutrients, a dog playing with a stick or a student solving a problem in physics (Maturana, 1978). To better explain this, two basic observations are made. The first is that cognition is inseparable from action, and the second is to the fact that individual organisms co-emerge and co-evolve with one another and with their environment.

The first aspect means that what a person knows and what that person does cannot be understood in isolation from one another. Of course, there are some important, qualitative differences between various forms of knowing, such as innate/reflex response (depending on the structure of the organism), the knowing that arises from recurrent interactions with the environment (e.g. more or less immediate response to stimulus) and those taking place in a semantic sphere (e.g. knowing in the form of linguistic distinctions of linguistic distinctions) (see Maturana & Valera, 1987). But in all cases, knowing is always doing, and consists in behaviors that are consistent with what is an adequate manner of living for the organism. Succinctly captured in the statement “all knowing is doing and all doing is knowing” (p.13), mathematics education researchers increasingly draw on this idea in the way that they attend, for example, to

students' mathematical activity (e.g. Simmt, 2000; Begg, 2009; Namukasa, 2004; Thom & Pirie, 2002).

The second observation indicates that knowing and doing do not belong to the individual, but co-emerge as *coordination* between the person and his or her socio-material world, a view on knowledge and reality that sets it apart from other theories such as constructivism in its various forms (Proulx, 2008). The epistemological posture adopted here is not, for example, a case of “adaptation and accommodation” as in Piaget’s (1968) genetic epistemology, which finds the need to pose development stages and mental structures. Here, attention remains to the co-action in and through which knowing is realized, the moment-to-moment dynamical process by means of which organisms manage with the situations in which their very actions contribute. In addition, the conception of an “embodied mind” inspired by Merleau-Ponty’s phenomenology, a central importance to the body and the concrete experiences in and through which one comes to know (Varela, Thompson & Rosch, 1991). This aspect was of particular influence on the seminal work of Lakoff & Núñez (2000) about the embodied nature of mathematics: grounded in the human body, mathematics naturally develops in the course of culturally specific everyday experiences.

From such a view, the social (which involves, here, all forms of coordinations between living organisms, *including* the very special human contexts of culture) is then inseparable from individual knowing. Although not limited to such examples, mathematics educators often observed this in the case of the classroom, where knowing or learning mathematically means to be able to act in ways that others (e.g., teachers, students) consider mathematical (Lozano, 2005). From that perspective, primary importance ought to be given to the dynamic, responsive dimension of the actual acts by which students and teachers “bring forth a world a mathematical

significance” (Kieren, 1995, p. 2) with present or distant others.<sup>1</sup> An important consequence of such a view is that knowing “is to be found in the interface between mind, society, and culture rather than in one or even in all of them” (Varela, Thompson, & Rosch 1991, p. 179), a radical departure from the conception of knowledge as an object to be sought after, acquired, possessed, and used (Davis, Sumara, & Kieren, 1996). Should it be from the perspective of the students, the teachers, or the researchers, knowing is something lived, continuously embodied and enacted, and at the same time it is a historically conserved/conserving mode of coordinating ourselves with our environment, including others (Maturana & Verden-Zöllner, 2008). All the knowings we enact and all that we learn as researchers, teachers or students are (actual) human relations, forms of being that embody both oneself and the other (Kieren, 2004).

The biological theory of cognition (part of what is referred to as “enaction” or “enactivism”) is quite explicit regarding the tight interrelationship of the researcher with the objects, the people, or the situation he/she observes (Reid, 1996):

Enactivism, as a methodology, a theory for learning about learning, addresses several levels of the activity of research. The level most familiar to most of us will be the interrelationship between researcher and data, in which we find ourselves learning new things within a context which is partially of our own creation (p.3)

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<sup>1</sup> In the studies that follow, I will offer more distinctions between this approach to mathematical knowing and others theories. As the reader will see, most of these distinctions rest in the object of study and, thus, in what is expected to be the outcome of researching a given situation. For example, in Radford’s (2002) theory of objectification, attention goes to the “socially and culturally subjective situated encounter of a unique and specific student with a historical conceptual” mathematical object or way of doing (2009b, p.51). In contrast, an approach inspired by the biological theory of cognition maintain its focus on the dynamic co-production of such situations, to appreciate how those mathematical ideas come about as a mode of coordination between teachers and students.

In the day to day of researching, we not only observe, but affect whatever we turn our attention to. When we observe people, they adapt to our presence, when we look at video recording, new aspects of the data continuously emerge. This affects at the same time a changes us. We coordinate our actions with what happens or comes into view, and we begin to think and act differently. Through this process, researchers observe themselves observing (Brown et al., 2009). Attention is given to how distinctions are made, which requires not only repeated observations, but also phenomenological attention to, and the unfolding of, the observational process itself. One can then realize how actions are multimodal observable forms of knowing involving ways of talking, gesturing, orienting one another, producing physical organization, and so on. On the other hand, the second key feature of research from this perspective rests on the idea that research is not merely “about” the phenomena of which models and theories are created. Research is conducted *for* people, theories have a purpose, and part of this is in the research endeavor itself, where doing research is also disrupting the normal course of action (Brown et al., 2009) so that it can be conceived as a site for learning, and hence transformative of both the individual and the collective (Sumara & Davis, 1997). Knowledge here exists in the possibility for joint or shared action in the complex fabric of relations in which everyone, and everyone’s action, intertwine with all else: in doing research, collaborating with teachers can impact practitioners in a positive fashion (e.g., Dawson, 1999).

In that second dimension, enactivism offers opportunities to address complex socio-cultural situations including *ethical* dimensions of education as a whole (Davis, Sumara & Kieren, 1996). For example, mathematical activity (and its teaching and learning) can be seen as a particular manner of living preserver across generations through “co-ontogenical drifting” (Maturana & Varela, 1987) in which each act is one of co-existence constitutive of the human

world. In teaching and researching situations, knowing, knowers, and the knowable co-emerge and the essence of the relationships between them is ethics (Begg, Dawson, Mgombelo, Simmt 2009): something in which we realize (as in ‘produce’) our very humanness. This is not the type of ethics that rests on principles described in codes of conduct, but instead in the embodiment of an ethical *know-how* (an expression Varela (1999) borrows from Dewey, but can also be found in Heidegger) for immediate coping with situations. Inasmuch, enactivism makes us attentive to how individuals concretely participate in their socio-material world, and emerge with/in larger systems (e.g., the classroom, the society) through common actions that contribute to the very conditions that, in return, situates them, others, and the more-than-human world (Sumara & Davis, 1997).<sup>2</sup> From an enactive perspective then, one exists simultaneously in and across the classroom, the school, the educational system, the society, and so on, participating (in the day-to-day, moment-to-moment of researching, teaching and learning in mathematics education) in the collectively and individually (trans)formative process of being, doing and knowing.

The biological theory of cognition hints at a social and historical understanding of everyday experience, but this aspect remains under-developed. Human social life arises from and is realized through the languaging acts individuals produce as a mutual, consensual coordination (Maturana & Varela, 1987). Historically, this manner of living appeared among human tribes a few million years ago, and evolved in tight relation with human biology, for example with the conservation of neoteny (extension of childhood) and the expansion of female sexuality (Maturana & Verden-Zöllner, 2008). In the day-to-day, moment-to-moment, our ways of

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<sup>2</sup> Sumara and Davis use this very nice expression, I will also use from time to time, from Abram, D. (1996). *The Spell of the Sensuous: Perception and language in a more-than-human world*. New York: Vintage Books.

experiencing the world with/in our body are socio-cultural and historical through and through.

### *Cultural Historical Activity Theory*

It is, in fact, from this very different starting point that Cultural-historical activity theory (CHAT) develops, from the work of Lev Vygotsky (e.g. 1986, 1987), its holistic perspective on human activity. Taking inspiration in Marx and Hegel, CHAT looks at relationships such as that of body and mind, or subject and object, but centers attention on their cultural aspect, rather than on their biological ground. Activity theory articulates human cognition as being situated in and distributed across the whole socio-material environment. This environment results from complex cultural and historical development, which in turn is fundamental in nowadays' human cognition.

Hence, CHAT turns its attention to activities, schooling being one example, and examines them as historically, culturally, and socially situated phenomena (Leont'ev, 1978). From such an angle, people's actions (like researchers', teachers' and students') are not to be reduced into psychological or sociological terms, but instead need to be considered in relation to the educational activity as a whole (e.g., Chaiklin & Lave, 1993). Hence, here the term "activity" does not denote something like a school task, or the things that a person does, but refers to collective, socially motivated action. Participants in an activity system are considered "subject" of this activity whose actions contribute to the realization of overall goals in and through the achievement of the given "objects" of the activity. For example, teachers and students contribute to the realization of mathematics education by accomplishing the task of sorting objects according to geometrical properties. That is, from such a cultural historical perspective, events cannot be reduced to any one aspect of an activity system because they all are codependent, and the system as a whole becomes the "unit of analysis."

The theory articulates around the concept of *mediation* (Roth, 2007) between the diverse elements constitutive of activity systems. For example, a person's concrete actions are understood as relationships between a subject and the object of the activity, which is mediated by the entities that are constitutive of that particular activity (Engeström, 2001). In a sorting task, the teacher and the students are subjects whose actions are mediated by a certain division of labour (they have different roles), but also by the material devices that they use (e.g., carefully crafted and selected plastic blocks representing geometrical solids). At the basis of CHAT is then a materialist dialectical approach in the tradition of Hegel (e.g., 1977) where dialectical thinking is central. When considered in its entirety and at the same time from the perspective of actual, concrete actions, an activity system, like schooling, appears somehow contradictory. On the one hand, the actions of researchers, teachers and students seem to result from the system's functions. Students are expected to learn mathematics with the help of the teachers, themselves, and are supported by the work of the researchers; and all of them do what they do because they are students, teachers, and researchers. At the same time, it is clearly the actions of the researchers, teachers and students that reproduce schooling as an activity. Mathematics education is what it is because individuals and collectives realize it in such ways. Indeed, activity systems are typical of those "chicken-and-egg" situations that are very difficult to figure out with traditional logic. Conversely, dialectical thinking can be easily used here because it always considers the two terms of the equation (the chicken and the egg, the structure and the agency, the stability and the change) as constitutive of one another, and therefore conflates them into one dynamical unit. This plays out in CHAT in the observation that subjects of an activity do not only produce outcomes, but also produce and reproduce themselves as a part of the system. Actions are functions of the entire system, and the system functions on the basis of those actions.

This also explains why actions play a special role in activity theory, since this is what people do and observe. In actions, teachers and students concretely (re)produce classroom mathematics, and those are what researchers observe in and through concrete ‘researching’ acts. CHAT then allows us to account for various activities that students, teachers and researchers are involved in (simultaneously or at different moments), giving means and meaning to discuss the doings of students, teachers or researchers not on the basis of assumptions about what they think or know, but from the perspective of what they create with and for one another. People’s actions are placed in relation to the researcher/observer so that the very process by which one comes to know is also made visible. Such perspective contrasts with mathematics education research in which the independence of observer and the observed phenomenon is taken for granted. With CHAT, one does not reduce teachers and students to objects of research and does not try to get into their head as if there was something (or things) “in there” to be sought after and to be uncovered. Focusing on actions (instead of persons or thoughts) allows individuals to reflect on the fact that students, teachers and researchers are social beings in relation with one another and the activity as a whole. Actions are fundamental because they are inseparable from the activity and, thus, of the presence and actions of others, and because they are precisely *that* in and through which people also make sense of the activity to which they contribute, and of the doings of others.

This goes back to the founding work of people like Vygotsky (1978, 1986), who explained how “mental functions” are first social before they are somehow internalized, and Leont'ev (1978), who insisted that the study of individuals’ “inner world” has to go through the study of their activity in its dialectical relation with the society that enables it. At the core of such perspectives are the observations made by Bakhtin (1993; 1986) and his colleagues (Voloshinov

and Medvedev) that culture exists in the form of activities (especially in language) embodied in moments of real, concrete actions that enact historically develop ways of doing and being (such as forms of speech like the ones used to produce greetings, teaching, or writing a dissertation). In CHAT, a discursive, semiotic, multimodal perspective dialectically articulates the relation of the individual to the social in the processes by which personal sense is produced with/in cultural meaning (e.g. Roth & Lee, 2007).

Cultural-historical activity theory enables cogent conceptualizations of the day-to-day, moment-to-moment of strong relations between researching, teaching and learning in mathematics education, and of the special role of “knowledge.” We can see different activity systems articulating to one another inside the education system as a whole: being a student, a teacher or a researcher presents specific tools and rules, but also exist in relation to one another, and in that view are not separated. This being said, it is important here to explain that CHAT is not some sort of a “master theory” aimed at explaining everything about social life. Rather, CHAT must be seen as a *tool for raising doubt* (Roth, 2005), to become more reflexive and aware of the way human activities realize themselves. Miles away from telling researchers, teachers, or students how to improve whatever they are doing, CHAT invites us to engage in trying to understand what is actually taking place on a very local basis. Making a difference in mathematics education, then, becomes a matter of bringing about new ways of thinking with the intention of expanding action possibilities. In this, CHAT does not prevent us from contributing to the literature, but instead demands a real openness to the activities (researching, teaching or learning mathematics) on which claims are made. To be sure, this approach sits very well with the overall intentions I exposed in the previous chapter, and the open ended structure of my project.

*An Uneven Topography (for an Uneven Topology)*

Despite my clearly stated intention, articulated in the beginning of this chapter, not to “discuss, prove or develop theory” but to “*upset* the reader so that, in reading the following studies, he or she can better appreciate the struggles of working the ideas,” I was suggested by my committee members to explain how the two theories presented above “mesh” or not with one another, to be more “critical” and consider the problem of networking such theories. But doing so naturally requires playing with ideas, forming claims, appropriately using relevant literature, and, thus, doing the kind of theoretical theory-work I want to avoid. For this can still be done in a relatively concise fashion, I will add a few words, in this subsection, about the biological theory of cognition and the cultural historical activity theory, but only to continue questioning the need and possibility for “talking theory.”

A problem in itself is the question of what is a theory. That is, not only to once and for all delineate what covers the terms “CHAT” or “enaction,” but the very definition of the word “theory” (as all attempt to do so brilliantly confirms). For many, a theory is mostly an established set of propositions (as in “the dual theory of light,” or “the set theory”), whereas other, like philosopher Richard Rorty (1982), defines it as a *genre* (and Bakhtin would agree). In this view, a theory is not a thing in itself, but a dialogue, the composition of many “theoretical utterances;” what Maturana would characterize as a person’s “explanations” of “his or her experiences as a human being” presented in a particular, recognizable way (Maturana & Verden-Zöllner, 2008, p. 147). It is possible, of course, to offer more structured definitions such as Radford's (2008a) suggestion to see a theory as “a way of producing understandings and ways of action” (p.320) based on a system of basic principle, a methodology, and a set of research questions. This, however, precisely poses the problem of defining what a (given) theory is: which utterances

should count as part of the biological theory of cognition or the cultural historical activity theory? Theory as a genre avoids having to answer this impossible question, and also permits to appreciate the well-known shifts and ruptures so typical of scholarly work.

I will give but a few examples, and at the same time hopefully satisfy the reader's curiosity about potential linking or clash between the lines of work I have presented under the two labels. Among the contributions I relate to the biological theory of cognition, little is said about specific socio-cultural processes such as what we commonly call living with/in "institutions." There are, however, some attempts to do so, especially in the work of Maturana. For that matter, Bateson too published considerable amount of work in which he makes uses of his (biological) insights about patterns and relations to understand socio-cultural experiences such as that of schizophrenia (e.g. Bateson, 1972). Maturana and Varela, however, did not develop those aspects so much in their own work. More so, although the two Chilean produced together a very important book, a theoretical utterance, in the "enaction conversation," they also parted and, in the following years, wrote about very different questions.

As for CHAT, it is interesting to know that the "Russian school" in which most of the first utterances were produced, did give some interest to the biological root of cognition. Hence, Leont'ev produced a detail account as to how human "psyche" emerges from evolutionary stages starting from chemical reactiveness (Leontyev, 1981). Blending well, in most of its aspects, with the work of Bateson, Maturana or Varela, Leont'ev nevertheless suggests a rupture in human evolution. He poses that the apparition of *division of labour* resulted in an emancipation from our biological evolution: "with the transition to [hu]man... the psyche began to be governed by laws of socio-historical development" (p. 204). Unlike the biologists, the Russian psychologist found it inconceivable not to establish a clear (theoretical) cut between human and other living

organisms. But beside this (some would say anthropocentric) division, the qualitative difference he articulates in many points resembles Maturana and Varela's proposition. Whereas one starts from the concept of *labour* to define the arising of semantic social-societal *meaning*, the others keep focus on the concept of *distinctions* to articulate human *linguaging* as our distinctive activity. However, the CHAT tradition's maintain its specific interest in that (thin) layer of historical and cultural dimensions of human existence (we are still animals, we still produce chemical reactions, etc.), whereas that biologists focused somewhere else. Also, needless to say, CHAT is too an evolving conversation. For example, when Klaus Holzkamp developed his critical psychology, he hardly challenged what can be heard in the Russian texts as a "supra-historical" logic of science (Leont'ev, for instance, repeatedly insist on the search for "objectivity"). He also somehow moderates the posed separation between human activity and that of animals, suggesting that in the realization of collective motives, humans mostly contributes to the survival of society and therefore to their own survival, hence also mainly acting in relation to their vital, biological needs (Holzkamp, 1991). Finally, Holzkamp also finds some of his inspiration in the phenomenology of Merleau-Ponty, which Varela places at the origins of his work.

Within the genre, theories exist in conversations, in networks of utterances (which also, in return contribute in defining the genre). Whether these conversations can be connected on compatibility or to stress radical differences is a matter not of the "theories" themselves but, of course, depends on us as conversationalists. Hence there are numerous strategies one can use to discuss ideas coming from different traditions (e.g. Prediger, Bikner-Ahsbahr & Arzarello (2008)). If such is my intention, I can try and set up hermetical boundaries around the theories that I use. With this, I can place them in conversation, so to speak, but doing so essentially

consists in taking part in those conversations, re-producing theory-talk as a genre and rendering a certain version of certain networks of utterances *I* construct as this or that “theory.” Most importantly, this also means that theories do not exist as things in themselves. A theory is something that I do, it is an act of conversing, not a conversation in the sense of a finish script like in a play. It is a networking activity, and any attempt to step back and look at the theory as some *thing* is always already conversing again: producing theory-talk in and through talking theory.

Going back to my original text (what I had written in this section before my committee gave me comments on that specific issue), it may now make more sense to put the biological theory of cognition and the cultural-historical activity theory as *emerging* (as a verb) from very different traditions and scholarships. They use a different language to talk about diverse things, and they both present the ambitious undertaking of developing a broad view on cognition and human activity while giving complete attention to the fullest detail of the situations they describe. The former focuses on everyday experiences from the perspective of human organisms, which coordination enacts and embodies to the human social life. The latter examines the social nature of everyday activities in terms of production and reproduction of particular ways of doing (e.g., mathematics, or mathematics education). As a result, one may feel uneasy, experience tensions, and/or suspect contradictions in reaction to the fact that the two traditions are at the same time similar and different. Truly, it is quite an uneven landscape that I have drawn.

This bumpiness is not, never was, problematic to me. On the one hand, both perspectives articulate the question of “knowledge” as a concrete act constitutive of and constituted by (here) human experience. Both suggest embracing the challenging conciliation of the fundamental bodily nature of knowing, and its primary social aspect. On the other hand, being rough is precisely

what makes a surface adherent. Navigating through these perspectives, differences here are not obstacles: they give grip. Again in the spirit of Bourdieu, discrepancies keep me *vigilant* in the actual process of understanding concepts. Discrepancies always push me to break with the pre-constructed, and objectify my objectifying actions. Furthermore, they serve me to construct (instead of taking for granted) the theoretical ideas that I am using. I do not experience tension, a necessity to “stretch” (tension if from Latin *tensio*, coming from *tendere* ‘stretch’), but *texture*. The fact that scholars from both theoretical streams do not define or approach knowledge the same way, do not give it the same meaning, is part of what makes this landscape interesting. At each step, one can feel resistance, so to speak. Hence a need to make everything explicit, carefully look at the idea themselves, work from the *text-ure* specific authors present, and as a result develop theoretical frugality (*une économie théorique*) by avoiding the use of ready-made concepts from either “theory.”

I have no intention to try and create a unified theory or framework of human cognition and activities, but explore different aspects of how and what it means to know from actual moments in the doing of mathematics education. With this intention, a smooth and flat theoretical landscape would be much less appealing and useful to me, because it would already set me up to make observation from within that frame, rather than keeping me on my toes, and in the inbetween of what is already known about “knowledge.” Of course, the knowing and learning in the cultural context is not the same as in a biological one, but on the other hand, human beings *concomitantly* exist, know, and learn both ways. The biological and the cultural are two dimensions in which actions and situations simultaneously and irreducibly unfold. Like independent vectors defining a plane, they can neither be reduced to, nor overcome one another. What the two theories have in common is something in the range of “being a vector,” of offering

direction, and the ability to combine with other “vectors” to create multidimensional spaces in which certain types of possibilities for thinking and acting emerge. But again, my intention with this dissertation is not to make theory-talk, let alone in a few paragraphs and in a somehow abstract manner. Moreover, providing a synthetic, perfectly articulated view of this landscape (making a framed picture, a frame-work) would also be taking away my very experiences over the past 3 years, the heartfelt, open engagement with ideas ‘in the wild’. Pinning this down like a bug would not only be untrue to my actual researching process, but also contradictory to the methodological stand I now develop in the following section.

### **Walking the Walk**

In the previous sections, I sketched the theoretical landscape forming the back-ground of the studies I present in the following chapters. I now want to do something similar concerning the way I “walked” that landscape and gathered the data. That is, I aim to unfold the practical, methodological underpinnings of my work. Yet, perhaps even more than for the theoretical aspect, I *first* need to place this question of method in parentheses.

#### *(Method) Against Method*

The search for method, Vygotsky (1978) explains, is the most important problem of the entire enterprise of understanding because method is simultaneously a condition and a product of research, a tool and an outcome of any study. It is an easily accepted idea that method should be informed by theory. When a theory is seen as presenting a set of basic principles, it naturally poses minimal requirements within which the method has to show operability and coherence (Radford, 2008a). A methodological design first serves to decide what will be collected as data, and then ‘helps’ identifying relevant aspects in it. But what if we take on the observation that a

theory is not a thing in itself, but something that we do, and appreciate how it actually develops inasmuch as we research? The challenge of method mentioned by Vygotsky then truly comes to the fore. If theory is a doing, an ongoing networking of ideas and principles, method cannot be “informed” once and for all. And when one, like I do here, opens the research process to what whatever one’s observation *will make relevant* for talking theory (rather than the other way around), method is then also open to fluctuation. Furthermore, method here is not only “informed” by theory in a one-way fashion, but in fact informs theory as well. Theory and method are, too, like in a conversation. From study to study inasmuch as within each one of them, dwelling upon an idea invites me to look at the data differently. But since not all that I notice (in a video excerpt, for example, or when talking with a teacher) already fit with or to with the theoretical concept I have in mind, theory is also called upon in a similar way: different ideas are needed, or need to be understood in a different way. Such perspective strongly stands against research inspired by early-identified research questions supported by a neat theoretical framework, and to be answered by a well-defined method.

But if method (just as theoretical principles) develops together with the observations and the theory, this does not take place in a vacuum, but again in a certain landscape of researching practices, which I call methodological resources. It is in the same spirit that Bourdieu (Bourdieu & Wacquant, 1992) argues against the separation of theory and method to acknowledge the fact that what counts as evidence is not evident whatsoever. Opposing what he calls methodological monotheism (commitment to one predetermined research approach), Bourdieu favors methodological flexibility in and for the construction of each one of the objects of research. When concerned with actual social life (as I am), we cannot define this object once and for all in an inaugural theoretical/methodological act because we “cannot return to the concrete by

combining two abstractions” (p. 225). One must remain with the primary experience of their observations, contextually (and rigorously of course) making use of various research techniques as they come about with and for an understanding of why and how one understands.

As in the case of theory, one could suggest to write the method after the fact, so that we *then* appear to follow a strong and well-defined methodological framework by mean of which the scientific validity of our findings have been secured. In the past, I have myself felt for such an artifice, and heard many young colleagues with similar stories. We apologetically do this because we feel the requirements of the methodological watchdog and we come to embrace the fairy-tale of science. This is precisely where philosopher of science Paul Feyerabend makes a convincing argument in favor of an “epistemological anarchism” to oppose the dogmatism of science-like research and its methodological pretensions. In his plea *Against Method* (Feyerabend, 1993) writes:

Scientists do not solve problems because they possess a magic wand - methodology, or a theory of rationality - but because they have studied a problem for a long time, because they know the situation fairly well, because they are not too dumb (though that is rather doubtful nowadays when almost anyone can become a scientist), and because the excesses of one scientific school are almost always balanced by the excesses of some other school (p. 302).

Congruently, this “method” section does not present a set of tasks or grids I use to classify my observations. Rather, I introduce some of the means by which, in each study, I invite the reader to “walk the walk” of mingling data and theory, in reading and writing the day-to-day, moment-to-moment, of mathematics education. It is in that spirit that I offer, in the next subsections, some factual information on what we did so that the reader can get a broader?

understanding of the research project whereto my work contributes. I even explain the willful orientation toward “what emerges” at its basis. I also discuss how I conceptualize my approach to researching, teaching and learning in mathematics education in and through real, concrete actions of a few researchers, teachers and students. This leads me to explicate some of the means by which, with the support of others, I developed the understandings of the *senseable* (what can actually be seen or heard) that I present in the following chapters.

I do this briefly, however, for there is, in my work, a constant effort to move away from abstract discussions, to consistently explain ideas and use methods in a very concrete, and illustrative fashion. Because my interest here is not so much *in the theory or the methodology* of how researchers, teachers and students put up with one another to produce mathematics education, but instead in the *day-to-day and moment-to-moment* of researching, teaching and learning. It is important to me to go back to the data, or unfold ideas and method from the data itself. I always want to show how each concept pertains to actual actions or aspects of a situation. A general discussion of method is, in that sense, of limited interest to me: I do not want to read and write methodology alone, but I wish to use it with and for some concrete observations in and through which make sense with one another. That is, I take a stand against method precisely to save methodological resources from losing their meaning and sharpness, from making a mess of the complexity of human life by trying to pin it down, simple and clear (Law, 2004).

### *The Participants, the Schools, the Project*

This dissertation rests on a dataset constituted for longitudinal research on elementary school students’ geometrical knowings. In the first year, the study took place in a K-5 school located near the university (n = 430 students). This school serves an ethnically, socio-

economically, and academically diverse student population. The 24 second-grade students (9 girls and 15 boys) featured in this dissertation (including the students I call Eugene, Bert, Chi-Chi, Oscar, Maeve, Sonia, Jade, Jordan, Tobin, Alison, and Collin) do not all speak English as their first language but are comfortable and have a good command of the English language. Each of the 15 lessons were collaboratively planned and co-taught by the regular teacher (Tara) and the principal investigator (Jennifer Thom).<sup>3</sup> These lessons and the lessons that were collaboratively designed for the second and third year of the study involved the children in several two-dimensional (2D) and three-dimensional (3D) geometric tasks: looking for shapes in their everyday environment, describing or sorting figures according to geometrical properties, and producing or reproducing images through the use of tangrams, geoboards, plasticine or other manipulatives. Typically, after one of the teachers introduced the lesson, the students worked in small groups and participated in whole class discussions.

The students who participated in the second and third year of the research project and in particular, Nadia and Nate (who are featured in Chapter 7), attended an inner city school with a population of 231. This school serves an ethnically, economically, and academically diverse community (60 % ESL and of those, 50% are Aboriginal, with a total of 16 different languages spoken by the students). The 23 children (10 girls and 13 boys) who participated in the study (with their teacher, Martha) were enrolled in a first and second grade class and in the second year of the study, during the students' second and third grade of school. Like in the first year, lessons were designed to include the geometric concepts specified by the Ministry curriculum in geometry.

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<sup>3</sup> Importantly, the sessions of course aimed at enabling the students' development of concepts and conceptual thinking as specified in the current provincial curriculum.

Throughout the data-collection, there were teacher assistants who participated in the study and provided support for individual students and as well, a varying number of research collaborators (including Miki, who plays an important role in Chapter 6) who joined the classroom to record the sessions (at any time, up to 3 cameras were used simultaneously). All data, including students' productions for the three years of the study were collected in a consistent manner<sup>4</sup> whereby everything was digitized, securely stored, and made available to all members of the research team. Naturally, I took part in all aspects of the research, from gaining ethical approval to contributing to the design of the lessons, to teaching.

### *An Orientation Toward what Emerges*

These data-collections were based on the assumption that something interesting to research always emerges with/in a prolonged engagement with people and, later, with the data. This orientation develops in the tradition of ethnomethodological inquiry (e.g. Coulon, 1995), in which research does not consist of testing hypotheses or capturing “realities” delineated beforehand. What is important here is the possibility of bringing to light significant and often unsuspected aspects of everyday practices within a given domain of observation, as it appears in and through peoples' actions. For example, one of the first studies published on the project in which my research takes place investigates the surprising question of how children who never had formal lessons concerning geometric concepts can make this practice emerge through their bodily actions (Roth & Thom, 2009b). The question slowly came about as the data-collection and the analyses unfolded. From notes and discussion on primary observations, it appeared

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<sup>4</sup> Of the 5 studies composing the core chapters of this dissertation, only the last one presents data coming from the second school/research location.

interesting to look into more detail how the teacher and students managed to pull off a very first lesson in geometry. Analyzing the recorded data from that lesson, the means by which teachers and students introduced the new topics, and then consistently re-oriented themselves while providing resources for the articulation of geometrical ideas became apparent.

Congruently, a similar orientation toward what was to emerge is also at the basis of the teaching and learning activity we realized. Our team did not impose upon the teachers ready-made lessons as researchers often do: the lessons were always collaboratively designed with the teachers. At the same time, one of our special contributions to the design of those lessons was precisely to try and make it so that we could also see the children's mathematical knowings emerge, rather than presenting them with tasks in which specific outcomes are expected. Of course, as I mentioned earlier, a proper presentation of this approach goes far beyond what can be said in a few paragraphs, and requires the examination of concrete, actual moments of collaborative design and classroom activity.

*Mathematics Education In and Through the Day-to-day, Moment-to-moment*

More importantly, I believe, is to articulate how such an approach, consistent with the theoretical landscape I have portrayed, brings me to the heart of the re/production of mathematics education (and without the need for a specific research design to which participants must comply). Naturally, I contributed to the research project I described with my general interests for researchers', teachers' and students' work in mind, without a closed agenda or any special requirements. Working with them in the day-to-day, moment-to-moment, touches on the whole of mathematics education because their doings concretely realizes mathematics education in addition of being its very product. With the next figure, I borrow from fractal geometry this

very illustrative way to describe the situation.



Figure 2.1: The relation between the branches and the tree is *similar* to that of any mathematics classroom session to mathematics education as a whole<sup>5</sup>

In this figure, we see branches obviously produced by the tree, but also constituting the tree itself. Each arm is a concrete realization, the embodiment and the enactment of the whole, and each offshoot simultaneously produce and reproduce the tree. The branches also have another special relation to the tree as a whole: self-similarity. That is, we can focus on one part, one moment of the tree, and develop some understanding of the global structure. Clearly, the branches are not exactly like the tree, and all are different. However, they each illustrate what Bateson (1979) calls “the pattern which connects” parts and the whole. A small set of branches is sufficient to show how the tree re/produces itself, and at the same time reveals its dependence on the surrounding ecosystem.

I take moments and the day-to-day of researchers, teachers and students to have a

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<sup>5</sup> Photograph by N. Dilmen, licensed under the *Creative Commons Attribution ShareAlike 3.0 License*. Source: [http://commons.wikimedia.org/wiki/File:Kucuk\\_Camlica\\_06465.jpg](http://commons.wikimedia.org/wiki/File:Kucuk_Camlica_06465.jpg)

somehow similar relation to mathematics education as a whole. Giving attention to patterns and relations, attending to how moments are actually produced, what I learn goes far beyond the particulars of my observations: it speaks of the living structure of which they are a part. It is from such an angle that this dissertation develops around the minute analysis of few short episodes in the everyday of *some* researching, teaching and learning in mathematics education. Not to claim that all researching, teaching and learning similarly unfolds, but instead to turn attention to some fundamental dimensions, processes, and relations through which it concretely realizes.

### *Understanding the Senseable (with Others)*

Beginning with first-hand observations, notes and discussions produced during the data-collection itself, a first stage in my work with the data is individual. My attention is then on what is senseable (i.e., what is actually observable in the videos). I watch and re-watch the recordings, taking notes as to what seems interesting to me, sometimes transcribing a piece of conversation and taking snapshots of gestures, body positions or orientations, and so on. This approach is in the tradition of authentic research (Guba & Lincoln, 1989), which assumes that there is no fixed reality to seek through repeated experiments, but rather trustworthy interpretations developed in progressive subjectivity. Through prolonged engagement with the data and persistent observations I produce thick descriptions, glossing over video excerpts and using negative case analyses to contrast my observations. I raise questions for myself, linking ideas and theoretical constructs. From this work, excerpts slowly aggregate, raising enough interest for me to go back to the literature and see how the work of other researchers connect, contrast, or complete my own. Keeping track of this progression through the data and the literature, a consistent idea takes shape.

When analyzing data is a matter of seeing what generally remains invisible, the interaction of multiple perspectives on singular events is extremely useful. For that reason, I also adopt a collective approach to data analysis inspired by a method called *interaction analysis* (Jordan & Henderson, 1995). When an idea finally comes together, I normally organize a meeting involving the entire research team, other graduate students and/or post-doctoral fellows associated with our group. I select one fragment of data, and present it to my colleagues with my interpretation. We then examine the episode in detail, discuss what is happening in the video, and review the segments several times (at different speeds) to explore various possible understandings. Those discussions are recorded, and then serve as a basis to continue my analysis. This approach is particularly rich in that personal interpretations are discussed, and evolved into collective ways of making sense of the data. I am using these collective interpretations to go back to the data, before engaging again with my colleagues (especially advisors) through the writing of a study itself. This iterative process illustrates why the actual “framework” with which students’, teachers’ and researchers’ actions are to be analyzed cannot be determined a priori, but arises as a result of an ongoing analysis.

In collective analysis (as much as in my individual work), a special effort is made to remain in the realm of what is actually observable in the video. That is, I refrain from making assumptions as to what I think the researchers, the teachers and the students might want to do, think or know. Rather, I keep focusing on what they actually do or say, how they move, the objects they use, and so on. Moreover, I develop analyses of how the participants themselves account for one another and for the situation. Grounded in *conversation analysis* (Sacks, 1995; Psathas, 1995), I study social actions in and as they are located in everyday contact between people. Phenomenologically oriented toward ordinary language and communicative actions,

conversation analysis is the tool of excellence of ethnomethodological, looking closely at how individuals produce situations with one another. One classical example of conversation analysis is that of Garfinkel's (1967) experiment in which an experimenter responded to questions from everyday conversations ("How are you?") by repeatedly creating "trouble" (e.g., asking for clarification: "What do you mean?" or "How am I in regard to what?"). Examining such situations lead to the analysis of the incredible amount of know-how people use in everyday situation to coordinate their actions (e.g., Atkinson & Heritage, 1984). For example, we can look at how people create room and provide affordances for one another in managing turn-taking in conversation (a seminal study in that matter is Sacks, Schegloff & Jefferson, 1974); or examine vocal and non-vocal actions (e.g., talk, interjections, laughter, gaze, body movement, and so on) as appropriate and coherent fit to the requirements of the situation (Drew, 1995). An important benefit of conversation analysis is that actions (and most clearly communicative acts) cannot be taken in isolation and analyzed on the basis of their (semantic) "content." The actual, functional meaning of what a person says or does in a given situation does not belong to that individual alone, but occurs as a response to another: it depends on what the utterance is responding to, and how others are going to respond to it (Bakhtin, 1986).

In terms of method, my research similarly proceeds from the requirement of the situation, the data, and what I find in the literature. In researching, I am not trying to jump from point A (some initial questions) to point B (a clear answer). I am walking a walk, and as is the case with walking, I am constantly balancing, adjusting myself, with my feet standing on the ground, looking at where I am going. Bateson often uses the metaphor of a tightrope walker, who constantly not only moves forward, but also changes the angle of the rod he uses to keep his balance. One could see the theory-method conversation(s) I mentioned in the beginning of this

section as such a balancing of the “tool” I use to help me walking the walk of my epistemological journey.

### *Like Roads in the Country*

In this section, as in this chapter in general, I describe features of the landscape in which the following studies take place. Like roads in the country, I open paths for the reader, point in directions, naming places (e.g., “mediation,” “coordination,” “conversation,” “gestures and body position”) giving the reader a general idea of the particular direction or place entails they just got themselves into, but not much more. Against traditional forms of theory and method, I set up the reader for their re-construction with/in the studies themselves by throwing them out here in an upsetting way. In doing so, in and through my resistance to present this dissertation in a conventional fashion, I am actually taking one step further in my epistemological journey, producing my work as a whole utterance in which I offer a *reflection* upon what I have done over the past three years.

Fundamental to my journey in the day-to-day, moment-to-moment of mathematics education is the manner by which I make my observations. Writing this dissertation and those studies, I *am* myself researching and, in that manner, contributing to the re/production of researching (*and* teaching, *and* learning) in mathematics education. In a sense, the studies I present here do not report “results” or conclusions one can quickly jump to (result is from the Latin *re-*, expressing intensive force + *saltare*, the frequentative of ‘to jump’). Research “findings” here are not separable from *method* in the true meaning of the word: from the Greek *meta-*, expressing development, + *hodos*, ‘way.’ Research is truly re-search and finding *is* developing a way, coming to look, so that we see certain things in the phenomena we observe.

As Maturana (2002) puts it, “I would not ask what is, but I would ask myself what criterion do I use to validate my claim that something is what I say that it is?” (p.1). From that angle, technical details regarding the length and number of lessons, or the sociometric description of the research participants may have their importance, but not as much as the actual, concrete actions that I actually observed.

Consequently, the work I am doing here, playing out elements of methods while resisting method as we typically conceptualize it, is in two ways a reflection of my doctoral work. First, I am developing here “serious thought and considerations,” thinking aloud, with and for my reader, about my researching process and my use of concepts and means of observation. I do what Bourdieu names *participant objectivation*: not a “textual reflexivity” in which I contend to observe myself as an observer and produce a narrative of my researching activity, but the undertaking of an exploration of the social conditions of possibility, of its effects and limits, and of my experience as a knowing subject (Bourdieu, 2003). I express and demonstrate that researching in mathematics education is not as pictured in the fairy-tale of science decried by Feyerabend. It proceeds, and I show, from the mingling of theories and method in dialogue with people, actions and situations. Even better, I reflect this interlacing in the sense that I embody it in those pages. I am not only producing text about the challenging, erratic, and often implicit work of researching in mathematics education. I re/produce it within the text itself. Writing these lines today, and as an additional step in my journey, I reflect it here like in a mirror, transforming what could otherwise appear as a disparate set of studies into yet another consistent, heartfelt, unsubmitive, and publicly open engagement with people, ideas, and data.

## **A Path Laid Down in Walking**

Where are we going with all this? I wanted to learn about mathematics education, and especially to learn about knowing, about what it means to know, and how it is that we know at all. When I recognize an object as a cube and call it or describe it that way, when I recognize a student's actions as instances of knowing about cubes, or when, as a teacher I knowingly engage with these students to bring about similar geometrical observations, how is it actually and concretely that I—that we—know? Progressing through this epistemological journey, I had no map, no itinerary, and no guide. I have laid down a path in walking. Taking notes, drawing large portions of the scenery and documenting with minute detail some interesting fragments founds here and there, five studies, like pages from a travel journal, emerged from the “expedition.” In this section, I briefly introduce them to the reader.

### *At Walker's-eye View*

In the next chapter, *What Makes a Cube a Cube? Contingency in Abstract, Concrete, Cultural and Bodily Mathematical Knowings*, I take a first look at how geometric knowings come into being in a second grade classroom. The chapter articulates how learning mathematics is about learning to explore the world and ideas with our body in a way that one can culturally associate with the field of mathematics. I discuss how physical body and cultural practice are a constitutive part of each other, and illustrate how this plays into the geometrical knowing. Analyzing a classroom discussion between a teacher and her students concerning “what makes a cube a cube,” I examine different forms of geometric knowings. I articulate the double ascension of the abstract and the concrete and the interplay of body and culture in relation to the nature of mathematics/geometry. Hence, I show, in this chapter, how concrete and abstract co-emerge

from children's everyday knowings, and, importantly, in transactions with the material world and with others. More precisely, I discuss (a) the process of translation of sensorial, cultural, concrete, and abstract experiences and (b) articulate how geometric knowings come into being by communicating with others, but also (c) the crucial role of cultural artifacts in these processes. In the end, I conclude with a reflection on the co-existence, co-emergence and co-evolution of abstract, concrete, bodily and cultural nature of geometric knowings as it appears from the observations of teacher and students' day-to-day, moment-to-moment actions in mathematics education. A first version of this text was published in 2009 in a book entitled *Mathematical Representations at the Interface of the Body and Culture* (Maheux, Thom & Roth, 2009).

In chapter four, I take a more critical look at the production of "knowledge" in and through the analysis of such classroom episodes from video data. That is, I follow up on the previous chapter's conclusion that "recognizing the contingency of abstract, concrete, bodily and cultural knowing stresses the need to be particularly mindful and attentive to what teachers and students bring into being." As an effort to better acknowledge and value the complexity of this interplay through which students are and become mathematical in the very process of doing mathematics, the chapter, *Looking at the Observer: Challenges to the Study of Conceptions and Conceptual Change*, situates itself within a debate between sociological and psychological perspectives to educational research. Taking up Maturana's (1988) observation that the most central challenge faced today is the (epistemological) question of reality, I articulate a perspective of the observer and the implications that result from taking a psychological or a sociocultural approach to educational research. Drawing on an excerpt in which three second-grade students discuss a cone's ability to slide and roll, I enter the epistemological domain to articulate the differences in how sociological and psychological approaches define the relation of the observer (i.e., student

or researcher) and the observed. Second, I illustrate implications by showing how conceptions are better understood as students' discursive co-productions. Inasmuch, I conclude by elucidating how, in analyzing data, the observer-observed interdependence foregrounds a certain understanding of the relation to the Other and his or her otherness, ethics, as an important dimension of research in mathematics education. In particular, I discuss the special responsibility that we have, as mathematics educators/researchers, to produce and reproduce mathematical activity as a mean of coexistence in which we celebrate the other's alterity. In that respect, I stress the need to understand how students' explanations contribute to different, *potentially mathematical*, conversations and in what way they are responded to in and through the knowings they brought forth in making (mathematical) observations. A different version of this chapter was featured in a book published in 2010 (Maheux, Roth & Thom, 2010).

Chapter five of this dissertation takes on some of the observations made in the two previous ones, and develops what I came to call the *relational* nature of (geometrical) knowing as observed in teacher-student communication. That is, I take one step further the contention of the previous chapter concerning the attribution of "knowledge" to students based on what they say or do, the observation of mathematical activity, as I observed it, is something teachers and students do with and for one another, and my conclusion on our special responsibility in delineating from student's contribution mathematics as a (conversational) praxis. The chapter, *Teacher-Student's Knowing-With: The Relational Dimension of Mathematical Communication*, articulates the irreducible, dynamical nature of mathematical knowing through communicative activity that is always knowing-with another and therefore constitutes an ethical relation. My goal is to analyze this in how teachers and students communicatively produce geometrical knowing. I use a case exemplar involving two teachers and a second grade classroom to

conceptualize and illustrate the inherently problematic analysis of classroom activities from the perspective of the individual (e.g., from students or teachers isolated utterances), and introduce the essential role of the Other in knowing mathematically. That is, I show how this relational dimension by which knowing is always knowing-with is visible through teachers' and students' moment-to-moment mutual coordination in the co-production of communicative actions of knowing. I then present a discussion on how this relationality of teacher-student mathematical communication permits one to engage with central, and otherwise hardly addressed, *ethical* considerations to teaching and learning. The knowing-with of teacher-student communication takes us to consider how classroom conversation allows mathematical knowing to co-emerge in a dialogical rather than monological way. That is, to embrace, in the concrete, moment-to-moment experience of wondering and wandering (e.g., describing and comparing mathematically) the other's perspective as to what nourishes geometrical exploration and creates an ethical demand for the teacher to teach, and for the students to learn. In a brief coda, I finally fold back and reflect on my own work and the kind of knowledge produced by (my) researching. I articulate how writing research always already places the author in an ethical relation with the reader because the language we use is the basis of consciousness in the very sense of this term, from the Latin *con-*, "with" + *scire* 'to know.' That is, knowing by researching is also always knowing-with, and therefore something in which we are responsible. At the moment of writing these lines, a first version of the text was submitted to the journal *Educational Studies in Mathematics*.

Arising from my reflections on ethical dimensions of researching and my observation of the day-to-day of the data-collection process, my sixth chapter, *Researching-in-the-Middle*, takes yet another look at our roles as researchers, and that of "knowledge production," in regards to mathematics education. I first observe that mathematics educational research generally develops

what researchers can find about student' and teacher' knowing and learning. In contrast, researching is hardly ever thought about considering that what is observed during data-collection is fundamentally affected by the fact that it is produced for and within a research project. Furthermore, I also note a general claim that participating in research positively affects teachers and students, although this is not examined in terms of the concrete actions of researchers, teachers and students during research. Thus, in this chapter, I investigate the question of what kind of learning might arise for students and teachers from the presence and activity of researchers *as such* with/in a classroom. Analyzing some excerpts of the day-to-day of mathematics education (research) in a moment-to-moment fashion, I find that researcher's implication with teachers and students creates opportunities for them to their knowing of what constitutes mathematics teaching and learning. This happens not because researchers cause teaching or learning to occur differently, but because they enact new forms of teaching/learning with teachers and students. Inasmuch, I conclude by discussing the ethical significance of maximizing researchers' transactions with those who participate in their studies, even if it jeopardizes what they first intended to learn about teachers or students through the research study. A different version of this text is also under review, at the *International Journal of Qualitative Studies in Education*.

Finally, in Chapter 7, the last study featured in this dissertation somehow returns to the first one, and presents a more elaborated understanding, from the students' perspective, of what it means to know geometrically. In *Rethinking Knowing: The Threefold Nature of Relationality*, I extend the idea of the knowing-with of teachers and students articulated in chapter 5 to the question of mathematical/geometrical "rationality" as a basis for mathematical knowings. The study develops the concept of *relationality* as an alternative to current conceptualizations of

mathematical knowing, and overcomes reductions that tend to separate the knower from others, the world they know, and themselves. Discussing an episode in which two second grade students look for shapes in the environment, I articulate knowing geometrically as arising from and constitutive of a threefold *relationality* between knowing a subject and: (a) the material world, (b) others, and (c) with itself. I then conclude with a reflection on how relationality frames *knowing-with-in* as an essential condition for any kind of rationality to take place, and open a reflection around the ethical responsibility for oneself, for others, and for the more-than-human world that comes with knowing mathematically. A version of this study is under review at the journal *For the Learning of Mathematics*.

### *Maps, Not the Territory*

From chapter to chapter, common themes emerge: Places I went and things or events I have seen or experienced during my epistemological journey. That is, although I have laid a path in walking, I can make a coherent story of all the landscape traveled that essentially “holds it all together.” These themes have not been there at the outset of my walk, but emerged as a result of my exploration, and relate together with one another to as expressed responses to the overall question of “How do we know” in mathematics education.

A first theme is around the *attribution of knowledge*, or non-attribution of knowledge. It concerns the way we look at students and the possibility of looking at what is created and responded to in moments of knowing rather than trying to figure out what is going on in their heads (e.g., Chapters 4 and 5). It also speaks of researchers/educators’ attitude towards teachers’ knowledge in their everyday work, and how researching can go beyond trying to find out what they do (only to later try and change it), and becomes an occasion to learn (e.g., Chapter 6).

Secondly, and related to this first theme, is the observation that *knowing in mathematics education is done with and for another*. This becomes salient in response to the necessity to avoid making assumptions as to what is known, and attend to the communicative acts in and through which are realized knowing mathematics, knowing how to teach, or knowing about what is taking place between teachers and students (e.g., Chapters 3, 5, and 7). It also concerns the very fact that researching itself is done with and for (the purpose of) teaching and learning (Chapter 6), and that we are “all in this together.” Mathematics education is not simply imposed upon us by other people or by history: It is we who orient one another toward what we recognize as mathematics education across culture and history. The third emerging theme, *Knowing in mathematics education is coordinating oneself with/in the sociomaterial world*, naturally articulates, at the same time condition and result, from the two previous ones. This theme refers to knowing mathematically as a particular kind of transactions<sup>6</sup> with people and objects, a way of knowing that participate in/of the culture and history of mathematical thought. We see it a student’s mathematical activity and in his/her relations with a peer (Chapter 7) or a teacher (Chapter 5), as they become acquainted with culturally and historically developed ways of exploring the world and ideas (Chapter 3). It is also the case of the researcher who works with people, data or concepts (Chapters 4 and 6) and by this contributes to his/her community by responding to ongoing scholarly conversations relevant to mathematics education, and on the

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<sup>6</sup> I prefer to use the term “transaction” to “interaction” to mark an important difference between actions thought of as going ‘through’ (the prefix *trans-*) rather than ‘between’ (the prefix *inter-*) two persons, or a person and an object. The distinction, as made by Bentley and Dewey (1949), specify both the inseparability of the subject and the object in the act (which occurs with/in them rather than as an intermediate thing that does not incorporate them), and their irreducibility (the action does not simply go from one onto the other, but necessitate both in a radically different manner –one’s agency and one’s passivity).

basis of his/her concrete observations (Chapter 5).

Hinging on all that precedes it, a fourth theme surrounds the observation of *various forms of knowing and the delineation of what counts in mathematics education*. Knowing mathematically takes on different forms, at the same time abstract, concrete, bodily, cultural (Chapter 3), and taking on that complexity is fundamental to researcher's knowing (Chapter 4) and to teachers as well (Chapter 5). This is important because whether or not we value such forms of knowing contributes to define what it means to “do” mathematics education, and to know in the day-to-day, moment-to-moment of researching, teaching and learning. Important also to better understand how these different ways come together and evolve in and as the re/production of mathematics education (e.g., Chapters 3 and 6). Finally, and clearly intertwined with the four previous ones, a fifth theme concerns *knowing as an ethical relation*. We are rooted in the observation that knowing is knowing-with another, and the special responsibility that comes with this in how researchers, teachers and students address and respond to one another. This theme highlights the necessity of the otherness of the other, of the differences between us as students (Chapters 4 and 5), teachers (Chapters 5 and 6) and researchers (Chapters 4 and 6). Our necessary relations to otherness come with the (cultural, historical) possibility and the responsibility for knowing mathematically (Chapter 7); responsible to ourselves, to one another, and to the more-than-human world.

As a token, and simply to help the reader situate him/herself as he/she situates each one of the studies with/in the whole of my Journey, I offer the following “road map” (Figure 2.2) in which the chapters are represented as the main routes they offer between some of the themes. However, I present it with the precaution of reminding the reader that *the map is not the territory*, and that this applies to the following figure as much as to each one of the following

studies, which are themselves like (textual) maps of the broader territory I explore... which is but a fragment, and so on:

We say the map is different from the territory. But what is the territory? Operationally, somebody went out with a retina or a measuring stick and made representations which were then put on paper. What is on the paper map is a representation of what was in the retinal representation of the man who made the map; and as you push the question back, what you find is an infinite regress, an infinite series of maps. The territory never gets in at all. [...] Always, the process of representation will filter it out so that the mental world is only maps of maps, ad infinitum (Bateson, 1972, p. 230).

Not all the paths walked in the studies are represented on the figure, and one can see that different chapters connect the same themes in their own, unique way. These are again very important aspects of my work that I want the map to highlight. Themes are not final destinations, they are not units of “knowledge” about how we know in mathematics education I was seeking, and finally found. What matters here is not so much the finding heard as a noun. The interesting part is the finding as in the verb: the process, the exploration, the walk. Again, the roads as I see them are to represent exhaustively; neither are they objects of knowledge I was trying to reify. Changing one’s gaze is not about looking at different *things*, my own “vision” of the day-to-day, moment-to-moment of researching, teaching and learning (should it be in terms of themes or roads). It is a question of changing the way one *looks* at the world. It is about looking, really looking (concretely, actively, here and now!) differently. This is the challenge, and the opportunity, that I offer here to my reader, and also bring to light with the following map, and those I later draw (with more detail) to help introduce/situate each chapter.

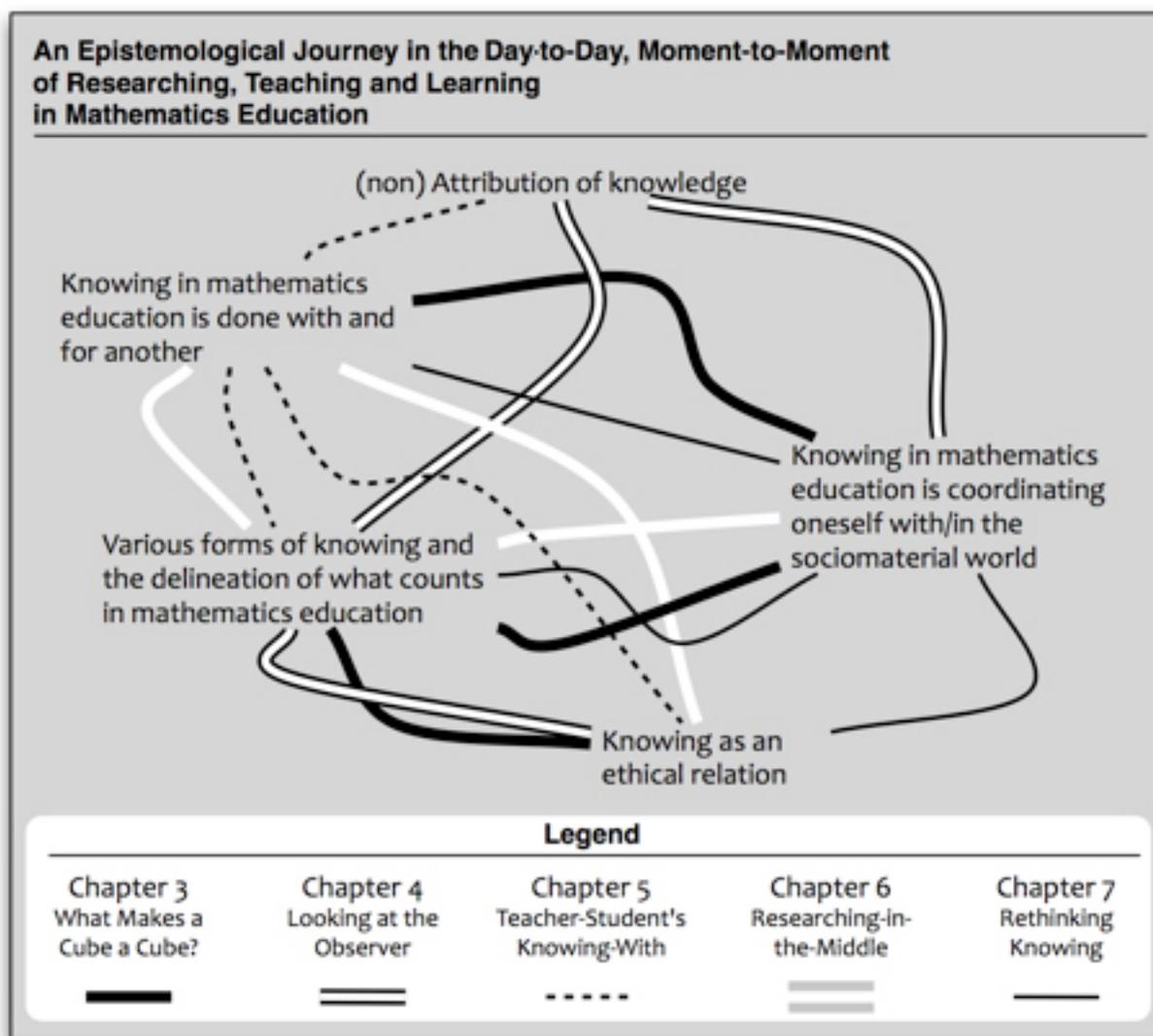


Figure 2.2: A map of the main routes traveled during my epistemological journey

In the very last chapter (Chapter 8, Going Back: There's No Place like Home), I will return to this irreducibility of the territory, the roads, and the maps (in text or drawings). For now, I invite the reader to “walk the walk” of mingling (with) data and theory.

May you join me, through pages of this traveler's notebook, on my epistemological journey.

May you take these roads in the country, reading and writing with me in the day-to-day, moment-to-moment, of researching, teaching and learning in mathematics education.

## CHAPTER 3

### What Makes a Cube a Cube? Contingency in Abstract, Concrete, Cultural and Bodily Mathematical Knowings

#### Preface

Learning mathematics is something we do with/in our body, and also with others, as we explore the world and ideas in a way that one can culturally associated with mathematics. I began writing this chapter on an invitation to contribute to a discussion on how physical body and cultural practices play out in the learning of mathematics. I was particularly curious about the idea of *abstraction*, which receives considerable attention from mathematics educators (Dreyfus & Gray, 2002; Hazzan & Zazkis, 2005). Indeed, although we variously define abstractions in our field, for example in terms of unfamiliar (Wilensky, 1991), complex (Ohlsson & Lehtinen, 1997) or general (Staub & Stern, 1997) ideas, it is not clear how we see body and culture take part in the real, actual moments in which one makes or uses abstractions.

With this observation in mind, I went through the data collected by the research group in the Spring 2007, and explored the literature, to finally realize the importance of thinking abstract, concrete, cultural and bodily not as separate “factors” influencing the learning of mathematics, but as absolutely *contingent* on one another in mathematical knowings. This observation came about from my continuous back and forth between the data –I shortly selected a brief episode to focus on—and the literature both in mathematics education, in the cultural historical line of thinking, and in enactivism.

From there, I worked (with my two supervisors) to articulate the ideas, and organize the text itself, composing and editing about twenty different versions of the study. In its final form, the chapter draws on an episode from a second-grade geometry classroom to articulate the

reciprocal dimension of abstract and concrete in bode with an understanding of the nature of mathematics that includes both the body and the culture. I give a special attention to how concrete and abstract come about when students and teachers engage with the material world and with one another. This takes me to examine the manipulative they use to enact/embody mathematical ideas. As a result, I observe the co-emergence of abstract, concrete, body and culture in and through knowing in mathematics. A first version of this text was published as:

Maheux J.F., Thom, J.S. & Roth, W.M. (2009). What Makes a Cube a Cube? Contingency in Abstract, Concrete, Cultural and Bodily Mathematical Knowings. In Roth, W.M. (ed), *Mathematical Representations at the Interface of the Body and Culture* (pp. 201-219). Charlotte, NC: Information Age Publishing.

To help the reader orient him/herself in the reading of the study and appreciate its relation with the other chapters through the five themes, the following figure presents a more detailed “map” of the path I laid down in walking this chapter. As with figure 2.2, my intention here is not to be exhaustive, but simply highlight *some* of the ways in which the themes emerge in this study, and connected. That is, although it provides a certain “entry point” to the chapter, the map really takes its value when read against the text and, reciprocally, the study reach to a new level when read together with the map. To be clear, the chapter does not explicitly addresses these themes: They result from an a posteriori reading of the study with/in the dissertation as a whole.

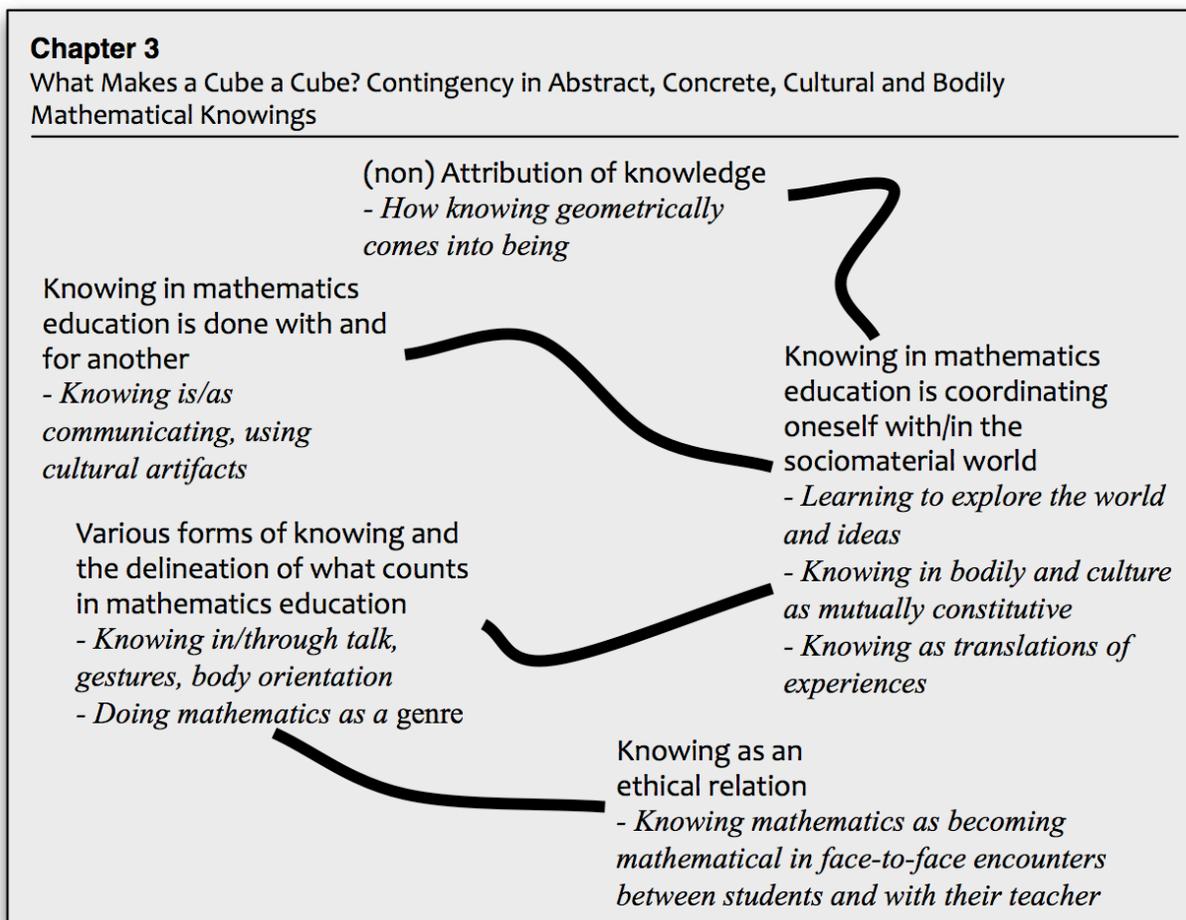


Figure 3.1: A map of my walk along “what makes a cube a cube”

## **What Makes a Cube a Cube? Contingency in Abstract, Concrete, Cultural and Bodily Mathematical Knowings**

### **This is Not a Cube**

When presented with the image below (Figure 3.2), a person will more often than not immediately see a cube. This happens despite the fact that the drawing *is not* a cube at all: like the French painter Magritte coined many years ago, the representation of the object is not the object itself. So how is this experience of a cube in a drawing possible?

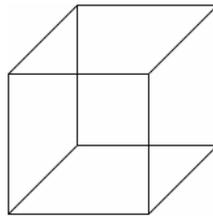


Figure 3.2: This is not a cube

From the viewpoint of phenomenology of perception, to recognize the figure as a cube, one must experience it bodily (looking at it, eying the image from every angle), and connect the perception to the culturally defined (mathematical) idea of the cube (Merleau-Ponty, 1962). Contemporary phenomenological cognitive science, too, asserts the physical body as the center from which all knowing emerges, while it defines culture as an open network of ways of dealing with the material world and with others, preserved from generation to generation (e.g., Gallese, 2003). In this view, bodily and cultural knowings are mutually constitutive. It is our cultural knowledge that tells us the drawing is a cube, as we recognize in the image a representation typical in Western culture. However, by looking at the image, we not only can recognize a cube,

but we also experience three-dimensionality by bodily knowing the relationship between the ink traces on the flat paper and a block on our desk. Indeed, when presented with a similar figure, young children express their puzzlement: “That’s not a cube, there’s no triangle on a cube!” For the cube to be seen, the physical body needs the work of culture, and vice versa. Here, I want to argue, the question is not whether knowing a drawing geometrically originates in the (biological) body inscribing itself in a cultural practice, or on the opposite, that knowing is first in the social, cultural, historical structure of a world progressively inscribed in the body (Radford, 2003).

Observing what makes a cube a cube from a different angle, that of actual acts of knowing, I suggest attending to bodily and cultural dimensions in terms of a “structural dance” (Maturana & Varela, 1987) – involving people and objects – between bodily actions and cultural meaning.<sup>7</sup>

When it comes to name the experience in which body and culture are at play in seeing a figure as a cube, the phenomenon of co-emergence takes place. It is common for us to consider that the cube and its representation are essentially different kinds of entities. In geometry, a cube is an ideal (transcendental) object, whereas its representation bears all the imperfections of its materiality. What we are less familiar to recognize is again the co-emergence between these two dimensions of the experience of a cube. To emerge means to rise out or up, as something coming out of a blend, e-merge. The ideal mathematical object is never experienced *per se*, but always realized in some tangible instantiations: a drawing, a block, a word, a set of coordinates, or even a mental image. In the same way, a block or a drawing are not in themselves related to cubes. The figure displayed above can very well illustrate a certain pattern composed of a square, two

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<sup>7</sup> I am grateful to Alfredo Bautista, a post-doctoral student working in our group and developing a new line of work involving various aspects of musicality in mathematical cognition, for turning my attention to this metaphor.

triangles and four trapezoids, and the concept of a cube need not be present at all. Indeed, a cube is for us simultaneously abstract (general) and concrete (particular), and these two dimensions need to draw on one another to be consistent with what we recognize as the experience of a cube.

I take from these and similar observations that abstract, concrete, bodily and cultural knowings in mathematics all co-emerge in the practical activity of doing mathematics. Is this actually the case? In this chapter, I explore this question by presenting and examining episodes taken from a second-grade class in which students are to learn geometry. I begin by reviewing the concepts of the abstract and concrete, the body and culture, in relation to mathematical understandings and then locate and discuss the presence of these concepts in the children's mathematical activity. This takes me to understand how forms of knowing that emerge from the children's mathematical activity are, inherently and contingently, all at once concrete, abstract, bodily and cultural in nature.

## **On Forms of Mathematical Knowing**

### *The Double Ascension of the Abstract and the Concrete*

We just saw that a cube appears to us as simultaneously abstract and concrete. Recent work in mathematics education which draws on the writings of Lev Vygotsky have come to address this situation from a dialectical perspective in which concrete and abstract understandings co-emerge from one another and co-evolve as double ascensions (Roth & Hwang, 2006). Moreover, such ascensions are neither directional nor hierarchical. Rather, understanding evolves simultaneously from the abstract to the concrete and from the concrete to the abstract. This is represented this by the symbol “|” when I point to the *abstract|concrete* dialectic.

In its etymology as well as in its usage, the term “abstraction” is both a process (the action of drawing out from a situation) and an object (the product, the concept). In the ascension from the concrete to the abstract, a complicated phenomenon is transformed into a simple thinkable unit by constructing relationships amongst the situational objects. For example, to recognize a cube in Figure 3.2, one must focus on a specific part of his or her rich and complex environment. Taking apart the concrete aspect of the figure (black ink and white paper), we recognize the straight segments and a particular organization that the abstract idea of the cube encapsulates. Because it centers the thinking and simultaneously gives meaning to the complex phenomena of perceiving the image, the abstract and the concrete dimensions indeed co-emerge in the environment and in the mind.

Abstractions can also be thought as properties of representations that are interpreted apart from their referents as the lived experience is brought about in consciousness. The reader can now think about cubes without necessarily referring to the specific figures from which the abstraction was generated. But in the ascension from the abstract to the concrete, abstractions are also unfamiliar or complex ideas that are situatively concretized in the practical activity. The concept of cube is also rich and complex as it connects with an inexhaustible number of other concepts. The cube can be defined from a variety of perspectives that differ according to their foci (geometry, algebra, arithmetic, calculus, topology), and each time in more than one way (in terms of faces, of edges, of vertices, as a region in space, or as a polyhedron presenting certain properties). The idea of a cube is general and it reveals the potential to embrace any of different experiences and what the reader sees as a cube when turning to the figure is therefore a form of reduction of specification (Roth & Thom, 2009a). Abstract ideas thus become concrete, particular and arguably complex. The concept sheds light on what is experienced, as context-

relevant aspects are noticed. Moreover, in one of its classical geometric representation (Figure 3.2), the idea of cube makes room for the interpretation of the two-dimensional figure to be perceived three-dimensionally. This occurs in the moment-to-moment transactions<sup>8</sup> with, in this case, an object in the material world. Furthermore mathematical understandings (of a figure, for example) are realized by this movement in the conceptual space, when the *abstract|concrete* dialectic gives meaning to the practical activity of looking at the figure.

Coming across drawings similar to Figure 3.2 in various contexts also lays a path for the development of conceptual thinking. As part of that movement, the ascension from the abstract to the concrete plays out when what was first unfamiliar, unspecified, entangled in the generality of an undeveloped (abstract) idea becomes more concrete. Through a variety of experiences, abstract understandings turn to be more context-related, precise, rich in details, and so forth. Simultaneously, growth in understanding is also characterized by the ascension from the concrete to the abstract, when a person develops fluency in connecting ideas to one another and drawing out of contextual events (including sensorial experiences) what appears to be relevant to the situation (Roth & Thom, 2009a). As the thoughts gain in independence and generality, we reconstruct the developmental trajectory from concrete to abstract forms of thinking. For instance, the idea of a cube and the figure probably “work well” together for the reader, but it is not the case for young children (who’s early contacts with cubes are more connected to “square blocks,” “ice cubes,” and other more or less cube-like objects rather than orthographic

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<sup>8</sup> A term used in contrast to the concept of *interaction* to implies the irreducibility of a person’s experience and of the object of that experience. In that sense, the idea of transaction permits an analytical separation of the two as independent entities that cannot be reduced to one another. What exactly a person sees when he/she looks at an image is the image’s response in relation to the person’s action of seeing it.

projections). The process through which each particular experience contributes in the development of the general idea that embraces all these representation of cubes is a movement from the concrete to the abstract. Concurrently, one way to illustrate the movement from the abstract to the concrete is the children's common use of the word "cube" to talk about oblong rectangular prisms: they only later employ it more precisely, taking in account the cube's property of having six square faces. And only a reader with a rich background in western mathematics can (a) appreciate in detail the interplay of structure within Figure 3.2 and (b) the idea of a cube as an ideal object having identical edges and square faces, which is not the case in the drawing (four of the "faces" are parallelograms and we have two different sizes for the edges).

### *Body, Culture and the Nature of Mathematics*

Researchers concerned with mathematics education have been successful in establishing theoretical perspectives that substitute traditional views of mathematics as transcendental, external, and objective. These other points of view make salient the fundamentally lived-dimension of mathematics itself, and its conceptualization on the part of learners. The idea of a cube always comes to being in some context (like in reading this chapter or in a classroom conversation) because it is fundamentally something that we humans do. We learn to recognize the line drawing as a cube by observing the picture and discussing it, by drawing it ourselves and connecting all these experiences to others in which the idea of the cube is brought about. We also connect these experiences to a variety of individual or collective endeavors, inevitably situated in specific, singular, socio-material contexts. When looking at the figure in this context, the reader is again situated in an infinite (cultural and historical) web of relationships with people and

material things that specify why and how the figure is identified as a cube (Roth & Thom, 2009a). In the context of a discussion on patterns, for example, the same image would be more likely read as a composition of two-dimensional shapes (two triangles, four trapezoids, one square) because of the reader|writer's orientation.

Research on the embodied nature of mathematics explain that mathematics is fundamentally grounded in human experiences, naturally arising out of everyday experiences of the world as a result of normal human cognition (Lakoff & Núñez, 2000). In this sense, doing mathematics calls upon the body as to the way we deal with the material world, in the continuous flow of making sense of the situations in which we find ourselves. A phenomenology of perception of geometrical figures (Roth, 2009) suggests that when eyes move about the figure without even “thinking” about it (i.e., in an unconscious manner), the movement leads to the recognition the straight lines, the corners, and the organization of the line segments in which some are parallel whereas others are perpendicular to one another. Visualizing the three-dimensional cube also includes an embodied knowing of the spatialization of the cube's faces.<sup>9</sup> While comprehending all of this, we knowingly disregard the fact that some of the lines cross other lines in the drawing and in doing so, we “see” the faces as if the cube was a three-dimensional object. This knowing-how to view and interpret the image constitutes “embodied knowing,” that is forms of relationships with the world inseparable from what we can biologically perceive (line drawing, not molecules!), and in the sense that the eyes now know what to do, how to interpret the figure for us to see the three-dimension object (and not only to think about the idea of a cube). But it is crucial to note that these knowings are also

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<sup>9</sup> We enact such spatialization in everyday life, for example, when we recognize a person we pick up at the airport or train station after having previously seen only a photograph of her/him.

“empracticed” because fundamentally cultural (Radford, 2003). For example, studies with Uzbekistani peasants (Vygotsky & Luria, 1993) showed that some people who lived in very poor villages and never went to school do not recognize shapes in the same way that we do, having grown up and living in western societies. We are getting closer, here, to the metaphor of body and culture “containing each other (Namukasa, 2004, p. 381), but with an emphasis not to these as systems and, hence, as “containers.” Rather, with the dialectic of abstract and concrete, I adopt here the metaphor of re-production. This metaphor maintains the focus on the performed action – as in a dance – instead of characterizing and thus essentializing “knowers and the known” (Dewey and Bentley, 1949).

To explain this better, let’s consider how the figure is open for interpretation and needs the work of the body in the active perception of the ink traces on the paper. Focusing the eyes on the lower or the upper square shape can bring about two different cubes: one protruding to the bottom left, and one to the upper right. These kinds of optical experiences (studied, amongst others, by Vygotsky and his colleagues) are used in a phenomenological perspective to show how the body (movement of the eyes) plays out in what emerges to consciousness. Furthermore, identifying line segments and shapes, spatially coordinating them in either 2-D or 3-D and gathering that flow of visual information are examples of how everyday experiences of the world specify one’s mathematical experience (seeing or interpreting) of the figure. Body and culture are truly in a dance of mutual specification in and through the re-production of what they enable in one another, and *for* the realization of the mathematical knowing of the image.

In addition, because doing mathematics is not just any kind of sense-making activity but one that has a specific history (that students continue nowadays), culture plays a central role in how what a person does is recognized as (or turned into) a mathematical activity. Culture is an

open-ended, unfinished set of possibilities afforded and specified by particular artifacts, like materials objects, tools, language, and so forth. When these possibilities realize themselves in communicating, some of these possibilities have historically come to establish themselves as genres. Communicating (which includes talking, gesturing, and other communicative resources such as body orientation) is both an embodied and a cultural practice. This is so because communicative resources are intelligible bodily productions *for others*, who are presupposed to see, hear, and understand what is meant without thinking and reflecting. Indeed, if a complex genre like mathematics distinguishes some human undertakings from others, it also draws on and is realized in living communication. Consequently, even what seems to be the culturally independent nature of mathematics is actually at the heart of a mathematical culture (van Oers, 2001), a domain in which people like us acknowledge ways of doing things as mathematical (or not). If the idea of a cube potentially has the same (mathematical) consequences for everyone in the world, regardless of their culture in the general sense, it is because they can all draw on and contribute in a mathematical culture.

Culture is (radically) conceptualized as living. Cultures are continuously produced and reproduced in practical activities. In the transactions between individual consciousness and the outer social world, the individual and the social reflexively not only complement, but constitute each other: conscious always is consciousness-for-the-other as it is consciousness-for-the-self (Vygotsky, 1987). In this situation, doing mathematics is a culturally determined way of being in the world. It is historically preserved by introducing the students to a cultural way of doing things, and by inducting the students' activities as part of their culture. This explains the generational (or inter-generational) continuity, as well as the dynamic, evolving nature of cultures. We recognize a cube in Figure 3.2 by relying on a range of previous experiences of

similar figures: they were seen in textbooks, on working sheets or exams, on chalkboards, etc. Most of them were part of societally organized activities in which a number of elements contributed in associating these situations with mathematics. These activities were also going on as such because there were people to bring them into being in a particular way. On another level, looking at the figure within this book also illustrates how culture evolves in our continuous conversations: I offer here a new usage of the image in a discussion that contributes in producing, reproducing, and transforming mathematics as a subject matter. The same image also has a history in the development of projective geometry, including its recent utilization in computer vision. One of the most important consequences of this is that the culture of mathematics is not limited to what mathematicians draw on and do, but includes all mathematically related activities, like teaching and learning. All those who in one way or another contribute to the re/production of the mathematical genre, using mathematical ideas to achieve their tasks (should it be a classroom session, the design of a computer game or the reading of this chapter) contribute to and are part of that mathematical culture (Roth & Thom, 2009b). It is only in and through this continual production of mathematics that the field also reproduces itself.

To summarize, mathematics as a cultural domain evokes individuals bodily engaged in mathematical activities, in transaction with the material world and with others from which abstract and concrete (mathematical) understanding co-emerge. In the following sections, I develop and illustrate these ideas by looking at an excerpt from a lesson in a second-grade classroom. The lessons were designed to allow students to learn geometry. Turning to the teacher and students' transactions, I show how concrete and abstract happen to be part in the consideration of what makes a cube a cube. To that end, I first clarify the concept of abstract, concrete, bodily and cultural knowings in this particular context. Secondly, I examine the

students' transactions with cube-like blocks and how they realized these transactions with and for one another.

### **What Makes a Cube a Cube?**

#### *A Classroom Episode*

The following episode takes place in the second lesson of a series of 15 classroom sessions. That day, one of the teacher-researchers (Jennifer Thom) introduced the following task: Each group of 3 students received a block and then was sent on a search in the classroom to “try and find as many things as [they] can that are the same.” Next, they were to “share what [they] found, and explain how [the objects] are all the same thing.” Eugene, Bert, and Chi-Chi received a wooden cube-like block. As they searched the classroom, they collected various objects: A red cardboard box, some wooden blocks of various sizes, a big yellow plastic cube and 3 bundles of post-it-notes (not cube-like). In the second part of the lesson, once all the students were back from the search, Jennifer used a chart where mathematical names and pictures of solids were printed to ask each group to “figure out [the] object’s name.” A few minutes later, she asked the members of Eugene’s group to place their objects on a piece of paper. Going through the following episode, I exemplify how bodily and cultural knowings are inherent in the double ascension of the students’ concrete and abstract understandings as they, and the teacher, discuss what makes a cube a cube.

01 Je: Can you put all your things <sup>[a]</sup> on the purple mat? Okay and take a look at all these things that they say and... Bert’s group says that these are all?

[a] Eugene, Bert and Chi-Chi start placing their objects on the paper. They carefully place them, move them around and position the bigger ones in a way that makes the whole collection visible on the mat.

02 Eu: <sup>[b]</sup> Cubes

[b] Picks up a yellow cube, holds it in his hand, turns it around, and places it back.

Bert and Chi-Chi are still changing the positions of the objects.

03 Je:These are all...

04 Eu:Cubes

05 Je:Cubes, okay. [c] And everybody I want you to look at theirs and they [d] say they're all cubes and a little bit explain to us what makes a cube [e] a cube, so Eugene, do you want to start? [e] What's one thing that makes a cube a cube [f].

[c] Jennifer pauses. Chi-Chi stops moving the objects, Bert is still turning around a blue piece of paper.

[d] Bert places the paper on the yellow cube and then stares at the collection.

[e] Eugene turns his gaze to the collection, and then looks at Jennifer. Eugene still looks at her while placing his hand on the top of the big yellow cube, holding it by two sides with his fingers and the thumb. At the same time, Bert had turned his gaze to Jennifer and then quickly back to the collection.

[f] Eugene picks up the yellow cube.

06 Eu:[g] All the sides [h] it has

[g] Eugene holds the block and turns it around, placing his right hand flat on some of its sides. Chi-Chi stops manipulating the other objects and turns her gaze to the block.

[h] Bert turns his gaze to the cube Eugene is holding and turning around.

07 Je:Sorry? [i]

[i] The three students turn their gaze to her, and then back to the block Eugene is still holding.

08 Eu:All the sides it has, like all the same, they are all the same [j], each side

[j] Eugene turns the object, touching and presenting different sides.

09 Je:Okay [k] so you're saying all the sides are the same [l], what do you mean by same?

[k] Eugene shortly turns his gaze to the object. Chi-Chi turns her gaze to Jennifer.

[l] Chi-Chi turns her gaze to the collection of objects on the mat. Eugene is still rotating the object in his hands.

10 Eu:Well [m] each one is the [n] same size and each one is the same the same [o] like square, like, right

[m] Eugene stops rotating the object, holds it with his hand round it, and looks in the teacher's direction.

[n] Eugene touches one side/face with his finger, turns the object and touches another side/face in the same way

[o] Eugene slides his finger all around the edges of different faces.

11 Je:Okay so Eugene says they are all the same size and they are all square [p] Oscar?

[p] Oscar raises his hand. Eugene returns the cube to the collection.

12 Os:Hum [q] well the squares all have four corners and [r] this square has four corners [s] and all the squares here have four corners and so... and all they have four corners.

[q] Takes the yellow block from Eugene and look at it, the points to the collection and turn his glance back to the block, touching different vertices with the tip of his fingers.

[r] Puts down the yellow block in front of him, picks up a small one and touch an

edge.

[s] Return the small block and picks up the yellow one again, using it to points to the whole collection various blocks before he deposits it back onto the paper mat.

*Concrete and Abstract from Everyday Knowings in the Mathematics Classroom*

We are positioned in our everyday lives within an established system of experiencing the material world (e.g. seeing, touching) that links what emerges to the sense with a complex web of previous experiences (visual or tactile) and with a set of culturally possible ones (examining a drawing, manipulating a block). When brought about in the practical activity of manipulating objects and engaging in a classroom conversation, the idea of the cube simultaneously constitutes an abstract and a concrete knowing. When the students move around the blocks, touch and refer to them, they use concrete instantiations of a cube, materialized “rough approximations” of the ideal concept. At the same time, when the students use the objects, name them “cubes” (turn 02) and discuss what makes a cube a cube, they enter in the domain of abstraction. In this episode, Eugene does not attend to certain aspects of the blocks (color, material, size, functions in another context) but he does explicate other properties that appear to be relevant to the situation (e.g. the fact that they have square faces, turn 10).

We also see the movement in the *concrete|abstract* dialectic in this brief episode through the transactions between Eugene and Jennifer. Eugene translates the appreciation of consistency in the block (concrete) in terms of sameness (“they are all the same, each side”), an abstract, general idea. In turn 09, as Jennifer asked him to explain what he means by “the same,” Eugene moves forward, and the abstract idea becomes more precise, concrete: “each one is the same size and each one is . . . square.” We learn what is a cube as an abstract concept and in its concrete instantiations by engaging in transactions with material objects and by articulating ideas about

them. Growth of understanding is therefore something that occurs in the realization of our ongoing activities (naming objects, moving them on a mat, explaining why they can be called cubes). It is from everyday embodied knowings that the abstract mathematical concepts come to life, while these concepts are always connected to these everyday experiences within a specific cultural domain of interaction, a genre, like geometry.

The role of body and culture in the practical activity can also be observed in the episode. In geometry, a cube is an ideal mathematical entity, a regular solid, a region of space formed by six identical square flat faces orthogonally joined along their edges. But to know “cubes” is much more than being able to produce a definition. It is to demonstrate a competency to use words, interpret ink traces on a piece of paper or blocks, to act according to the mathematical properties of a cube as to a given activity, and so on. There is also a form of embodied knowing about cubes that comes with the experiences of the blocks. To know an object in its cube-like dimension is to know how it affects the senses, what these impressions reveal about the structure of the object and how they are consistent with the idea of a cube. In other words, it is to know how to hold a cube-like block, how it feels and how to manipulate it. Accordingly, we recognize bodily mathematical knowings when students bring about salient aspects of mathematical thinking as they use their body to carry out a task.

In the episode, the concrete experience of the block perceptually guided Eugene to develop the idea of a cube. Along with his first manipulations, Eugene indicates that a cube is an object that has sides (turn 02). In return, it seems that his conceptual understanding guided his perception, focusing his attention to the faces themselves, so that he could see and feel the conservation of distance between the edges in all the faces and conclude they are all the same size, and all square. Physical manipulation is the way that Eugene explores the object, and it

brings forth his articulation of the idea of what makes a cube a cube. It is the conservation of the haptic and the visual impressions of the object (in turning it around) that can give rise to the observation of sameness. To rephrase, experiencing different faces in different orientations reveals that they all look or feel the same, in whichever orientation they are. In the practical activity of dealing with the block, Eugene already realized the concretization of an abstract idea: the block he picks up (gesture [b]) is a member of the “object” category, something tangible that can be held and touched. In the same movement, turning his attention to the block, Eugene experiences a concrete, material instantiation of a cube while talking about it in an abstract way (explaining that what makes a cube a cube is all the faces that it has). The experience of the block then develops in both the concrete and the abstract dimensions, as Eugene feels and articulates more. Indeed, he touches and mentions other aspects relevant to the concept, talking about dimension (“size”) and shape (“square”) of the faces (turn 10). Perceptually and conceptually guided and guiding his transactions with the block, the abstract becomes more abstract, the concrete gains in concreteness, and not only simultaneously, but literally from one another. It is because Eugene is in a position to feel more in the concrete that he can articulate more in the abstract, and vice versa.

In examining the previous episode, we see that these embodied knowings are related to everyday experience of the world: It is from very early experiences in life that we develop a sense of space, boundaries, organization and three-dimensionality of objects. These are not a random collection of experiences, but rather, they are the result of an established system of experiencing the material world. And we need to remember that they are at the same time constitutive part of establishing that system from generation to generation. The students carefully pick up the objects and move them, limiting their exploration to the visual and haptic

dimensions, while they could have tried, for example, to taste them, smell them, throw them away, or even break them apart. Therefore, they demonstrate cultural knowing about how we generally treat certain objects and in certain contexts. In his contribution, Eugene draws on everyday knowing about objects: picking up the yellow cube and turning it around is not problematic (like it would be the case if, for example, the sense of touch was failing him). This form of knowing is culturally oriented: Eugene holds the block in ways that differ from how we would, for example, touch a living being. Further, these embodied forms of knowing (drawing on everyday experience of the world) allow the students to engage in a mathematical genre (a way to deal with objects and communicate understanding) by focusing on the description of the blocks, looking for regularities (like Eugene saying that all the faces are the same) or similarities (like Oscar explaining that all squares have four corners).

In other words, the episode exemplifies how embodied forms of knowing are fundamentally cultural (developing from earliest experiences), while all cultural forms of knowing are realized in the body (manipulating objects, writing, or reading a definition in a text book). Because it is impossible to recall and connect all these experiences (that might or might not be relevant to the activity), a concept is at the same time both abstract (what is possible) and concrete (what is actually brought about). These inseparable ways of knowing are not the result of some mind game, or of uncommon phenomenological experiences. It is something students, as all people, do at every moment in the practical activity of doing school and as an integral part of what learning mathematics is.

#### *Concrete and Abstract in Transactions with the Material World and with Others*

The episode exemplifies how, in relation with the material world, the cube as a

mathematical object is experienced bodily and that these experiences are culturally oriented. Articulating *concrete|abstract* observations realize culture in and through the body. But to go beyond these general assertions, a close observation of what is going on in the episode is necessary. To fully appreciate the complexity of students' mathematics, in the following sections, I (a) exemplify how sensual, cultural, concrete and abstract experiences are translated in each other by the students and (b) conceptualize how the student's transactions with the material world are realized in and for communication with others and (c) helps us understand how bodily, cultural, abstract|concrete forms of knowing in mathematics are all fundamentally contingent on one another. I illustrate these relations by examining the role of cultural artifacts in bringing about mathematical ways of dealing with objects and ideas.

#### *The Translation of Sensual, Cultural, Concrete|Abstract Experiences*

When students manipulate objects while discussing what makes a cube a cube, a translation takes place between bodily and cultural forms of knowing. To illustrate this, there are limits to what can be said about the visual experiences of the objects in natural settings: movements of the eyes, changes in focal point, and so forth are not accessible through the data I (and the research team I am working with) collected. However, to get a sense of how these translations are at play, I can turn to the students' gestures and examine how they manipulate the objects. Looking closely at the students' transactions with the objects allows us to develop the concept of genre as key to articulate the cultural dimension of the students' practical activities. In the analysis of genre in communication (i.e., forms of communicating), the utterance, which includes the response received on the part of the producer, was chosen as the minimal analytic unit (Bakhtin, 1986). Utterances are comprised of both speech and gestures. They are the rough,

lived material from which and through which a genre realizes and concretizes a particular way of dealing with the world and with others. Students' utterances are what brings culture into being and at the same time culture is what makes any intelligible utterance possible, that is, something a student will intelligibly do. Paraphrasing Bakhtin I might say that it is through concrete utterances that culture enters life and that life enters culture.



Figure 3.3: Four ways of touching the yellow block in gesture [g], [j], [m] and [o]

In the excerpt, Eugene's manipulation of the cube is rich and complex (Figure 3.3), unfolding with his verbal articulation of the properties of a cube. From a phenomenological perspective, the block in itself cannot be known, as one can only identify the effect that it has on him or her. As he reaches for the yellow block (gesture [b] and [e], Fig. 3.3), Eugene can experience volume and can feel resistance. In gesture [g] and [m] (Fig. 3.3), he uses his two hands to feel the surface of four of the six faces of the block which allows him to experience their flatness and their boundaries. Moreover, holding a cube requires the complex coordination of the arms, the hands, the fingers, and the continuous interpretation of sensory information. In picking up the block, Eugene shows an embodied form of knowing about the cube as a solid: a region in space bounded by faces.

Using a block to talk about cubes is also intertwined in the cultural dimension of mathematical forms of knowing. We know that there is a need to select among what aspects of

the object are felt and what is to be drawn out of that sensual experience about what is relevant in the cultural domain in which these observations are made. It therefore appears to us that Eugene translates his sensorial experiences within the cultural domain. The weight of the block, its color, and the particular volume it occupies are examples of the object (and therefore objective) characteristics that he does not draw attention to. These translations reveal a form of participation in a cultural/mathematical way of considering the material world. They realize a way of talking about (and manipulating) objects, to make sense of experiences that contextualize a cultural possibility directly related to what we define as doing geometry.<sup>10</sup>

The translation of sensorial and cultural knowing goes both ways. A cube, explains Eugene, is an object whose faces are all the same size and all square, which is something he can articulate without entirely knowing “all” that makes a cube a cube. From a rich and messy background of bodily and cultural knowing about objects and linguistic terminology—the words “cube,” “sides,” and “square” that Eugene uses without the need, at this point, to define them—surfaces the idea of a cube. Whereas what the body realizes is a cultural (mathematical) orientation, the genre itself takes part in orienting the sensorial experiences. For instance, cognitive phenomenological studies show that the fact that the light from an object falls onto the retina cannot guarantee that the person consciously sees something. Not only does culture guides bodily actions in the material world (and in the immaterial world of ideas), but it is also realized by the body.

Picking up one block amongst others to talk about the whole creates a network of representations in which the abstract and the concrete are at play in a specific way. The particular

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<sup>10</sup> In contrast, the meaning of the organization of a pattern, in some African cultures, is developed through the elements of a story rather than by describing it (Gerdes, 2007).

block represents the whole collection, and both are concrete entities in relation to the abstract idea of the cube. Manipulating and talking about the block, Eugene allows us to witness his search for “internal” relationships. From this perspective, saying what makes a cube a cube is to look for perceptual differences from a whole (the concrete bloc) to reconstruct the unity with others objects and thus access its abstract dimension in the mind and in communication. The idea of the cube, reified in Eugene’s talk (turn 02 and 04, as he answer “Cubes” to Jennifer’s question) to designate the collected objects, is concretized as Eugene explains what makes all these objects cubes. It is as if Eugene is translating his concrete experiences of the block in a domain of abstraction (and vice versa). An abstract understanding of cubes is demonstrated by concretely selecting a block from the collected ones to talk about cubes in general. Indeed, one assumes in such case that what is pointed to and felt (as essential elements of what makes a cube a cube) can be experienced with any one of the collection. The particular cube Eugene holds becomes the general cube in relation to the collection while at the same time it addresses the abstract idea of a cube. Similar analysis can be made from Oscars’ utterance in the last turn of the episode. Oscar manipulates several blocks as he explains that “all the squares . . . have four corners.” In this case, looking at different blocks to answer Jennifer’s question is about identifying a set of “external” relationships that address what is common to all the objects. As Oscar observed multiple objects, we seem him perceptually identifying similarities between distinct items to reconstruct their unity. Contributing in a significantly different way to find out what makes a cube a cube, his contribution adds to the concreteness of what Eugene said before. This is done in part by clarifying the metonymical relationship of the single block to the collection and with the abstract idea of the cube, and the abstract dimension of what is brought about which also deepens the process.

*Communication With and For Oneself and Others*

In the last section, my analysis of the children's mathematical activity draws on what can be found in both their speech and gestures. I look at the words they use and the way they manipulate the objects at hand. These are constitutive elements of communication engaging the whole body. For example, when Eugene and Oscar communicate ideas about what makes a cube a cube, they produce sounds with their vocal cords and the movement of their thorax, tongue, jaw and lips, and they also use their hands to touch and point to the blocks. However, the effective meaning these resources bring about in conversation cannot be detached from the immediate context in which they are produced. Just like the drawing of figure 3.2 can appear as a cube, a pattern, or even a version of the Renault logo depending on the situation and the people's orientation (within a given genre), what might look like a similar utterance ("I am doing fine") can have a very different meaning according to the situation (such as talking to a neighbor or to a doctor, in response to a greeting or an offer for help). For this reason, an utterance should not be considered in isolation, but rather in relation to what comes before and after it, in order to understand what it answers and how it is responded to (Bakhtin, 1986).<sup>11</sup> As a sense-making activity, communication also presupposes intelligibility, that is, the engagement of a listener (that can be oneself or another) for whom the utterance is produced, and whose response completes it. In addition, the cultural aspect of communication is obvious in both the utilization of words and in "body language."<sup>12</sup> For example, to pulling one's earlobe expresses appreciation in Brazilians

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<sup>11</sup> Only this way can we distinguish between the doctor-patient opening chat and the beginning of the medical consultation.

<sup>12</sup> Strictly speaking, the term "body language" is a misnomer, for most body movements (including gestures) have neither syntax nor semantics, the condition for a system denoted by the term "language."

culture, while it will more likely convey the idea of scolding, or perhaps pondering in North America.

Because each contribution is woven into the unfolding classroom discussion, Eugene's and Oscar's communication were created together with the teacher and the other students. And it is precisely in this collective dimension that the meaning of students' utterances reveals itself. As part of the classroom lesson in geometry, for the collective purpose of understanding what makes a cube a cube, each utterance is a response to what had previously been said or done. This is exemplified in the events that followed.

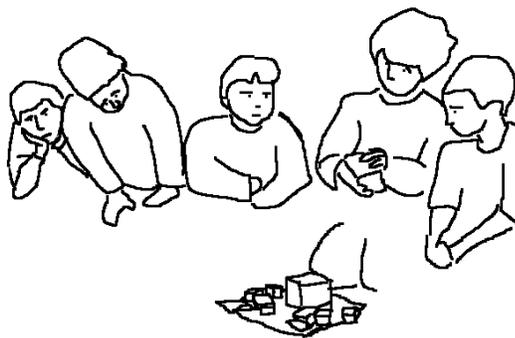


Figure 3.4: Some students' orientation while Eugene explains that what makes a cube a cube is "all the sides that it has"

As the objects are moved to the mat, Eugene turns his gaze to the blocks, and then to the teacher as she starts talking. Although Eugene, in response to Jennifer's question about what the objects are, seems to solely address his answer to her ("cubes," he says, looking in her direction), the teacher's utterances show that the conversation is not intended to be a one-on-one dialogue, but rather to engage the members of the entire classroom ("*everybody* I want you to look"). What we see happening here in communication is a re-orientation of the students' activity from the individual experience of the blocks to a collective endeavor of sense making about them. The orientation of the students towards the collective endeavor is proven successful by the change of

the students gaze during the episode. In the beginning of the excerpt, Chi-Chi and Bert are looking at the collection until the moment when Eugene actually picks up the big yellow block and starts talking (turn 02). At this moment, they turn their gaze to the object Eugene is holding (Figure 3.4). As the conversation continues, the students gaze then moves repeatedly from the object Eugene is holding to the blocks on the floor and back to the teacher. The change in the orientation suggests that the students are now attending to the classroom discussion. In its collective dimension, the task reveals itself as a cultural one, something the teacher and the students do together by taking part in the classroom event they create. Moreover, it is essentially through their body orientation that they realize this, bodily expressing their attention to the discussion over what makes a cube a cube. We therefore see that Eugene's communication about the cube is indeed realized for and with the others. While none of the students appear to participate otherwise at this time, the fact that they do not interrupt Eugene also counts as contributing to the conversation. Moreover, what follows in the classroom discussion confirms that the students actually were, at this moment, attending to what Eugene and Jennifer were communicating. And just like Oscar mentioning that all squares have four corners, other students will soon contribute and *add* something to what had been said.

As the conversation unfolds, the students and teacher's orientations confirm Eugene's contribution as a valid one, progressively entering the speech genre of accountably doing geometry. As a way of transacting with the material world and with others, the genre is not simply imposed on the students, but slowly emerges as such from the transactions between the students, the teacher, and the material setting. For example, a few minutes before the episode, Oscar and the two other students of his team were also asked to explain why their objects were all cones. Oscar answered the question by telling the story of his search in the classroom:

“Because these ones are the same, and then I found this one, and then I found this one and they all [look] like a cone.” In contrast, Eugene spontaneously addresses the question using the cube concept. Through each other’s response (including the teacher’s validation), the genre defines itself within the classroom in a way that is consistent with the broader cultural practice of geometry. We recognize this in the manner that sensual interpretations are translated in verbal or body form and subsequently are validated in communication. In turn 05, the teacher repeated the word Eugene used to name the objects (“Cubes, Okay”) and then used it four times to ask a new question about the collection (“what makes a cube a cube”). Similarly, it is the conversation that takes place in the collective that confirms Eugene’s interpretation, suggesting that his sensual impressions of the cube and their translation in language (“all the sides that it has,” “all the same size,” “all . . . square”) are culturally valid ones.

The same can be said about gestures that will shortly become a favored way of communicating about solids in the classroom, like Eugene’s flat-handed gesture to indicate a face. It is from these kinds of sensual experiences that abstract ideas are developed as a contextual way to make sense of the world – of what is felt, seen, and heard. Simultaneously, in the moment-to-moment transactions with others, embodied knowings become legitimated as specific ways of being in the world. They conceptualize in their similarities what is experienced in different contexts, such as a classroom search for “things that are the same” or a classroom discussion about what makes a cube a cube.

Because what happens in the classroom episode enters the realm of communication, an understanding of the concrete and abstract co-emergence needs to be closely examined from the perspective of how meaning arises from the interpersonal transactions at play. Lets briefly re-examine the following three turns in the conversation:

- 08 Eu: All the sides it has, like all the same, they are all the same,  
each side
- 09 Je: Okay so you're saying all the sides are the same, what do you  
mean by same?
- 10 Eu: Well each one is the same size and each one is the same, the same  
like square, like, right
- 11 Je: Okay so Eugene says they are all the same size and they are all  
square. Oscar?

My first observation is that we know what Eugene is talking about because his utterance, in turn 08, is offered as an answer to Jennifer's question ("What's one thing that makes a cube a cube"). Similarly, if we understand that Jennifer positively received Eugene's answer (as answering her previous question) it is because we have access to how, in turn 09, she responds to him ("Okay so you're saying all the sides are the same"). Secondly, this brief exchange between Eugene and Jennifer confirms *within the situation* the ascension from the abstract to the concrete. We interpret what Eugene's utterances potentially mean *for us as researchers* (in which case we detach what we call the student's mathematical activity from the activity itself, as lived that day, in that specific classroom), but also we can see that Jennifer and Eugene experienced it the same way, as the general idea of the cube becomes concrete in the specification of some of its characteristics.

Thirdly, it is again in a simultaneous movement that the concrete, complex experience reaches the abstract realm of thinkable units. The undifferentiated perception of sameness of the block is addressed through the abstract concepts of size and shape ("square"). These concepts emerge in conversation while Eugene answers what he understands as Jennifer's initial question and while Jennifer (in turn 09) responds to Eugene's utterance by stressing on one particular aspect of his talk ("what do you mean by same"). Eugene then produces an explanation in which, through Jennifer's reaction, we recognize an observation ("each one is the same size and each one is . . . square") of an observation ("they are all the same, each side"). It is because she

answers Eugene utterance the way she does (“Okay so Eugene says they are all the same size and they are all square”) that Eugene’s contribution takes part in the conversation as an ascension from the concrete to the abstract in mathematical thinking not only for us, but in the here and now of the children’s engagement. This is why we can argue that concrete, abstract, bodily and cultural mathematical knowings are inherently and contingently at play in the children’s mathematical activity as *they* experience it, and not only as *we* analytically conceptualize it.

### *The Role of Cultural Artifacts*

When we pay close attention to what is meant by bodily, cultural, abstract and concrete knowings in mathematics, and look at how they are brought forth in what a teacher and her students are doing, their co-implication become increasingly evident. The abstract is always culturally and bodily dependent, just as the concrete is. The contingent nature of these knowings are illustrated even more clearly by examining the role of cultural artifacts.

When students are doing mathematics, the material objects at hand are of great importance because they contribute to the orientation of the activity. Cultural artifacts are made to preserve and bring about some ways of understanding the world, even though they do not constitute knowledge in itself. However, artifacts do present cultural affordances. An object like the plastic block Eugene is holding is not just any object. As a concrete instantiation of the concept of a cube, its faces were crafted to be flat and smooth, its edges straight, sharp and all the same size, making as salient as possible the properties attributed to the abstract concept. On a larger scale, culture includes the means by which these blocks are produced (but again bodily produced by people of flesh and bones, using other tools and reproducing, too, other aspects of our living culture). Mathematics teaching and learning, as part of that culture, make it possible for the

children to enter in transactions with these objects. From there, the material properties of the blocks are experienced bodily by the students in order to enable the *abstract|concrete* dialectic we see in the episode, as body and culture work together in the orientation of the students in the task. In contrast, the foam cone used during the first classroom session, where students were sorting objects, is quite illustrative: The cone had an eroded tip, suggesting to the student the creation of a distinct group in comparison to pointy-top cones. This example also illustrates that cultural affordances only exist as long as they are part of a culturally oriented activity, and dependant on what the biological body can sense (in that specific time and place). For example, in a calculus class, the same cones could be used to bring about a discussion on differentiability; and in a topology class, both have two faces and a line (circular base), but only one has a vertex whereas the other has not. But all this is possible because the cones and its “pointy” or “rounded” tip are in scale with what can be bodily experienced: zooming in, we could see the roundness of something we call pointy (e.g. a needle), zooming further, we would see the rough and bumpy nature of the surface, and further again we could loose all sense of anything like an object, since matter mostly consist of void at the atomic level...

Nevertheless, it is important to note that not all artifacts have the same orienting power, even though they are all critical tools ensuring that not everything has to be re-invented from generation to generation. Some cultural groups do not use written language and their knowledge is still preserved through oral language and face-to-face intergenerational transactions around the utilization or the production of cultural objects. Nevertheless inscriptions, like the board with names and pictures the teacher used when she asked the students to figure out their objects’ names, are very powerful artifacts. For the learning of mathematics, such a phenomenon is widely taken into account by researchers studying technological environments like dynamic

geometry software. Eugene's choice to pick up a big yellow block among all the objects is not trivial but significant. Indeed, because of its size, the block was a good choice to sensually experience what is a cube and to communicate these experiences to others. Whereas this might seem to be an isolated event, it was not in this classroom. In the next few minutes, teacher and students would select this same object four times and used it many times during subsequent lessons. Similarly, the chart with names and pictures of solids also was not used for the last time. The teacher recursively invited the students to use it when words "failed" them, up to a point where the students themselves spontaneously turned to it, using it as a tool to enact memory allowing them to adequately engage in the mathematical/geometry genre.

All of our actions are, in one way or another, carried out through cultural artifacts, and language is probably one of the most observable. Language is something produced, reproduced and transformed by culture. Language exists independently of us (in the sense that it is a social creation that does not depend on any particular individual) while at the same time it only exists through each one of us as we use it (Leont'ev, 2005). Words like "cube" or "square" are part of the lexicon of the mathematical genre of geometry. They have a cultural meaning the students learn in the practical activity of employing them, even though they might not, in the beginning, have a clear idea of what they mean. This also illustrates the movement from the abstract to the concrete through (linguistic) artifacts, and the important role that they play in the conservation of our cultural ways to make sense of the world. Although, language is also (and always) experienced bodily as we feel things and name them, using our body to talk (producing sounds) about them with other people (describing what we see, what we feel). Speaking and writing are concrete actions by means of which we concretize abstract entities to make descriptions for ourselves and for others.

Ontogenetically, even our body—our whole sensorimotor system—is a cultural artifact. We learn to use our body in transaction with others, in what “forms the sensory composition of specific images of reality—currently perceived or arising in memory, relating to the future, or even merely imagined” (p. 14). We learn to exist as human beings, to walk, to use objects, because our body transactionally contributes to human social networks in which these ways of being are constantly produced and reproduced. In this sense, we are ourselves the most powerful of all cultural artifacts, and this explains why human mediation is so important. In turn 05, for example, Jennifer asks the students to look at the objects and try to figure out what makes a cube a cube. In doing this, she orients the students to the material objects, suggesting that an answer to her question might lay in the observation of these artifacts. Despite the fact that she does not mention how she sees them, Jennifer’s invitation to look at the blocks suggests that she has a certain way to sensually and conceptually perceive them. Students respond to the invitation, turning their gaze towards the collection (see Fig. 3.4). By looking at the objects, students notice relevant aspects, and the blocks serve as a focal point for the initiation of the concrete and abstract double ascension. But to look at something is not just to see it. The objects were already visible to all the students (placed inside the circle formed by the students), Jennifer then invited the students to focus their (visual) attention on the blocks in order to examine them. If (part of) what makes a cube a cube is, somehow, visible in the blocks, this needs to be discerned. By looking at the objects, the students are invited to see them differently, forming new images about them in a bodily, sensual, visual experience. Research in cognitive science has long demonstrated that these experiences of the material world shape the sensorimotor system, enabling us to operate with objects in a specific, culturally oriented way. Now, what we appreciate here is precisely this *dance*, this un-ending back and forth between body and culture,

between abstract and concrete, that realizes (in) the actual act of seeing in the material object something about what makes a cube a cube. Like in a pas de deux, features of the cultural artifact guide the orientation so that the transactions (perceiving the blocks, examining them further) bring about mathematical ideas (and mathematical ways of dealing with objects), and this engages the concrete and abstract dynamic of these experiences.

### **The Co-existence, Co-emergence and Co-evolution of Abstract, Concrete, Bodily and Cultural Knowing in Mathematics**

In this chapter, I introduce a perspective on knowing mathematically that is radically different, although in conversation with, other “theory talk” addressing question of body and culture in mathematics education. I resist, however, playing the game of theory talk by calling names and making it sound like theories are talking by themselves. Rather, I take on the talk myself and engage with ideas (and data). In so doing, I exhibit the double ascension of concrete and abstract and how this double ascension directly links bodily and cultural forms of knowing. Recognizing and inquiring into the embodied and cultural theoretical and focal concerns, the practical activity shows us how these are nested and seamlessly connected phenomena. Whereas distinct – and therefore irreducible to one another – body and culture are actually dependent, inseparable, and emerging from one another in the transactions with the cultural-material world. That is, I present in those dimensions how mathematical knowing can actually be seen as this “ongoing structural dance ... of events which, even in retrospect, cannot be fully disentangled and understood, let alone reproduced” (Davis, Sumara & Kieren, 1996, p.153).

Examining teachers and students practical activity in a classroom, concrete and abstract knowings reveal themselves as forms of bodily and cultural knowings. When Eugene uses a

singular object to give access to the cultural domain in which a cube is defined, his manipulations and talk contribute bodily to the cultural genre that defines geometry as a discipline. In the same movement, geometry as a genre provides Eugene an entry point, a way to transact with the block. These transactions – in the relationship they establish with the other blocks and with the idea of a cube – illustrate the *abstract|concrete* dialectic. From the concrete to the abstract, the singular block brings about the general idea of the cube, whereas from the abstract to the concrete, the idea of the cube is what all these objects are leading him (and us) to consider from that specific instantiation about what it is that makes a cube a cube. Briefly, then, teachers and students use their bodies for contributing to and making use of the math cultural domain. They do this in communication and when dealing with cultural artifacts resulting in abstract and concrete mathematical knowings.

If a person looks at Figure 3.2 and sees a cube, it is because bodily knowings work together with cultural knowings in the social context that reading this paper brought about. In this re-cognizing action, the abstract and the concrete co-emerge, and each cognizing instance is a step forward on the path of understanding in which abstract, concrete, bodily and cultural forms of knowing develop. As something we can observe in the everyday work of students in school, (mathematical) meaning (from the Latin *medianus*, “middle”) results from the co-existence, co-emergence, and co-evolution of these forms of knowing in the practical engagement of doing mathematics. Recognizing the contingency of abstract, concrete, bodily and cultural knowing stresses the need to be particularly mindful and attentive to what teachers and students bring into being. And we must acknowledge and value the complexity of these interplays through which students are and become mathematical in the very process of doing mathematics.

This is essential because it touches a very important dimension of human life: *ethics*. Doing mathematics is about *becoming* mathematical. Individual and practice mutually constitute one another as person-in-practice-in-person for whom “language and other forms of communication [are] critical in the possibility of an individual becoming a human being” (Lerman, 2001, p. 93). Contributing in the mathematical genre with and for one another, teachers and students come “face-to-face” and find themselves responsible for one another as human beings. We all have, at any moment, this responsibility that lays in the observation that we owe our own being as human to others (Levinas, 1981). For sure, mathematics educators are generally not very explicit, in their practices, about how they embrace this ethical dimension of teaching and learning. In my view, observing the co-existence, co-emergence and co-evolution of knowing in different forms (abstract, concrete, bodily, culturally) invites us to become more attentive to one another, and at least recognize the importance of various, co-dependent, forms in which knowing comes about between students and with their teacher as fundamentals of an ethics of becoming mathematical.

## CHAPTER 4

### Looking at the Observer: Challenges to the Study of Conceptions and Conceptual Change

#### Preface

The first thing my work with Michael and Jennifer taught me is that I never finish giving attention to whatever data I am working with, and that in so doing, I also always need to be extremely critical in view of my own interpretations. Reflecting over my experience in the writing of the previous chapter and connecting those thoughts with my reading of Maturana (1988) on what he calls the biology of the observer, I wanted to better understand how this was affecting my own work as a researcher in mathematics education.

It is about the same moment that I was again invited to produce a paper, this time in the context of a debate around the study of “conceptions” and “conceptual change” in educational research. It soon appeared to me that this was the perfect occasion to deepen my understanding of how researchers come to know about the students they observe, and also root my understanding of how and why one could go about making interpretations from collected data. The study on conceptions and conceptual change is quite popular among mathematics education researchers (e.g. special issue in the *International Journal of Science and Mathematics Education* edited by Mintzes and Chiu (2004) or, even more recently, Levin’s (e.g. 2009) work on teachers and students shift from pre-algebraic to algebraic problem solving). I noted that educational researchers often adopt (although perhaps unknowingly) perspectives that conceive themselves as observer separated from their objects of observations, whereas it has been long known that observers are constitutive of the phenomenon they observe.

I decided then to explore this observer-observed relationship both from the perspective of analyzing data, and in how students themselves arise as observers making mathematical analysis. Once again going through our database, I selected a lesson in which students were to decide whether solids such as cubes, cones, or cylinders, could “slide, stack or roll.” Developing my analysis in contrast to what we typically do in research on students’ conceptions, I then articulated a case in point in which I measure a sociocultural perspective on the observer with a psychological one. In this, I gave a particular attention to how students’ actions were not simply the fact of individuals, but made with and for one another. From this angle, I could articulate new possibilities for researchers to go about interpreting such actions to produce “knowledge” about the students’ situation. Placing, as Maturana (1988) suggests, objectivity “in parenthesis,” this laid out what emerged to me as an ethical ground for mathematics educational research.

Interestingly, because the study was written in the context of a debate between “social and cultural” theory and more “psychological” ones, I developed the paper in the genre of “theory talk” in which comparisons are made on the basis of principles and methods to compare, contrast, connect or oppose different perspectives on a given situation or phenomena. I do so, however, by trying to keep my focus on the actual practice of theory, that is the use researcher make of those ideas. As a result, little of the epistemological nuances and underpinning are presented, and the picture could leave more acquainted readers quite unsatisfied. But my intention here, as it slowly reveals by the end of the chapter, is to focus on our actions and responsibility as educational researchers. Were I to write this text today, I would certainly proceed differently, but I leave it in most of its original form, as a memento of my very first steps, depicted in the following map, in the journey of researching research.

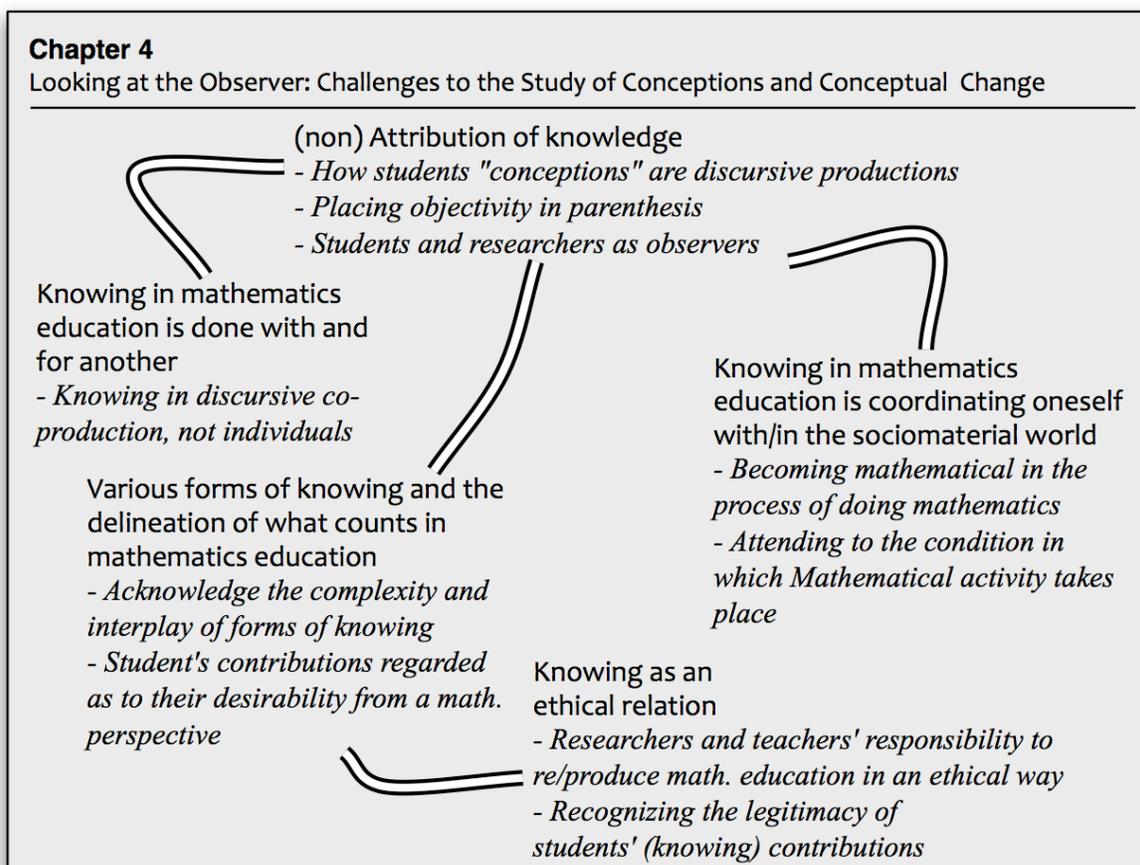


Figure 4.1: A map of my walk when "looking at the observer"

A different version of this study was published as:

Maheux, J.-F., Roth, M.-W., & Thom, J S. (2010). Looking at the observer challenges to the study of conceptions and conceptual change. In W.-M. Roth (Ed), *Re/Structuring Science Education: ReUniting Sociological and Psychological Perspectives* (pp. 201-219). Dordrecht, The Netherlands: Springer.

## **Looking at the Observer: Challenges to the Study of Conceptions and Conceptual Change**

### **The Observer and the Observed**

In a typical study of students' conceptions and conceptual change, researchers generally analyze what a student does or says in a classroom or in an interview and recognizes ideas that match or do not match their own understanding of the topic. Attributing the perspective they recognize in the student, those studies support the idea that a conception is the way by means of which an individual *intrinsically* conceives (of) a given phenomenon. They then hypothesize the existence of some mental structures that can be theoretically and objectively re-constructed based on what is observed in a student's performance. Thus, researchers studying conceptions commonly assume that the observer and the observed are separate entities. However, even in the most theoretical and hardest of all sciences, physics, the independence of the measured object and the measuring subject cannot be taken for granted: Light, for example, will present itself as waves or as particles depending on how we examine it. The artificial sense of separation from the object(s) of study found in many accounts on students' conceptions makes irrelevant the relationship that exists between the observer and the observed: an interdependence and co-emergence of the observer and the observed. This tight relation exists because each participants not only reacts upon what others say but also acts upon the reactions that his/her own actions give rise to. With this situation come epistemological, practical, and ethical implications for those researching in mathematics education. Positing or questioning the existence of an objective

reality mediates how we accept or reject another human being and the worldviews s/he develops. It provides a rationale that guides our actions. This is especially important when it comes to teaching and learning at a time where the ability to deal with the plurality and diversity of human culture have emerged as significant referents for our social behavior.

The most central challenge we face today is the question of reality (Maturana, 1988). With respect to the relation of the observer and the observed, Maturana suggests that there are two postures to reality and objectivity. One assumes that an observer's actions and knowledge does not affect the object of observation. The other posture recognizes that the observer is constitutive of the observed phenomenon, particularly in his or her ability to distinguish different aspect of a situation. This framework helps us understand how psychological and sociocultural perspectives distinctly define the observer. That difference in nature can be captured the following way.

Whereas the former attributes conceptions to the students, the latter situates conceptions in the actions of the observer who identifies them. The significant epistemological divergence entailed by these two postures has practical implications. Rejecting the observer-observed contingency necessarily leads to the confrontation between exclusive interpretations, for their validity is founded on the posited objective reality. If we pretend to know how things really are, then other interpretations are objectively wrong. This affects not only the work of researchers, but also promote a certain attitude toward the others, the students. In that perspective, we forget that students, too, are observers, thereby examining performances solely in the light of a given (objective) concept and judging them on the base of their compatibility with that single idea. In contrast, the ethical responsibility we have for the others is to fully recognize, and encourage, the legitimacy of various possible understandings. Appreciating how an explanation contributes to different conversations opens the discussion of how desirable these are. Dissimilarities then

become invitations “to a responsible reflection of coexistence, and not an irresponsible negation of the other” (p. 32).

In this chapter, I articulate a perspective on the observer and the implications that result from taking a psychological or a sociocultural approach to educational research. Drawing on an excerpt from a mathematics lesson in which three second-grade students learn geometry, I argue in favor of a sociocultural approach to conceptions in a two-folded argumentation. First, I enter the epistemological domain to articulate the differences in how sociocultural and psychological approaches define the relation of the observer (student or researcher) and the observed. Second, I illustrate implications by showing how conceptions are not the mere figments from the students’ minds but that students’ performances can be better understood as discursive co-productions. I conclude by elucidating how the observer-observed interdependence foregrounds ethics as an important dimension of research in mathematics education.

### **Two Perspectives on the Observer**

In this first section, I articulate how sociocultural and psychological approaches differently define the relations of the observer (student or researcher) and the observed. I introduce the relationship of the observer and the observed and then examine what it specifically tells us about each approach. For each one, I discuss (a) the observer-observed relationship and then (b) what entails the reading of students’ performances. The case analysis that comes after will follow a similar organization.

In the mid 1960s, Humberto Maturana became conscious that as a biologist he had no means to make any claim about objects, entities or relations as if they existed independently of what he was doing. That led him to realize that the most central question in any scientific debate

about the existence and nature of a given phenomenon implies the nature of the observer. To explain a phenomenon demands delineating the position of the observer in relation to it. When we explain a phenomenon (like a student's utterances during a conceptual change interview), we propose a reformulation of the particular situation that we are attending to and simultaneously define the extent in which that reformulation is taken as valid. Fundamentally, we can conceptualize how the object of observation is considered in two different ways: as independent or as contingent of the observer. Indeed, the fundamental operation of an observer is one of making distinctions and creating descriptions. These descriptions partially take up the infinite complexity of a situation and organize themselves to provide a reformulation, an account of what is taking place. As researchers we tend to focus on certain aspects of a phenomenon to answer particular questions. We select data and examine them with a specific theoretical lens. We know that a significant aspect in any researcher's work lies in its personal involvement with its research object.

#### *The Observer from a Psychological Perspective*

Many researchers working on students' conceptions position themselves as if reality (a conception) exists independently of the observer and the act of observation. This trend is particularly present among researchers who assume "psychological" perspectives. Take this example:

[W]hen we, as radical constructivists, focus on analyzing children's schemes, we work as first-order observers. Although a first-order observer makes a concerted attempt to assume the position of the child and think as the child does, the observer's ways and means of operating are left implicit, and the observer does not intentionally analyze the mental

structures of the child relative to his or her own mental structures. However, the first-order observer does interpret the interactions of the child and by this means tests the interpretations for their viability. . . . When we focus on analyzing the mathematical learning of a child . . . in both actual interaction and retrospectively, we focus specifically on explaining the child's learning relative to our own purposes, intentions, and contributions to mathematical interaction. (Steffe & Thompson, 2000, p. 202)

When researchers position themselves as if they can observe a phenomenon by putting aside their action as researchers, the observed inherently take a texture of objectivity: something that is not (fundamentally) influenced by the observers' perspective, means of observation, feelings and theoretical framework. In the first part of Steffe and Thompson's quote, conceptions really appears to be examined as if they existed "objectively" and as if they had, as researcher, the special ability to put themselves in place of the child (!), regardless of their reference to radical constructivism in which such endeavor would be seriously questioned. That is, no matter if this objectivist attitude rests on a positivist/realist epistemology or not, taking "conceptions" as independent of the operations by which they are identified leads to assume that students' understandings can be examined by analyzing what they do (e.g. in the classroom) or what they say (e.g. during an interview) and theoretically deconstruct and reconstruct their thinking. In this trend, it makes sense to discuss why students think that way and how they could be prompted to do otherwise. Students hold conceptions and undergo conceptual changes that the researcher-observer pretends to simply report. Such an approach makes irrelevant the ways and means of operation of the observer, and ignores the relationship of her or his own understanding relative to the children's. The preceding quote is clear about this: When analyzing children's "schemes,"

researchers leave implicit their own operation and do not consider the children “mental structures” relative to their own.

This corresponds to what can be called the path of transcendental objectivity. Blind to his or her participation in the observation, the observer here “implicitly or explicitly assumes that existence takes place independently of what he or she does” (Maturana, 1988, p. 28). The observer accepts his/her cognitive abilities without questioning how they work and influence what is observed. Accordingly, entities like mental structures or interactions can exist independently of what the observer does. In this perspective, even though one might acknowledge that the observer’s perception or reason is limited and sometimes fails, what is striven for is an objectively (‘objectively viable’, would I need to say in the case of constructivists) account for the observed event. For that reason, researcher-observers naturally find in the common agreement of each other’s interpretations a support for the belief that they validly account for an event. In this, two important facts are put aside: (a) that any observation is secondary to the observer’s experience of the world and (b) that agreement among observers cannot determine the validity of a claim that none of them can make individually.

These are fundamental epistemological implications and they are partially recognized by some researchers in the psychological tradition. Some research complements the search for cognitive structures with an attempt to take into account their own research endeavor by, as Steffe and Thompson, “explaining the child’s learning relative to our own purposes, intentions, and contributions to mathematical interaction.” However, there is a fundamental contradiction that comes with this. If the account of students’ learning depends on the researcher’s purposes and intentions, then the conceptual changes *to be observed* in the students depend on those motives as well. Thereby, how schemes undergo cognitive restructuring is based on theoretical

entities defined by the researcher and not something intrinsically characteristic of the students. Those mental structures, however, are conceptually defined as something that belongs to the child, not the observer. But clearly, researchers cannot divorce themselves from their objects of observation when analyzing children's schemes. The consequence is this: not only (a) can students' conceptions and conceptual changes not simply be reported and (b) observations cannot merely indicate what the students are thinking and why, but also (c) it also makes little sense to deduce from these conceptualizations any form of pedagogical prescriptions (in order to lead students to perform differently).

Intertwined in the separation of the observer and the observed is the posited existence of a problematic (objective-like) reality. Looking for universal features of development in the child, a typical study in the psychological perspective presents a researcher's analysis of students' performances and theoretically attributes his/her interpretation as being that of the students. The researcher then discusses why students "hold" or "acquire" those conceptions (generally referring to students' previous in or out-of-school not observed experiences) and suggests how they could be "changed" or "replaced." Belonging to the students, conceptions are then seen as the way in which an individual *intrinsically* conceives a given phenomenon. Those conceptions are thought as the rendering of mental or cognitive structure made implicitly or explicitly available to others by students' talk or actions. A conception is something that the individual possesses, a "cluster of internal representations and associations evoked by [a] concept—the concept's counterpart in the internal, subjective 'universe of human knowing'" (Sfard, 1991, p. 3). By adopting such view, one assumes that conceptions are imprinted in the mind (some will even say: in the brain) and later acted out when called upon. Thereby, what is (objectively) presented to the students is cognitively re-presented by them and, in their performances, re-

presented again for the researcher to examine. Through some sort of reverse engineering, the researcher searches for “schemes” or “mental structures” that s/he validates in the observation of an ill-define/objective reality. An example of this can be found in the first part of the quoted text: Researchers turn to an objective reality that bears the possibility to “tests the interpretations for their viability” through perception or reason. We recognize again the path of transcendental objectivity. Researchers here ultimately validate their explanation by referring to entities like mental structures and interactions: transcendental referents to which the observer reduces both his/her observation, and the observed. Such approaches therefore require a single reality (a conception) that explains what was observed by the observer, even though that observation might only be considered “viable.”

This is problematic because cognitive scientists now widely reject the existence of cognitive structures in which representations of the world can be embedded. An additional difficulty with this perspective concerns the constitutive part the socio-material environment for what is observed in students’ actions. Nowadays, educational researchers generally recognize the situated nature of what students bring forth. It is agreed that we need to consider that what students say or do is closely related to the specific context of the performance. Problems and phenomena are not addressed in the same way when encountered in mathematics classroom versus everyday life. Because they have dissimilar goals, means and rules, and because they are in different relations with different people, students do not always do things the same way. The perspective I develop here therefore invites us to consider this by examining coordination of behaviors to understand how “each individual is continually adjusting its position in the network of interactions” (Maturana & Varela, 1987, p. 192) that forms the collectives and situations. Being themselves observers, students coordinate with their societal-material world and with

others, contributing differently into diverse activities, with and for distinctive “communities of observers.” Those coordinations are not simply seen as mental constructs belonging to individuals, and that researcher can seek after: they are the very locus of what researchers in the psychological tradition want to call “conceptions.”

Most researchers who study students’ conceptions are aware of the problem of the social, and have tried to answer it by acknowledging the constitutive role of the context in the realization of what they observe. However, this poses another epistemological contradiction. If the context is recognized as shaping individual cognitive structure(s), then it is impossible to assume that what an individual does is based solely on the conception(s) in his/her mind. When what students say or do cannot be isolated from the context in which it exists and is observed, then it cannot be assigned to a conception that an individual *has*; it is not a characteristic (internally) belonging to individual students, but also is marked through and through by the context.

### *The Observer from a Sociocultural Perspective*

Researchers in a sociocultural perspective do not normally talk about conceptions and conceptual changes, because their work does not focus on the individuals but on the situations these find themselves in, and on the meaning they create with and for each other. In this section, I briefly outline (a) how the constitutive role of the observer and what is observed is taken into account in sociocultural perspectives and (b) give some insight concerning how students’ performances are made with and for the other(s).

“Everything said is said by an observer to another observer that could be him - or herself” (Maturana, 1988, p. 27) and the observer and the observed arise together, emerging from one

another in the act of observing. A sociocultural approach to mathematics education research defines the observer in a very similar way. To begin with, this approach clearly asks researchers to consider how the object of observation is constituted. Instead of striving to objectively assess what a student knows or can do, or to identify transcendental features to the human psyche (on the basis of talk or actions), the observer studies the way cultural traditions and social practices appear to be at play. The observer not only recognizes his/her purposes and intentions, but also delineates a unit of analysis that incorporates “goals, needs, affect, and cognition while locating the individual in the cultural life that precedes all of us” (Lerman, 2000, p. 211).

This corresponds to a path of constituted objectivity. In that path, the observer accepts that s/he is constitutive of the phenomenon s/he observes, recognizing that the observer’s observation depends on his/her cognitive abilities. I may use as example our inability to distinguish between perceptions and illusions to reject the independence of the observer and the observed. Existence “is constituted with what the observer does, and the observer brings forth the objects that he or she distinguishes with his or her operations of distinction as distinctions of distinctions” (Maturana, 1988, p. 30).

In sociocultural approaches, the mutually constitutive role of the observer and the observed is recognized both in the researcher and in the student’s activity. As Radford (2006), quoting Bakhtin, puts it, the observer has no position outside the observed world: all observations are constituent of the observed and this is why all “objectivity” is always subjective, and needs to be open to change and discussion. On the one hand, researchers turn their attention to the elements from which what was called a “conception” can be identified and try to understand how this happens. In an interview, there is a collective activity from which any data spring forth. From a researcher-observer perspective, this should mediate our understanding of what was achieved. In

other words, conceptions from this perspective are not something that belongs to the children, but to the situation that produces the talk from which conceptions are abstracted by means of multiple reductions.

On the other hand, a sociocultural approach stresses the constitution of consciousness through discourse, which includes all forms of using language such as gestures, written text, and so on. This helps us see how the student's activity is also conceptualized as the work of an observer contingent on its object of observation. In this view, students do not present conceptions stored in some cognitive structure, but participate in mathematical discourses by which they learn to distinguish different aspects of a situation. In other words, they become mathematical observers by creating mathematical objects of observations or by attending to what they observe in those particular ways. Communication includes more than the words a person speaks and encompasses all perceptuomotor activities exhibited to the researcher. Distinctions and distinctions of distinctions are operations in language in which the observer and the observed co-emerge. In the process of making distinctions, the observer is affected by what he or she observes, but simultaneously responds by selecting what is relevant and sensorially accessible to him or her, and therefore affect what is observed. Being an observer implies both agency and passivity because we are both observers and observed:

I am not a neutral factor. Together with others, we researchers both constitute the situation and are constituted by it. . . . I have no transcendental position, but neither is my theorizing, as a mathematics education researcher or as a teacher, on a separate level from my work on mathematic. Sociocultural theory does not need the separation of levels of analysis required by Steffe and Thompson's model. (Lerman, 2000, p. 224)

These reflections stress the need to avoid attributing conceptions to individuals. Examining how we make sense of things and situations, a sociocultural approach came to situate knowledge in the social. Researchers from that perspective thus characterized knowing as participation in an activity, and especially turn their attention to examine how meaning is discursively constructed in communicating with others. Sociocultural researchers do not negate the potential existence of some “mental plane” in which conceptual development might take place for the individual. But recognizing the inescapable dependence of the observer’s position and what is observed, they deliberately orient their undertaking elsewhere, that is to the inter-subjective plane forming the conditions in which that mental plane is formed.

This perspective is consistent with the type of observations we make as educational researchers: While it is not possible for us to see inside a student’s mind, we can observe what is made available to us by that student’s verbal or physical actions. Here, the explanatory domain, in which the observer observes the student, is based on what discourse and actions make available, not on the student herself. The essence of communication is the coordination between an individual and his/her social and material environment. What counts here is not only the content and the form of people’s talk and actions, but also, and more importantly, what they contribute with respect to the coordination of actions they bring about (Maturana & Varela, 1987). In that sense, a researcher-observer is interested in understanding what a student’s contribution in/to an activity reveals about the conditions in which s/he coordinates him/herself with the societal-material setting. The choice of focusing on these aspects is not incidental, but reflects the observation that, as observers who can contribute to the formation of such conditions, this is precisely a domain of possible action – unlike those mental structures which, were they ever to be found, researchers and educators would certainly never directly act upon!

In this realm of observables, close at hand contributions can take different forms: speech, gesture, action, or any combinations of these. With its origin in an individual (e.g. a student), a contribution nevertheless only makes sense in the relation it establishes with others, and to an ongoing activity: together responding to a particular situation and affecting its unfolding. In the perspective I adopt here, these contributions are not assumed to be the result of an individual's conception, but taken as created with others and for oneself and others. We recognize in this a path of constituted objectivity in which everything is said by one observer/observed to another observer/observed.

### **An Historical Take on Observation**

In the ancient Greece, history tells us, an important epistemological rupture occurred when Aristotle, opposing his view to Plato's, promoted, in order to produce knowledge about the world, the practice of observation over that of what we could call theoretical inquiry. In that gesture, Aristotle put aside the impossibility to directly attend the essence of anything through concrete observation (the argument at the core of his teacher's position), and maintain that one comes to know not by means of Socratic remembrance of pure ideas, but through our senses and, thus, in the actual experience of the world. A most fascinating aspect of the philosopher's system of thought is that whereas it explicitly poses the existence of an external, objective reality and that of universal truth to be discovered in scientific observations, the domain of observations relevant to the knowledge of something in that system includes not only material aspects. In addition to those, Aristotle includes the production and use of whatever is known (what he calls the efficient and the final causes), along with the agent who realizes this.

Aristotle's views profoundly shaped scientific scholarship through the Middle age and the

Renaissance, only to be gradually replaced by a new practice of observation. Making observations gradually became a practice of experimentation. More structured, systematic, and repetitive, and shifting from aesthetical appreciation to “rigorous” argumentation, this rupture is well known in the history of science (with Galileo, Kepler, Brahe, Newton, etc.) and, for example, medicine (with William Harvey, Andreas Vesalius, Antonie van Leeuwenhoek, and so on). Came with this an increasing specialization among the “Natural philosophers” by means of which the realm of scientists’ observations got significantly narrower, limiting his (rarely her!) object of interest. In this process, concerns for the contribution of the creator or the user of any “thing” known to be know was put aside in favor of a positivist/realist epistemology.

Since the beginning of this scientific revolution, the actual practice of observation reveals, however, the difficult nature of attending and creating models or explanations for what one notices. Tycho Brahe is known as an exceptional observer, meaning that others were far less precise. We know that Galileo “cheated” on his observations to prove his idea, and that various mixture of aesthetic appreciation, instrumental precision and skepticism in the difficult “art” of observation are also at the heart of, for example, the famous Martian canal controversy where Italian’s telescope plays a central role.

With this brief historical take on observation, what I want to do here is to clearly render the practice of observing as we imagine it today is not self-evident, and free from cultural or historical determination. Making observations and coming to see specific features of a situation contributes to the reproduction of particular ways of seeing. Looking at an object mathematically or looking at a student’s observation to hypothesize on her conception(s) is to realize certain ways of being in the world and with others. In a sense going back to Aristotle perspective’s on knowing, this entails that one’s observation is not merely made about the observed object itself,

but somehow always already includes the actual conditions (and their historical development) of the observation and of what is known. To know (mathematically, or in terms of conceptions) means inserting oneself into a certain praxis that provides a particular way to engage with(in) the world (Radford, 2008b). Various (and new) practices of attending are possible; some are dominant, and others are marginal (or still in the birth), each one realizing a different world view in the actual act of observing in such or such a manner.

### **How the Observer and the Observed Co-emerge**

The contrasts between sociocultural theories and psychological perspectives have been well debated over the years, particularly in mathematics education. My own work took me to articulate the problem in terms of how the observer is considered. What I want to do here, is to illustrate this position by (a) showing how the observer and the observed co-emerge from the student and the researcher's perspective and (b) illustrating how we can examine their talks and actions as discursive contributions. The following excerpt, in which three students talk about a cone, serves to exemplify this. The episode was videotaped during a lesson in which the students examined whether various solids could stack, slide, or roll.



Figure 4.2: Sonia, Jade and Maeve experiencing with the cone

01 Je: You can go ahead and record your predictions, okay.

02 Ma: My prediction [is].  
 03 So: [Slide]  
 04 Ma: do you remember that if you put it side ways and then it will roll like that? ((Maeve place her pen on its side, pushes it and finally rotates it on the table, see Fig. 4.3))  
 05 Ja: It will slide.  
 06 Ma: So it rolls actually.  
 07 Ja: But if you put it on its side it will roll, but if you put it up straight it kind of slides.  
 08 So: Roll means-  
 09 Ma: But, but if-  
 10 So: Ah Maeve, roll means when it goes back, back. ((Sonia rotates her hands backward one around the other))  
 11 Ja: Or keeps on going like in circle. ((Jade rotates her hands one around the other like Sonia is doing, but moving forward))  
 12 So: Yea like this. ((Sonia transforms her backward rotation in a forward one similar to Jade's))  
 13 Ma: So which one do we write?

To clarify how the observer, the operation of observing, and the observed simultaneously arise, let's examine Sonia, Jade, and Maeve's perspective on whether a cone can roll or not. Asked to make a prediction before they first try whether a cone will stack, slide, or roll, Jade first suggests that the cone will slide, and then nuances (elaborates on) what she first offered: placed up straight or on its side, it will either slide or roll. Contrastively, Sonia then insists that a cone could not roll because its movement is not linear, which Jade quickly supports. The three girls discuss some more and Maeve finally concludes, "Oh yeah, because it goes in circles, it's not going down."

As observers, the students make distinctions in what affects them. From those distinctions, they develop an explanation for what they frame as the phenomenon to be examined. This comes from focusing attention on only some of the infinite number of aspects of an event to define what counts as a phenomenon. For instance, the girls here delineate that deciding whether a cone rolls or not has to do with its orientation, the movement and the trajectory of the cone, not its color, the temperature of the room, or the angle of the inclined plane on which it would be positioned.

In this, they select some aspect of the situation to attend to, and also define what it means to roll (as rotating with a linear movement). A selection of what the observer looks at is not only necessary for any explanation to be developed, but is constitutive of what it means to observe (from its Latin root *observāre*, ‘to attend to’). Observing the motion of a cone down an inclined plane thus cannot be realized without adopting an observer position, making distinctions that delineate what count as the phenomenon. For example, in Jade’s utterances in turns 05, 07, and 11, the orientation of the cone (placed on its side or upright) and its trajectory (circular or linear) consecutively appear in the explanation of the thought experiment she conducts to make a prediction.

The same happens when students experiment with material objects. Soon after they made their predictions, the three students observed the motion of a cone as it was going down an inclined plane (Figure 4.2). After the cone is released and ungainly moves down the plane, Jade and Sonia observe the phenomenon and briefly pause before they conclude, “Yea . . . but it rolls. Yea it’s okay.” The observer’s position they adopt now leads them to see the cone as rolling. However, a few minutes later, they reproduce the experiment. Whereas Sonia affirms beforehand that the cone would roll, when she releases it to look at what happens, she turns to Jade, and the two girls decide, “No, not really, just put slide.”

In these two observations, it seems that the trajectory of the cone, taken as an important part of the phenomenon, is no longer included in their observation. Focusing on the revolution of the cone around its axis or considering the path down the plane defines the phenomenon in two different ways. What is observed in each case is different, and that observation arises together with the observer’s position with respect to the phenomenon. Because not everything can be attended to and accounted for, the observer and the observed are contingent on one another. Even

if here, for us, the articulation of two dimensions of the movement of the cone is not highly problematic, there are still other aspects we can (not so easily) take in consideration. For example, if a cone were to be released on a large and long enough plane, would its overall trajectory be linear or not? Are the wooden block and board adequate approximation of a cone and a plane? These questions are directly related to the means of operation of the observer, which include both his/her disposition and the setting in which the observation is realized.

This simple example also illustrates how common agreement in different observers' interpretations does not guarantee objective validity to a claim. The girls can differently position themselves as observers, and thus dis/agree about what they perceive and how to report their perceptions. Moreover, they can transform and overcome their agreements or disagreement by changing their observing posture. This is why looking at the observer-observed relation demands considering aspects such as goals, needs, and affect. The movement of the cone on a plane is not absolutely foreign to us. We experience something very similar with objects like screws or pens, when their points of contact with a plane (e.g. a table) make them similar to cones. However, because they are part of very different activities, we do not observe them in the way students might do in a mathematics classroom. A student observes similar phenomena in different contexts. In each case, we ought not be astonished if the observations have very little in common even if a psychologist claims that the situations are structurally identical. We thus see why the mutually constitutive role of the observer and the observed is in itself secondary to the observer's experience of the world in the very moment of making an observation.

It is clear that Maeve, Sonia, and Jade situate themselves as observers making certain kinds of observations. Doing so, they define an observable domain by specifying what counts as a legitimate observation. Similarly, what allow us researcher-observers to analyze bears family

resemblance with what the three girls experience. Introducing a piece of data (like a transcript) we delineate an approach in which only some aspects come to the fore, and we make observations from which we emerge with an observed object. What is captured on video is a small part of the students' lives that we frame and isolate. Within those limits, we can portray the students' achievement in school or interpersonal relationships as a lens for our analysis, or describe students' gestures in mathematics communication. By all these means, we create an object of observation by situating ourselves as observers making certain types of descriptions. Inasmuch, the text I present here is indeed a description that reifies my researcher's experience with the students and the material, it is a commentary co-produced by my observer's activity and what the world is offering me.

The foregoing highlights the fact that the observer cannot be removed from the phenomenon under observation, and, thus, that our observations are something we create in the practical activity of observing. Observers can look for instances in which they recognize that students draw on primitive knowings (everything that they bring to bear on the task at hand), notice properties, or formalize observations. Seeing them as moving from one dimension of mathematical engagement to another, we can evoke growth in mathematical understanding when, for instance, students create new mathematics questions, or new concepts. In this case, we can use this framework to assert that the three girls draw on an intuitive understanding of what a cone is and what rolling means because they do not define them at first. We recognize the moment in which they make distinctions in rolling trajectories or possible positions for the cone. Similarly, it appears to us that students together articulate these observations as properties relevant in the context ("if you put it on its side it will roll"). In the same way, we can recognize a form of generalization, as the students do not only discuss a particular cone, but also cones

generally (making prediction as to what will happen in the case of a particular object their never experimented with). The association of all these elements within the conversation could also be described as forecasting the articulation of formal theories, being like “theorems in the talk.” For example the three girls appear to develop an understanding of the task of making a prediction under the law of excluded middle: (a) Since the cone must either slide or roll but not both and (b) if to roll means having a linear movement, then (c) the cone does not roll (from [b]), and therefore (d) it slides (from [c] and [a]).

Such a reading is but one possible approach to the students’ activity. It is one in which we do not leave implicit our means of operation, our own observer cognitive abilities, our intentions, and so on. Considering our researcher-observer engagement, it is a perspective in which regardless of what the data might “objectively” present, the researcher looks into how people and material things affect his/her perception and in/form the interpretations of what is happening. It is an approach in which researchers are observers observing themselves and others (students). This is one of the reasons why, in contrast to psychological approaches, sociocultural perspectives would attribute conceptions not to students but to the actions of the observer who identifies them.

### **Students’ Actions Are Made With and For the Other(s)**

Considering the co-emergence of observer and observed calls attention to the problems that come with the attribution of conceptions to the students. To exemplify this, let’s examine Maeve’s utterance in turn 04 and pay close attention to how a researcher develops his/her own understanding from the student’s engagement with the societal-material world. Saying, “if you put it side-ways and then it will roll like that” (turn 04), Maeve gestures with her felt-tip pen

(Figure 4.3). She first places it on its side (G1), then gives it an impulsion and follows its rolling motion on the table (G2) and finally grasps it to describe a circle (G3):

04 Ma: [do you remember that if you put it side-ways]G1 [and then it will rolls]G2 [like that?]G3

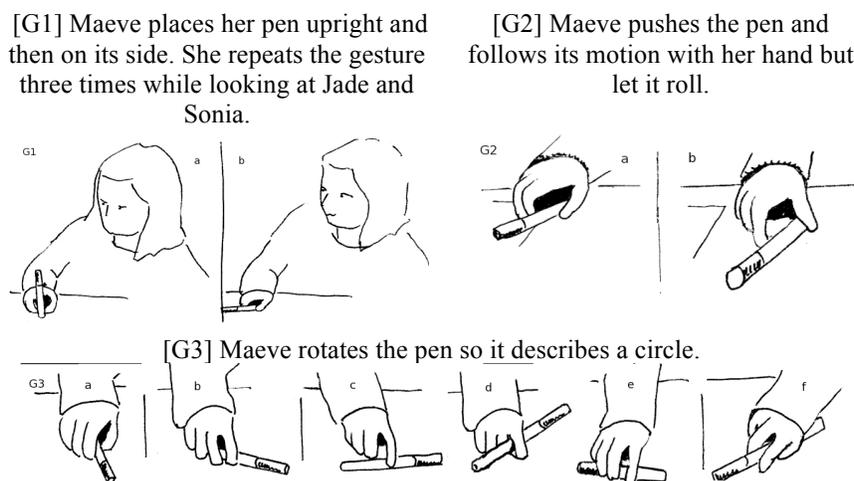


Figure 4.3: Maeve's gestures in turn 04

If we want to identify a conception by discussing what Maeve means with her utterance, we need to associate speech and gestures and assume they represent a specific understanding of the situation. But here, we face a problem. As she pronounces, “it will roll,” Maeve pushes her pen which moves linearly. Maeve then seizes the pen and rotates it to trace a circle, emphasizing “like that.” The change in her gesture and talk contrasts two rolling motions. But does Maeve expect the pen to roll in a circular way, and then observes that its motion is not the anticipated one, or does she realize from the movement itself that what she has previously observed with a cone differs? In terms of how students' conceptions are traditionally analyzed, these two interpretations are profoundly different. On the one hand, Maeve would have a “correct” conception of how a cone rolls, and simply adjusts her performance according to that understanding. In the second case, she would have an “incomplete” conception and, through a cognitive conflict generated by the observation of a phenomenon, went through a conceptual

change by expanding her understanding (to take into account the direction of the rolling object).

As observers, we can only make inferences based on our transactions with the data and make conclusions based on what appears to our eyes. We are here in a situation in which we examine a student's talk and gestures (not a cognitive structure). We see Maeve producing an observation, but we have no means to ascertain what is effectively happening for her. Even if we try to "assume the position of the child and think as the child does" (Steffe & Thompson, 2000, p. 202), there is no way for us to ensure the validity, or even the likeliness of one interpretation over the other. The impossibility to overcome such a simple dilemma (concerning a single utterance of a student in a conversation) captures the limits of an approach in terms of students' conceptions. If we want to explain what a student does or says based on some mental structure, we are forced to impose our own understanding as being that of the student. And if we posit the objective existence of such a structure as belonging to the student, we have to ignore the fact that it is by no means accessible to us.

A sociocultural approach to the work on conceptions foresees the impossibility to objectively decide what a person knows or thinks. The difficulty exemplified in Maeve's utterance also similarly applies to Jade and Sonia's talk: how would we explain the change in what they state about the cone (roll or does not roll) if such affirmations were the sole reflection of their conceptions? As researcher-observers, we also have to admit that what students say or do only partially reveals what they are capable of, and that what is enacted just as much reveals a socio-cultural possibility. Indeed, there is no limit to the number of ways of expressing a concept, and it is the infinite set of possible applications of a given idea that constitutes the concept in the most general sense (Roth & Thom, 2009a). To go beyond these limitations and contradictions, sociocultural perspectives suggest turning our attention to the observable

coordination between individuals and their context, and examine student's performances as contributions made in a *discursive domain*. I exemplify this by returning to the episode to see how Maeve, Sonia, and Jade brought into being their activity with and for each other by situating themselves as observers.

The communicative functions of language stress that it is the listener's response to an utterance that completes it and, thereby, reveals the actual meaning of what was first contributed (Bakhtin, 1986). Maeve's contribution ("Do you remember that if you put it side-ways and then it will roll like that" [turn 04]) turned out to become functionally a statement about the cone only following to Sonia's and Jade's responses. Had Sonia and Jade asked about the particular event Maeve was referring to, the conversation might have taken a different course and not focused on the properties of the cone. In other words, students' contributions are not simply individual productions, but what they are worth is made with the other(s). Moreover, students also contribute to an activity not only because it makes sense to them, but also because they assume their contributions will be intelligible for others. For instance, in turn 06, Maeve repeats her conclusion. This affirmation signifies that she has heard Jade and Julia's contributions and that she interprets them as different from hers. It is to this difference that she draws attention.

Students do not simply make "neutral" and independent responses, because each participant is oriented to responsive understanding from the others. They contribute to the conversation for their interlocutors' and their own benefit and thus do not merely represent their personal conceptions. In turns 08 and 10, Sonia offers a definition of what it means to roll. This consideration marks the departure of the conversation from merely deciding whether the cone slides or rolls, to examine and clarify what these two motions denote. The emergence of that concern, however, cannot be solely attributed to Sonia; we do not know what precisely she

means by rolling. Sonia's utterance is in the first place a response to what was contributed before, including Maeve's speech and gesture in turn 04. Indeed, Sonia's contribution is made in contrast with what Maeve has brought forth. It is thanks to Maeve's suggestion, in its manner and its moment, that Sonia can oppose an alternative view. Therefore, what would appear—from a perspective centered on the individual—as the expression of a conception is in fact developed from turn to turn, at the very heart of the conversation. Each utterance takes an active position in a chain of utterances, made for oneself and with and for the other(s), each one connects to what precedes and what follows. It is in that succession that not only do students reveal their observer's positions, but also that questions and observations acquire their productive, functional meaning. Such a view strongly contrasts with perspectives which consider social and/or cultural dimensions in terms of an individual student engaged in cultural forms of thinking or using cultural objects without attending to students' (and teachers') contributions as always, fundamentally, and irreducibly in relation with one another.

In addition, this collective dimension unfolding from students' contributions is not free from external influences. Together they take place in and from a discursive domain closely related to the ongoing activity. Maeve, Sonia, and Jade produce observations about the cone and by doing so, they define an observable domain by specifying what counts as a legitimate observation. It is not surprising to us to find evidence of mathematical thinking in the girls' activity because they are part of a societal situation in which certain forms of talk are valued over others. In search for compelling arguments, mathematical forms of observations provide the students (and us) with both structure and agential possibilities. Classroom mathematics demands to talk about certain things and in certain ways. Because they correspond to typical

communication situations, concern particular themes and attribute specific meanings to communicative resources (such as words and gestures) in relation to the circumstances in which they are used, these forms of talk guide the students (and us) in observing. Indeed, what happens between Maeve, Jade, and Sonia is not accidental: at the same moment, Jordan's group is facing a similar dilemma and (after discussion) turns to the teacher (Je), who asks Jordan to repeat the question for the whole classroom:

Jo: Do we gotta put it like this or like this? ((placing his pen upright and then on its side))

Je: Okay, that's a good question. You are gonna be working with your group and you are gonna put the object any way you want down the ramp.

Both at the individual and at the collective level, a discursive domain emerges because students contribute with and for others, and because those contributions take place in the same sociomaterial context, with similar resources, similar goals and similar rules. It is as part of that emerging geometrical discursive domain at the classroom level that Maeve then concludes clearly addressing her partners: "Look! It will slide and roll!" This we take as, to her, the most relevant way of examining what happens with the cone in that situation. In another context, she might have maintained her coordination with the others who give rise, with her, to that socio-material world by characterizing the cone as a rolling object, stressing for example what is most remarkable about it (e.g., being the only solid that rolls in a circular way). In this, she would have enacted a different understanding of the question, making a different contribution in a different activity, and perhaps realize a different (nevertheless culturally historically available) practice of observation (i.e., maybe not a mathematical-empirical one). A researcher who would not take into consideration and include in the analysis this decisive aspect coming from the context could easily and misleadingly make inferences about Maeve's "conception" of the cone.

Examining how participants in an activity create meaning indeed reveals that students bring into being a discursive domain for and with each other, a domain that might, or might not, promote their observations as mathematical. Against narrowing an evaluation of student's conceptual development to what is observed in a specific situation, I show here that a sociocultural approach focuses on the conditions in which certain aspects of a cone are discussed when the students produce classroom events.

### **Research and the Ethical Ground**

We see in the previous sections that psychological and socioculturally inspired approaches differ in the path of objectivity they take. This distinction naturally unfolds in an ethical reflection on the legitimacy of the other and of what he or she brings forth in making observations (Maturana, 1988). Objectivity is often associated with the absence of value and responsibility for what is said. By simply stating how things are, what students think or know, we do not appear to take a position, unlike when we say that something is right or wrong. The change of discourse over the years in the psychological approach on conceptions captures this very well. Researchers nowadays generally refrain from talking about “misconceptions” or “naïve conceptions” and rather use adjectives like “non-mathematical,” “everyday,” or “incomplete.” Such an attitude is typical for the path toward transcendental objectivity. Through reason, observers claim access to an objective reality and the validity of the argumentation that are independent of the researcher. However, presupposing an ultimate source of validation also leads one to define a single reality in which only one interpretation is acceptable. In this view, claims about knowledge are demands for obedience. Because the observers do not take responsibility for their explanations, others are then implicitly or explicitly forced to accept what

is said to be true and are not legitimate in their own understanding. I clearly show, however, that observers have the possibility to develop different explanations. For example, a researcher examining a video excerpt will account for different things by examining isolated utterances of a student, or considering how each utterance is a response to what was said before. Observers are thus responsible for the explanations they give, which is why researchers in a sociocultural perspective call to the examination of the implicit elements and the limitations of the theories they, and others, adopt. Blaming difficulties to learn on cognitive immaturity or underdevelopment, like a psychological approach conveys, blind the examination of the social, economic, and cultural dimensions of knowledge, learning, teaching, and what it means to succeed or fail. This has ethical implications because in and through language, explanations or discourses, individuals position themselves and others. Saying that students hold conceptions, their perspectives are not recognized as legitimate explanations because the researcher-observers keep the focus on their understanding. Thus, they (rightfully) note that it would not be acceptable to them to say or do things that way and see students' performances as "non-mathematical" in contrast to what they personally delineate as mathematical, and "incomplete" in comparison to their personal understanding.

Although not all explanations are equivalent, they are all equally *legitimate* (Maturana, 1988). To take responsibility for their explanations, observers are not to decide which explanation is right, but to understand how *desirable* each one might be with respect to the goals the observers set themselves. We can take on that responsibility if, instead of attributing conceptions to the students, we question how as researcher-observers we recognize forms of mathematical thinking. Making such observations does not lead to conjectures about what is going on in the student's mind, does not posit the existence of some static mental structure and

does not require the acceptance of a single, transcendental reality by means of which our observations can be validated. It opens room to legitimate students' performances and discuss how they do, or do not, bring forth a mathematical discursive domain by positioning themselves as mathematical observers. This applies not only to the researchers, but also to the students as observers. From a sociocultural point of view, objectivity is replaced by something dynamic, discursively constituted in and by experiences: praxis. The concept of praxis entails the adoption of an ethical attitude toward the other because it recognizes the value and the validity of what students do or say, but also make possible to discuss why it is contextually desirable or not. Such an attitude opens up a "responsible reflection of coexistence" (p. 32) because it avoids reducing students' understandings to another observer's interpretation (like the researcher's) and thus accepts students as legitimate others by valuing their contributions for what they are. Discussing how and why different understandings are equally legitimate, but not necessarily equally desirable, allows us to situate knowing and thinking not in a student's cognitive structures, but in the action itself, which includes both the individual and his/her context.

Traditional perspectives focusing on cognitive or psychological aspects of learning are unable to value the uniqueness of each student because they treat difference as a derivative of sameness. Talking about students' conceptions is trying to identify something that would be essentially the same about them. It is to create an object of observation that reduces what is brought forth in conversation to singular, well-defined ideas that represent universal features of children's development. The positions such researchers adopt make differences indifferent to difference instead of valuing the heterogeneity of personal experiences. At best addressing students' discrepancies, these perspectives look for standard procedures to approach students' understandings and "fix" them. Against an instrumental orientation of mathematics education, a

sociocultural approach places the students' uniqueness at the center of the ethical relation to the other. Aligned with the acceptance of the other and his or her perspective characteristic to constituted objectivity, researchers adopting sociocultural frameworks count that individuals are always more than what they offer in a single moment, or in a collage of isolated utterances. This "surplus of humanness" comes with any encounter with others, and asks us to "take the performed act not as a fact contemplated from outside or thought of theoretically, but to take it from within . . . in all its concrete historicity and individuality" (Bakhtin, 1993, p. 28). Rather than seeing students' contributions exposing context-free conceptions belonging to the students, these contributions are, for example, to be examined as once-occurring situated attempts to maintain coherence with the environment. Including what is observed, but also including the others with and for whom observations are made, students' contributions are not widows of the mind, but a moment in a process of becoming.

To undertake our ethical responsibility for the other as non-indifference to differences, a fully developed sociocultural perspective is opposed to the assessment of what students, as individuals, knows or can do as incomplete, naive or inappropriate "conceptions." To support students' learning, a sociocultural perspective examines how they contribute to a situation the way they do, and what it is that they create in doing so. Accordingly, researchers will define themselves as a certain type of observer whose intentions are not, in the end, to get the students to a correct or a complete conception (or any predetermined understanding of a situation). Non-indifference to differences exists when we support students to enter in a certain kind of discursive domain, or in other words, by helping them position themselves as observers in a mathematical way. Inasmuch, difference is theorized in and for itself: It is because they are different that students ought to enter in a shared discursive domain, and they can do so precisely

because they are different and, thus, have something unique to contribute. Taking in account the surplus of humanness inherent to the encounter with another, a sociocultural approach sees difference as prerequisite for and constitutive of dialogical engagement and participation with others (i.e., with other's differences). In addition, examining what students say or do informs us about how they position themselves as observers. Consistent with an ethical orientation to the other, what is assessed from this are conditions with which students coordinate themselves, not their ability to do so. If Maeve, Sonia, and Jade discuss the cone and its properties in a mathematical way, it is not merely because they have the appropriate cognitive structures imprinted in their mind. It is most importantly (and observably) because of the societal-material conditions they are in and that they change with their actions.

Such a perspective also allows us to take an ethical stand as to what is going on in a classroom without positioning students and their performances negatively. On the contrary, we value students' differences, give them attention and draw on them to revisit our own understanding of what doing mathematics is about. Because mathematics is something we societally (institutionally, culturally, and historically) define, it is justified, for researcher-observers, to discuss whether the situations in which students find themselves lead them to create what corresponds to our vision of a mathematical activity. Moreover, we have a special responsibility, as mathematics educators, to produce and reproduce, to define and redefine, what is a mathematical activity as a societal phenomenon. We thus assume our ethical accountability by examining not only the conditions, but also the explanations (the discursive domains) in and by means of which students are positioned and position others and what is brought forth in making observations. In contrast, such an ethical ground cannot be found to support making judgment on what students know or do not know.

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The path of constituted objectivity is in essence welcoming a variety of world-views, and thus recognizes the path of transcendental objectivity as a legitimate one, because even though one pretends to make observations in transcendental objectivity, the human praxis in which these observations are made is still a path of constituted objectivity. Sociocultural approaches are able to welcome conceptions and conceptual changes perspectives as one possible way to examine students talk and actions, but do not see them as compatible or complementary with their own effort. They are possible alternative views that reveal different assumptions (the existence of cognitive structure, conceptions), undertakings (identifications of what students knows or can do), and focal points (individuals). But, according to the goals we set ourselves as researchers or educators, conceptions and conceptual changes frameworks are not the most desirable way to examine students' talk and actions. The reasons for this lie in the problematic attribution of conceptions to the students in a way that neglects the contingency of the observer and the observed, and in the ethical implication that comes with this contingency. We recognize here a comprehensive, yet oriented ethical approach in which the observer takes on responsibility to discuss how and why different understanding, although equally legitimate, are not equally desirable. This is especially important today because our ability to deal with plurality and diversity guides social behavior. Western culture has long been characterized by separation and universalism, dividing the world from the person. Such division and dislocation leads to the systematic negation of the existence of the other by applying the same cultural logic to all people, as if everybody and all contexts were essentially the same. Although this principle has led to some positive outcomes, it is generally for the benefit of those who are already culturally well positioned, and to the detriment of the disadvantaged, like indigenous or working class

peoples, women, immigrants, and so on. Similarly, an approach that reduces language use to the individual is rooted in an ontology that underlies “all inequities, including those along the lines of gender, culture, socioeconomic status, class, and age” (Roth, 2007, p. 742). By challenging the assumptions made in research on conceptions and conceptual changes in mathematics education, I offer here a practical answer to the urging of those who ask us to address this situation.

## CHAPTER 5

### Teacher-Student's Knowing-With: The Relational Dimension of Mathematical Communication

#### Preface

Writing the final sections of my chapter on the observer, the question of *ethics* became increasingly important to me. I then decided to spend some time reading and reflecting about ethics, and as I usually do: pushing my theoretical understanding against what I can observe in teachers and students day-to-day, moment-to-moment. Reviewing the literature, I was quite surprised to find that many, in mathematics education, incidentally express concerns about the ethical dimension of what we do, not only when we research, but also when we teach and learn. For example, some point to issues regarding educators trying to transform teachers' beliefs or practices (Goos, 2008; Grootenboer, 2006), of adopting an instrumental orientation to mathematical knowledge in the classroom and its effect of teacher's ability to be in ethical relations with their students (Taylor, 1996; Neyland, 2005; Ernst, 2007), or in respect with social justice and mathematics education commitment to humankind for the wellbeing of all of us (Davis, R., Maher & Noddings, 1990; d'Ambrosio, 2007; Atweh, 2007). But I also noticed that despite this interest, researchers seldom attended to this ethical dimension in the realm of *concrete acts* in teaching, learning and researching.

From this angle, I read philosophers articulating what is called an "ethics of alterity" in which self and other are understood as fundamentally dependent on one another, and in a relation

that is the very ground for any form of knowing to take place. In the mean time, I naturally went back to our data and began to articulate a reflection and an analysis around the way teachers and students engage one another in the classroom to bring about a mathematical way to attend to objects or situations. In December 2008, I started to write a first version of this study. A year later, I had produce countless versions of the text reflecting my readings, analysis, and also the very process of writing and focusing my work. From a broad interest in the ethics of concrete acts in and through which teachers and students produce mathematics education, I highlighted the need to consider how knowing always already is *knowing-with*, and therefore is an ethical relation. I call this the *relational* dimension of teacher-student communication. Finally, reaching out for the other fundamental theme of this dissertation, I conclude the study with a reflection on the knowing-with of teacher-student communication in research, and articulate what I take as my own ethical responsibility for the reader.

The following map illustrates how some of the elements I encountered came about on the path traced by this chapter, of which a different version is currently under review at *Educational Studies in Mathematics*.

**Chapter 5**

## Knowing-With: The Relational Dimension of Teacher-Student Communication

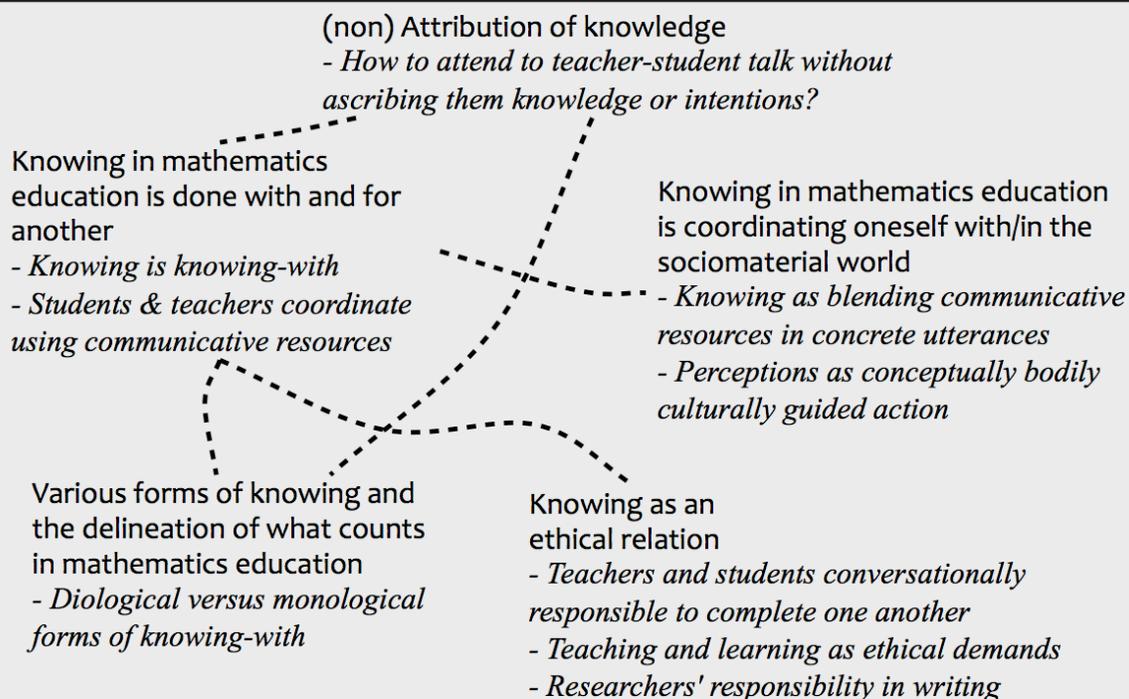


Figure 5.1: A map of my walk in “teacher-student’s knowing-with”

## **Teacher-Student's Knowing-With: The Relational Dimension of Mathematical Communication**

### **Mathematics and Communication**

It is generally agreed that communication is of fundamental importance for human thinking, and, accordingly, to mathematics teaching and learning. Considering the active and contextual creation and use of signs from which mathematical meaning emerges (Van den Heuvel-Panhuizen, 2001), researchers nowadays explore avenues in which language is not separate from its use (Ongstad, 2006). Hence, attending to communicative acts by which teachers and students bring forth, embody or enact mathematical ideas or way of doing (Radford, Edwards & Arzarello, 2009) enables us to conceptualizing the open, dynamical nature of mathematical cognition.

There is, however, another aspect of knowing and communicating mathematically that awaits development in our field: that mathematical activity is always and fundamentally produced and understood in relation to *others*. The collective nature of teaching and learning is now commonly recognized in research, for example in terms of classroom mathematical practices and negotiation of knowledge (e.g. Cobb, 1999). But those perspectives have a “taken-as-shared basis for mathematical communication” (p. 13) and cognition that isolates the individual within the collective and leads to reduce teachers or students’ utterances to isolable “elementary building blocks” (Bjuland, Cestari & Borgersen, 2008, p. 280). In that view, knowing and communicating mathematically seems to flow transparently from the individual into the collective, where it is simply made available to others. Moment-to-moment analysis of

mathematical classroom communication shows, however, that knowing is brought about conversationally (from *con-* ‘with’ + frequentative of *versare* ‘to turn’, forming *conversari*, ‘to keep company with’), i.e. produce with and for another. This *relational* nature of knowing (in general and mathematically in particular) is well observed in the dialogical dimension of communicative acts, where everything we say is always (a) a response to what has been said before and (b) an anticipation of what will be said next (Bakhtin, 1986). Hence mathematical cognition cannot be reduced to the individual. By this I mean that mathematical knowing, especially when thought in terms of communication, has to be attended to transactionally, in the actual back and forth between utterers. Going further that seeing the utterance as the fact of, say, an individual student producing a “personal iconic orchestration” of previous utterances (Radford, 2009b), the relational dimension I suggest attending to demands to consider such communicative action *always* in the light of the “polyphony” of voices whereto they contribute. More than a “merging of voices” which, with their own tone, would resonate like strings forming a chord, we are to hear in this the dynamic attunement and responses of musical improvisation (Neyland, 2004a) in which knowing mathematically is realized in socially/culturally recognizable forms. In this view, knowing mathematically necessarily involves the presence and actions of others, and knowing through communicative acts is then really better conceptualized as *knowing-with* others. To put it another way, the Other is the condition for knowing and languaging, and this is fundamental to understand how students and teacher get on with one another in learning situations (Roth, 2007). Moreover, de-emphasizing the relational nature of knowing, the omission of the Other in mathematical activity makes it difficult to address one of the most important aspect of the actualized, dialogical relation with the other: its *ethical* dimension. Such ethics is fundamental to mathematics education because it makes thematic our

very nature as social creatures (Ernest, 2009), and can disrupt undesirable orientations and practices through the experience of our being with others (Neyland, 2001).

My purpose in this chapter is to articulate (a) an irreducible way of knowing that is always knowing-with and therefore (b) constitutes an ethical relation. I do this with examples from a moment-to-moment analysis of classroom fragments in which elementary school teachers and students articulate themselves in the “language” of geometry. I first show the irreducibility of knowing in and as teacher-student relation. I then exemplify how mathematics educational research can consider teacher-student communication in how geometrical knowing is always already knowing-with another. Finally, highlighting the ethical dimension of this relation and how two teachers differently receive a student’s geometrical contributions, I discuss the ethics of teacher-student communication in and for mathematics teaching and learning.

### **Knowing Always Already is Knowing-With**

Although educational research now generally recognizes the “social” or “collective” nature of mathematical activity, this aspect is rarely considered from the perspective of its the dynamical nature. Reciprocally, when we focus on the process of signification in and through which knowing mathematically takes place, we tend to leave in the background the dialogical nature of this activity. In this first section, I want to articulate the irreducibility of knowing in teacher-student communication that arises from the dialogical relation in and through which knowing is concretely expressed.

Little do we know about the history of actual practices of teaching and leaning mathematics. One of our oldest records of such practice is found in Plato’s *Meno* in which Socrates – already contrasting with some dominant practice – is presented teaching geometry to a

slave boy. With the episode, Socrates (who compares himself to a midwife) wants to demonstrate how knowing is “recalling” by having the slave first realize his own need to know, and then find out what he wants to know from within his own thinking. To do so, the philosopher presents the problem of duplicating the area of a square. Restlessly questioning the boy with simple questions, he guides him through realizing that doubling the side of the square (as the slave first suggested) would not give him an acceptable answer, and then observes that the diagonal of the original square provides a correct solution:

SOCRATES: And how many spaces are there in this section? BOY: Four.  
 SOCRATES: And how many in this? BOY: Two.  
 SOCRATES: And four is how many times two? BOY: Twice.  
 SOCRATES: And this space is of how many feet? BOY: Of eight feet.  
 SOCRATES: And from what line do you get this figure? BOY: From this.  
 SOCRATES: That is, from the line which extends from corner to corner of the figure of four feet? BOY: Yes.  
 SOCRATES: And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal? BOY: Certainly, Socrates.  
 SOCRATES: What do you say of him, Meno? Were not all these answers given out of his own head?  
 MENO: Yes, they were all his own.  
 SOCRATES: And yet, as we were just now saying, he did not know?  
 MENO: True.

Socrates defend the idea that knowledge is something held by the individual who simply inform others of what he/she thinks, and reduce his own part in the knowing that comes from his talk (and gestures: the full transcript is indeed quite difficult to read without the help of a drawing). He neglects, if not rejects, that his own utterances not only accompany the boy’s thinking, but are integral to it.

The practice of teaching as we inherited it, and that of researching teacher-student communication, in many ways maintains this kind of assumption, whether the knowledge is to be

situated in the teacher (who transmits) or in the student (who constructs). The (historico-cultural) difficulty to break with the concept of a knowledgeable self is deep-rooted not only in the modern cult of individualism but in the whole history of philosophical thought. The birth of formal mathematics (and, one can think, of a kind of mathematics teaching severed from technical use) coincides in Greek society with the invention of the “man of reason”, the critical thinker capable of judging for and by himself, and legitimate to democratically take part in the governance of his *polis* (Vernant, 1962). It is not surprising then to observe even in “socially” inspired theories of mathematics knowing, teaching and/or learning, the individual and his/her thinking as central preoccupation. The collective (as an entity) tend to be presented as a collection of individuals who at the best orient one another by “negotiating” knowledge/meaning just as in an agora where everyone speaks for (and by) him/herself... while the result of social action (existing practices) affects the individuals in return.

In order to better articulate this, and why and how a different approach to teacher-student communication is desirable, hence taking one step further our current conceptualization of mathematical activity (in/as communication), I will introduce a first segment of a classroom conversation. Providing a good entry point to discuss the difficulty of rendering teachers or students as holding knowledge about objects, and communicating (sharing) that knowledge, the following fragment features Jennifer (the lead researcher) and a class of second-grade students in the process of reviewing some of the geometrical ideas they explored the previous week. Jennifer (Je), after discussing how solids can be described according to their faces, continues here by raising the question of edges:

Je: What about edges, what about when we were talking about edges? Can we

describe edges as being different? Are there some edges that are different from other edges? Hands up if you think some edges are different from other edges.

Moves her hand up and down while pointing to a poster about solids, then picks up a cube and slides her finger back and forth along an edge (Fig. 5.2). Finally, raises her right hand.

Sts: (4 or 5 students raise their hands)

Je: Ok. How many people think edges are all the same?

St: (7-8 students raise their hand, including Tobin)

Je: Okay. Who had their hand up for some edges are different from other edges?

St: (A few students raise their hand, including Tobin)

Je: Tobin

To: (3 sec.) I don't know

In this brief fragment, a discussion comes about in which we recognize the geometrical concept of edges. Jennifer's communicative action contributes to this by *pronouncing* the word “edges” ten times, *pointing* to the word “edge” and to an illustration on a poster, *making* a linear (up and down) movement with her hand (providing a metaphorical representation of an edge) and also *using* a block and *touching* it by sliding her finger along one of its edges (Fig. 5.2). The students show that they follow her performance by raising their hand (or not). That is, the concept brings to bear in and through a process by which communication (language) is the substance whereof knowing mathematically materializes and, reciprocally, whereto the mathematical idea gives shape. We follow Jennifer's and the students' use of communicative resources to produce dynamical and imminent mathematical knowing in what they concretely express (Roth & Thom, 2009a). For example, using words and pointing to objects, Jennifer's first utterance forms a semiotic node (Radford, 2009a) in which those resources complete one another to bring forth the geometrical idea that none of them contains in itself. Communicating about edges, Jennifer engages with the students in a process of creating distinctions, bringing forth geometrical elements and their relations. When she speaks and gestures, the geometrical idea is enacted and embodied in and from the experiences that bring it about.

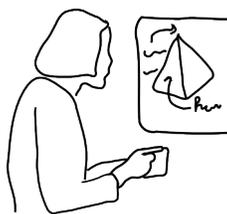


Figure 5.2: Jennifer slides her finger back and forth along an edge of the block. Next to her, there is a poster on the wall

On the one hand, from this perspective, we can hardly see teachers and students' utterances as re-presentations of pre-given ideas, for they result from a creative process, one that occurs in and through lived geometrical activity. In the first turn of the conversation, Jennifer voices a series of questions about edges: (a) what do the classroom members think? (b) what do the students' remember? (c) is it possible to describe edges as being different? (d) are some edges different from others? Only in the very end does she ask the students to "put [their] hands up" if they agree with the latter. In and through its realization the communicative action (talking, gesturing, using objects) changes thinking. The utterance grows as Jennifer defines for herself what she is talking about, and how she is doing so. That is, the utterance is dialoguing with itself, juxtaposing, over-inscribing and sequencing communicative resources (Ernest, 2006) in the formulation of several questions. Moreover, the students' reactions also naturally affect the course of the conversation which moves back and forth between edges being all the same, or different. Far from simply "carrying across" some abstract knowledge, Jennifer (like any speaker) can hear herself talking, becoming aware of the distinctions that she makes, of their affect on others, and adjust her own speech. Inasmuch, knowing is not merely a knowledge in passing, not the combination in speech of prepared elements, but an actual *deed* (Brandist, 2002) consisting in the *linking* and *blending* of communicative resources that come about in the lived

material of the utterance (Merleau-Ponty, 1962).

Another dimension to this process of knowing geometrically makes it necessary to step back from focusing on the cognizing individual to consider the collective dimension of his/her production of utterances. Jennifer articulates an idea. But clearly, her utterances are not merely for herself but are for the students. In her first utterance, Jennifer formulates many questions and articulates various communicative resources (words, symbols, gestures and objects) to bringing about the idea *for* the students. Moreover, the effect of her utterance comes to be known only in the students' subsequent actions (a few of them raising their hands). Had they laughed or complained, the functional sense of the utterance would have been very different because it would have lead to a different outcome. And similarly, Jennifer's response to the students raising their hand attaches meaning the gestures, producing them as contributions to a collective investigation of the question of edges.

From this angle, knowing comes to bear in the form of an irreducible relation in teacher-student communication. There is a back and forth between Jennifer and the students in and through which utterances concretely produce the mathematical idea. Jennifer and the students are in conversation, substantiating for one another the intelligibility of the emerging topic, their mutual orientation to one another, and to the topic itself. The comprehension of the question of edges is a geometrical endeavor in itself (especially, like in this case, with young children having their first experiences in geometry), a concretization of an instant in the continuous development of mathematical understanding. In her articulation, Jennifer makes available publicly, accessible to and by any one present, a cultural mathematical way of knowing objects, and of communicating geometrical ideas. But, because her talk is designed for the students, to whom what she says already has to be intelligible, this mathematical understanding is inherently in

common. Whatever she articulates, to make any sense of a lesson in mathematics, has to be viewed as Jennifer-knowing-for-and-*knowing-with*-the-students. The irreducibility of knowing in teacher-student communication arises from very necessity for *any* knowledge that is brought out in the open and, publicly performed to be intelligible by the other. This other is constituted by the students when the teacher speaks, and is the teacher and other students when a student speaks.

In the lived life of mathematical activity, the other is ever- and omnipresent. Knowing is collective because it takes place in, and realizes, a social relation and a social situation. Jennifer and the students are here producing mathematical talk, a classroom conversation. Jennifer speaks for the students, but also *with* them: responding and being responded to. The success of bringing about the question of edges in the classroom is entirely contingent on the active role of everybody else: If the students and the teacher manage to bring forth some geometrical knowing around edges, it is as a result of their dialoguing activity. It is fundamental to realize that whereas talk is with and for the other, this is possible *because* using language is also using the language of the other. For example, the word “edge” Jennifer pronounces has various significations, some of which are familiar to the students. In that sense, Jennifer returns to them something they know, something that belongs to them as much as it belongs to her. But there is also a geometrical reference, a special (culturally and historically defined) network of ideas, images, objects and relations with which the students are not acquainted, as it is precisely one of the learning purposes. The geometrical reference itself then emerges from with/in this language in common, through the use of other communicative resources that exist as such only because they belong to both. This is also why any utterance in and through which knowing mathematically takes place has to be thought as belonging to both the speaker and listener. That is, even before

his/her reaction to the talk, the other is already and irreducibly part of the communicative mathematical activity.

Neglecting either part of this irreducibility leads to very problematic situations one can easily see with this first fragment. On the one hand, forgetting the constitutive role of the other gives the impression that we can understand mathematical activity in terms of discrete, individual communicative actions just as the Greek democrat or Socrates' student are considered to part take (take away, hold on to something that belongs to himself only) in the situation as free, independent individuals. On the other, losing sight from the dynamical nature of knowing will suggest that we can take individual contributions to the collective activity as representations of something already present in their head and transparently communicated. In the very tradition of Western thought, we find here both the models of democracy (where rational voters clearly express well-formed opinion), and Socrates' view of an individual inherently possessing knowledge.

In both cases, attending to what happens with Tobin in the above excerpt can only be controversial. In the episode, Jennifer asks the students who believe that "edges are all the same" to raise their hands. In turn 04, Tobin's reaction, along with other students, is to do so. Jennifer then asks "Who had their hand up for some edges are different from other edges?" and Tobin also raises his hand. When Jennifer pronounces his name, Tobin responds by saying "I don't know." Research focusing on the individual or taking knowledge as a representation avoid presenting such excerpts because it is impossible here to ascribe a knowledge (or its absence) to the student, and difficult to anticipate what is going to happen next. At first, it seems that Jennifer formulates questions around edges, and that Tobin affirms his conviction that edges are all the same. But then Tobin appears to make the opposite statement before saying "I don't

know” (in response not to a question but to hearing his name!). What is happening? Is Tobin changing his mind? Was he really making statements by rising his hand? What is it that he doesn’t know?

Traditionally, a response to this type of situation is to make assumptions on what might be happening in the student’s head. To go about the episode, one draw on subsequent utterances, on the context, on their theoretical knowledge about students’ mathematical understanding or their personal experiences with them, to *presume* what the students might be thinking. Despite explicit warning against taking student’s contributions as “windows” of their mind (Lerman, 2001), many researchers still fall prey to the temptation to get into the students’ minds when in fact everything to know is right there in the open – and only there. One reason for this is perhaps a difficulty in abandoning the belief in so-called mental structures and functions,<sup>13</sup> and in the attribution of knowledge to individuals. But doing this forces us to assume that our own understanding is that of the students, and reduces the students’ and their understandings to the semantic content of their talk.

A different approach to Tobin’s actions is to keep the dialogical nature of knowing in focus. Communication by nature relies on the presence and active understanding of others whom speakers address and for whose benefit they speak. For that reason, we cannot attribute the utterance to the speaker alone but need to take into account the listener as well (Bakhtin, 1986). Knowing emerges not only for the self, but also with and for others. Therefore, we need to attend to how Jennifer and Tobin *together* produce utterances that exhibit geometrical knowing. From

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<sup>13</sup> Another incarnation of the Greek paradigm (the Socratic idea of individuals holding knowledge they need to remember) that keeps shaping the way we think about knowing and, thereby, look at teacher-student communication by trying to ascribe them knowledge.

that angle, it is not relevant to ask what Tobin knows or what he might want to say or not to say. Rather, we examine how Jennifer and Tobin mutually place themselves and are placed as the respective other and together produce a discussion in which geometrical knowings may eventually come about. One might be puzzled by Tobin's "conceptual path," but it is what he and Jennifer utter for and with one another that is available to us, which only renders the problematic aspect of talking about edges when edges are what students are to learn about. But proceeding "upon the ground that knowing is co-operative and as such is integral with communication" (Dewey & Bentley, 1949, p. 97), this is exactly why such episode is important and meaningful. Adopting a perspective on knowing radically grounded in the observation that knowing is always *knowing-with* another, the fragment opens up the question of *how* teaching and learning geometry take place in the double interlacement, whereby any speaker evaluates the preceding utterance (exhibits its effect) and sets up the succeeding utterance. Knowing geometrically in and as an irreducible relation between teacher and students will articulate the teacher-student communication as simultaneously a condition for and a result of a dialogical (geometrical) activity. A perspective very different from the accepted interpretation of Socratic dialogue, that of negotiation of knowledge, and other approaches attention to the individual's development or achievement.

With this in mind, I may now return to the classroom episode and observe how Tobin and Jennifer manage to develop a geometrical dialogue:

Je: (3 sec.) Tobin how do you describe this edge?  
 Takes the block she was previously holding and slides her finger (like in Fig. 5.2)  
 To: Well I [?]  
 Je: [I] ask you to run your finger along it. How would you describe it?  
 Moves toward Tobin, stretching her arm with the cube in his direction (Fig. 5.3)  
 To: Well I... because hum because this one have vertex

Grabs the block and looks at it, touches it, but does not slide his finger along the edge

Je: Yes

To: Yes

At this point in the lesson, Tobin and Jennifer take turns in the conversation and from their dialogue, a mathematical discursive domain emerges. Certainly both are speaking from their own angle, saying what they think or what they see, but also, and most importantly, Tobin and Jennifer are responding to one another. After Tobin says that he doesn't know, Jennifer reaches for the block she previously used, gestures to one of its edges, and asks: "Tobin how do you describe this edge?" Here, she *responds* to Tobin's affirmation of not knowing by offering him a new entry point to the question of edges. Similarly, Tobin also articulates his response in relation to what Jennifer says when he hears her utterance as a question and begins to formulate a response ("Well I"). The dialoguing keeps taking place as Jennifer moves toward Tobin with the block and a new request ("I ask you to run your finger along it. How would you describe it?"). A cognitive domain emerges from this dialogue where communicative actions, such as using certain words or pointing to specific aspect of an object result in coordination of actions *about actions* that concern this domain itself (Maturana & Varela, 1987). Tobin coordinates with Jennifer by remaining silent so she can speak, and by stretching in her direction to receive the block even though nothing was said about that particular action (Fig. 5.3). One another's reactions become mean and object of coordination in and as they produce geometrical knowings. That is, despite Tobin's affirmation that he does not know, the teacher and the student manage to bring about a dialogue in which they talk about edges from a mathematical perspective precisely by responding another in a geometrical way.

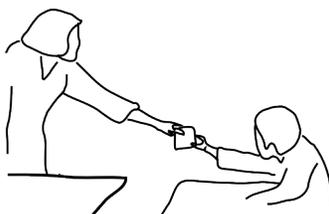


Figure 5.3: Tobin and Jennifer coordinating with one another as she hands him the block

Formally, I call this student-teacher dynamical and reciprocal process of signification the *relational* aspect of teachers and students geometrical activity. Dialogically attended to, Jennifer and Tobin's conversation reveals a braiding of persons, communication, and forms of knowing. Communicating is inherently relational (Bakhtin, 1986), and is a condition for the emergence of consciousness in the relation of Self and Other (e.g. Roth, 2007; 2006). In fact, etymologically the sense of the word 'consciousness' signifies relationality, from the Latin *con*, 'with' + *sciere*, 'to know'. To be conscious is to know-with another, and this knowing-with takes the form of coordinated actions (with others and the non-human world), and not that of isolated manifestation of individual thoughts by communicative resources independent/indifferent to the other. Communicative resources themselves presuppose intelligibility, and not only as culturally and historically shaped artifact, but in the face-to-face history of coordination, of conversation, Jennifer and Tobin had with one another, and with others before that. Without this communality, they would not be resources for communication; and this intelligibility comes from that our very capability to think is due to our common existence with others. Jennifer and Tobin actively contribute to a way of being mathematical/geometrical with one another by the recursion of their consensual coordination in and through the use of communicative resources. They relationally engage in knowing mathematically by the responding to one another's utterances, thereby gives life to geometry as a kind of conversation (a discursive domain) which, in return, also orient them to one another and to the emerging ideas and make their actions geometrically meaningful).

Moreover, this coordination is not only analytically available to us, but to the teacher and the student themselves, being an essential part of their knowing-with: Jennifer's and Tobin's "yes" (turns 13, 14) explicitly mark an agreement that reifies their mindful engagement in a process of mutual coordination. Emphasizing the togetherness of knowing, the relational aspect of teacher-student communication hence conceptualizes the communicative process of (mathematical/geometrical) signification itself as relational, and that knowing is always already knowing-with.

In sum, knowing-with stems from the reciprocal nature of communication, whereby each speaker has a double function: (a) evaluating/responding to the previous speaker, thereby making known the effect of the utterance, and (b) setting up the first part of the next utterance. Jennifer mentions "edges" and Tobin responds by pointing to the presence of "vertices," something often observed when young children begin to make distinctions between curved and rectilinear boundaries (Laurendeau & Pinard, 1970). In this, we see Tobin and Jennifer arrive together at geometrically attending to a plastic cube in a geometrical way *because* they respond to one another by addressing structural aspect rather than, for instance, color or material. That is, we describe how geometrical thinking arises as a domain of observation precisely because it is a way of knowing-with. In the next section, I further examine the consequences of acknowledging this relational nature of teacher-student communication by showing how it both requires and enables us to address one of its most important aspect: the *ethical* dimension.

### **From Knowing-With to Ethics**

A relational perspective on geometrical activity looks at how students and teachers

coordinate in knowing-with one another in and through the dynamical and reciprocal production of concrete communicative geometrical acts. But thinking in terms of dialogue does not mean that there is always agreement and that geometrical knowing emerge as soon as teachers and students talk geometry. Theoretically, every moment in communication is a confrontation of a diversity of voices (Bakhtin, 1986): Any utterance answers both the requirements of an individualized, embodied speech act, and the requirements of other voices because the utterance is an active participation in such speech diversity. In the Greek model that in many ways persists nowadays, this diversity is dealt with in terms of sameness: at the best, each voice has equal value in the negotiation of meaning, and it belongs to the teacher to find arguments (as in the form of “didactical situations” for example) to win students to his/her view. Or it might be the voice of history speaking through the teacher that justifies the introduction (and the “election”) of one voice among others. But from the perspective of knowing-with, there is, in that double requirement, a form of responsibility of a very different texture toward these voices and the other who voices them. To put it another way: because of its relational nature, knowing and communicating are never mere production or transmission of information, but a sharing of Being that precedes and constitutes the background for any utterance (Roth, 2007).

I now return to the episode where we left off (a) to discuss and illustrate the double ethical relation that comes with the effect one and the other’s voices in knowing-with, (b) to explain how teachers and students contribute to bring about *particular* forms of knowing-with, and (c) to articulate the ethics of teacher-student communication that accompanies knowing-with. In the following fragment, Tara (Ta), the regular classroom teacher, joins the conversation by addressing Tobin:

Ta: But here, instead of, if you start at one vertex, put your finger on that vertex. Your finger, point. One vertex. Now run it along that edge to the other vertex. What's that feel like what kind of edge is that?

Takes the cube from Tobin and points to one vertex, holding to cube in front of him. Then grabs Tobin's finger and places its tips on one of the vertices. Finally, runs Tobin's finger back and forth on an edge.

To: (*Runs his finger along the edge, then scratches his head in a 'confused' way*)

Ta: Ok, here, this time what does that one feels like?

Tara reaches out for another block. Jennifer gives her a cylinder. Tara again grabs Tobin's finger and slides it on it edge.

To: It feels like hum (3 sec.)

Tara keeps sliding Tobin's finger for a moment and then changes position, holding the two blocks side by side (Fig. 5.4).

Je: Does it feel the same or does it feel different?

To: Different

Je: What is different about those two edges?

To: Cause this one is round, this one is like a square.

Ta: If you just take one of them. What would it feel like? Not using a shape word, so you're not gonna use a triangle, a circle or square.

(2 sec.) Can someone help us out? He says this one is round, what will this one be? Alison

Al: Straight.

Ta: Do we agree? This one is straight, this edge is round. (2 sec.) So which one is the straight edge?

After her pause, she turns back to Tobin and presents him the two blocks as in Fig. 5.4.

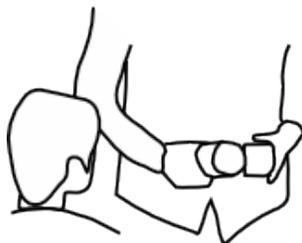


Figure 5.4: Tara holding the cylinder and the cube for Tobin, and the classroom

Tobin manipulates the block, looking at and touching its edges and faces, and somehow translates his experiences in communicative acts. Perception requires both the work of the body and a particular form of consciousness for the co-emergence of the perceiver and the perceived. This is because perception is not merely “passive” or receptive disposition to an objective reality, but an active contributing to the enactment of the surrounding world (Merleau-Ponty, 1962). When Tobin manipulates blocks and explains differences, he bodily and conceptually disposes

himself (and others) so that the light that bends around the edges of the object and projects onto his retina, or the haptic information that comes with touching the block, make sense. In perception, the proprieties of the object result from a conceptually guided participation of the body in the physical world. Our body has an active role in structuring our perceptions, and the meanings of our perceptions need to be interpreted in the context of our bodily situation. Hence, and clearly in this piece of conversation, distinctions between edges are not straightforward, but can be established based on length, curviness, sharpness, and so on. This also signifies that perception is something we do with and for others. We experience things through our body, but we call upon collective ways of knowing to determine the content, the meaning and the signification of our bodily experiences (Bakhtin, 1993). Tobin also attends to the block and communicates geometrical ideas around it not only for himself, but to coordinate himself with Tara in the realization of a social situation: a classroom discussion in teaching and learning geometry. Perception is thus not only by, and for, oneself. The meaning that comes with perceiving an edge necessarily involves the existence of others, even though its sensorial aspect is individually experienced (in and through the body). Tobin can “see” the edge, can even touch it from the moment Jennifer hands him the block, but, as it appears, not in the way Tara, as she keeps asking him to do it another way.

From there, another important aspect of knowing-with comes strikingly comes fore in this conversation: Teachers and students exhibit responsibility for one another. When Tara enters the dialogue between Jennifer and Tobin, she at the same time responds to their previous utterances, both in its saying and in what is said, and also, addressing Tobin, sets him up to answer her own utterance. In the lived life of language, any utterance is an act (conversation analysis use the term speech act, e.g. Ricœur, 1992), a deed that affects oneself and the other in those two directions.

Tara addresses Tobin by asking him to start at one vertex “instead of” whatever he was doing. In that sense, she is responsible for completing in her own way Tobin’s previous contributions. Demanding something different, and grabbing his finger and sliding it on the block in a specific way, Tara provides an evaluation of what Tobin has said and done before, and in this affects him. At the same time, Tara also affects Tobin by placing him in situation to answer her. Here, this effect is clearly visible in the student’s responses, as he places his whole hand on the block, begins to articulate what “it feels like” (turn 18) or remain silent as an indicator of his struggles to coordinate with Tara’s demands (e.g. turns 18, 23). This two-sided responsibility is of course also integral part of Tobin’s talk (and silences) as well. Because all utterances always take place in a chain of utterance, at any moment in a conversation both speaker and listener affect one another, and therefore are responsible for their acting. And most importantly in the case of teacher-student communication, those communicative acts are evidently those in and through which, as we saw, knowing is always already knowing-with.

Consequently, knowing (geometrically) is subject to ethics because, in its relational aspect, it is realized in acts that affect others and ourselves. Knowing geometrically is not only a matter of negotiation between rational agents expression “conceptions” and adjusting their own private models. Nor is it only a collection of individuals synchronically part-icipating (from *part-* ‘part’ + *capere* ‘take’) in some activity<sup>14</sup> as when:

Thinking is a re-flection, that is, a dialectical movement between a historically and culturally constituted reality and an individual who refracts it (as well as modifies it) according to his/her own subjective interpretations, actions and feelings” (Radford, 2008b,

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<sup>14</sup> See, for example, what would be left of Socrates’ demonstration if we were to isolate the slave-boy’s answer...

p.219)

When a person con-tributes (from *con-* ‘with’ + *tribuere* ‘bestow’) to a (mathematical) conversation, he/she not only produces statements, but also presents himself to others as the person making those affirmations (Ricœur, 1992). When Tobin talks about edges, his utterances are moments in the realization of his being and his becoming geometrical. Speaking and gesturing, he makes himself vulnerable in and through his saying, and rejecting or ignoring his contributions inevitably affects him in how he is accepted as a person. Just as Socrates first sets up the slave boy in ignorance (taking him to contradict himself) and then as a knower (while negating his role in this), addressing Tobin, Tara – like Jennifer and like any teacher responding to a student – affects “the relation between what [he] knows about geometry and what he knows about himself” (Todd, 2003, p. 24), thereby raising questions concerning how teachers engage with students as *others*. From the perspective of an ethics of alterity – one that recognizes that the otherness of the other is constitutive of self and that we are at any moment responsible for the other and his/her sayings (Levinas, 1981) – the responsibility teachers and students exhibit in a conversation is thus an ethical one.

Seldom taken into account in mathematics education, this ethical responsibility permeates teacher-student relationships and is fundamental to understand the very possibility for knowing, teaching and learning (Roth, 2006a, b; Zembylas, 2005). Such ethics is fundamental to mathematics education precisely in that it requires us *not* to conceptualize knowing in individualistic terms (Ernest, 2009) but as relationships with others. In knowing-with, teachers and students mobilize communicative resources with and for one another and inasmuch, their utterances are not only units of speech, but units of *communion* (Bakhtin, 1986). The

relationality of knowing-with is an inherently ethical one because:

There is an interlacement of participants and turns that does not allow us to reduce anything said to any individual; rather, each turn can be understood only as part of a collective production of which individual contributions are constitutive moments. ... It is in this double allegiance to prior and subsequent (discursive) performances that the ethical nature of Being can be found. (Roth, 2007, p. 728)

That knowing-with constitutes an ethical relation is of special interest to us not only as a fundamental aspect of concrete teacher-student communication, but also because this is how we bring about particular forms of (mathematically/geometrically) knowing-with. When Tara, Tobin, and all the others present in this classroom exhibit responsibility for one another, they also demonstrate a responsibility for their situation. Asking “what kind of edge is that,” suggesting round versus square as a difference, or calling out for (and offering) another adjective to contrast with round edges, the teachers and students confirm, from turn to turn, their responsible for the unfolding of the conversation as a geometrical one. The geometrical conversation, however, can take various forms, and assign very different role or position to those who realizes it, hence resulting in different ways of knowing-with.

For instance, struggle arises in the episode because, although knowing-with emerges from dialogue, conversations are not necessarily *dialogical*, sometimes allowing only one form of knowing to be produced. One idea, one form of knowing, marks a monological talk rather than the number of people that are involved – the late Socratic dialogues are monological because they only bring one truth into play (Bakhtin, 1986) as his famous geometrical conversation with

a slave clearly shows. Such monological forms of dialogues,<sup>15</sup> often marked by relentless questioning, makes little room for the students to engage (Todd, 2003). We know, however, that students' spontaneous orientation to objects and situations does not always align with what a teacher may expect. Here, we can see Tara and Tobin producing a conversation in which they insist upon a specific way to attend to edges. Both the student and the teacher are responsible for producing the situation in such a way, but students inescapably have difficulties to coordinate with/in such specific understanding precisely because they do not know what they need to learn. Tara asks Tobin to touch, to look, to compare, and to name his bodily experiences. Tobin completes those requests as such, but his utterances do not seem to satisfy Tara, as she insists that he would manipulate the block or address the situation differently. Even within a geometrical discursive domain, there is a large diversity in the ways we can attend to edges. In teacher-student communication, however, this variety is not always acted upon. When only one form of knowing is made acceptable, students' "inappropriate" contributions need to be tossed apart, hinder the process by which those communicative acts can become instances of knowing-with. In the fragment, Tara's (and Tobin) visible effort to obtain the round/straight distinction from the student(s) (rather than stating it herself) sets up limited possibilities and opportunities for the next speaker thereby making it difficult for Tobin to coordinate with her. And vice-versa, we can see in Tara finally turning to the classroom to produce the statement.

This is important because, from a relational perspective, such conversations are what concretely constitute mathematics and geometry as ways of knowing. Indeed, knowing geometrically only exist as an active, historically and culturally preserved mode of coordination

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<sup>15</sup> To put it yet another way, a monologue is a dialogue where there is no response.

among human beings in the form of a “body of knowledge [...] consisting entirely in the complex choreography of knowers” (Davis, 1995, p. 4) or, I would say, of *relational* geometrical acts. Thinking in terms of knowing-with precisely enables us to take on this radically relational perspective and examining how students and teachers coordinate to realize the practice of geometry in particular ways. For example, in the episode presented in this chapter, we can see Tara and Jennifer producing with the (same) students very different ways of doing geometry. Jennifer engages with the students on a personal, subjective mode. She speaks to the I, refers to personal experiences (as when she asks Tobin “how do *you* describe this edge?”), and draws on a history of actions in common (“we were talking about edges”). So doing, she sets up a wide range of possibilities for Tobin to answer, to coordinate with her, and for herself to pick up on what he says and continue a two voice conversation. Hence, this happens not *only* because of the kind of questions she asks or the subjective perspective she adopts, but also as a consequence of how she recursively responds to Tobin. When he faces a difficulty (“I don’t know”), Jennifer offers resources (giving a block, asking a more specific, yet open, question) that provide Tobin with means and space to coordinate with her in a geometrical way. Contrastively, Tara’s strong reorientations from Tobin’s perspective, followed by the student evaluation of those utterances in giving up was he previously articulated, strictly delineate (even physically, when she takes his finger and slides it) the means by which distinctions between edges can be made. These are two significantly contrasting way of teaching and learning in and as producing student-teacher communication. Actual, lived geometrical activity is what Jennifer, Tobin, Tara and the other students relationally create for one another because dialogues not only bring about geometrical

ideas, but also particular forms of knowing-with geometrically. It is this relational – transactional – dimension, very different from the idea of negotiation of meaning,<sup>16</sup> which is still to be accounted for in ongoing social or collective inspired discourse on knowing, teaching and learning mathematics.

From that angle, we can examine the central, and otherwise hardly observed, ethical dimension of teacher-student mathematical communication. Among the different ways of doing geometry (teaching and learning), can see approaches similar to Tara and Tobin's creates an important tension between what comes from the students' and the teacher's expectations. Tara and Tobin set themselves in a position in which they needs to choose either to be present to him (to what he says, to his perspective), or to ensure that a specific geometrical idea comes forth in the conversation. Moreover, when such tension occurs, it is generally the teacher's "technical accountability" that takes priority (Neyland, 2004b), that is: teachers and students tend to conduct conversation so that it produces the apparition, in the (supposed) dialogue, of the communicative resources use to signify curricular outcomes, but *without* fully engaging in the process of signification. This is possible because the community of being (the present continuous form of the verb "to be") that constitutes the background for teachers' and students' utterances make it so that they are responsible for the possibilities and opportunities they set up for one another. When Tara pushes him to examine the question of edges differently, Tobin finds himself responsible for knowing-with her, providing her with resources to continue the conversation. Because knowing is knowing-with and thereby actualize the self-other relationship, the responsibility for the other we find in communicative acts is, through and through, an ethical

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<sup>16</sup> If I am allowed one more play on words, I would say: closer to *meaning* negotiation, as in giving significance, rooting it in its essential relationality.

one. The last part of the conversation with Tobin makes this tension and its ethical dimension even clearer. Tara turns to Tobin and asks a question one might think he will have no problem answering, since the utterance essentially contains both the question and the answer (“This one is straight, this edge is round. So which one is the straight edge?”), and as the student respond:

To: This one

Slides his finger on the middle of a face of the cube

Ta: Is it down here? Put your finger down the straight edge. Now when you are doing this, you are going right through the face. Here's our straight edge. Now go through the round edge.

Grabs Tobin's index finger and slides it down on an edge, then presents him the cylinder

When Tobin indicates the cube as the object with straight edges, his utterance includes a gesture by which he slides his finger in the middle of a face. Tara responds to him by reacting to that gesture (thereby showing us that she considers it as an integral part of Tobin's answer), asking the student to touch the block differently. Insisting again on a particular instantiation of a specific geometrical idea (a straight edge marked by a finger sliding on an edge), they make it so that their coordination produces the appearance of signs that signify that specific geometrical understanding. However, we can seriously question the conceptual richness that comes, for the student, from such an enactment. And as the episode concludes (Tara then moves on to another topic), Tobin's original idea involving vertices is discarded, and the plurality of voices and ways of knowing about edges is reduce to a monological interpretation. Moreover, if the students (Tobin, Alison) may appear to be little more than a foil for the teacher's production of the straight/round distinction, this requires us to give attention to how knowing-with comes about in geometrical activity.

Knowing geometrically is something we embody as the eyes come to “know” how to look or the hands know how to feel. Articulating those experiences in language, they become part of what allow us to coordinate ourselves with others by asserting meaning to what we do. But there is, and simultaneously, a transformation of the senses into cultural-theoretical organs of perception (Radford, 2009a) as we recurrently coordinate ourselves with and for others. Mutually constitutive of one another, these two dimensions clearly proceeds from the relational dimension of knowing, while at the same time also produce knowing as knowing-with. To put it yet another way, geometrical activity takes place in and through acts of coexistence, and coexistence with our human fellows in part realizes in the form of knowing-with.

Immediately comes with this the fundamental importance of recognizing, in knowing, the legitimacy of the other. When Tobin, Jennifer and Tara engage one another (and the other students) around questions, description, or manipulations of edges, they bring forth a world of mathematical understanding (Kieren, 1995), but also produce the moment of their human existence as irreducibly social creatures. In their dynamical and responsive nature, every one of their actions (including those we see as typically geometrical) are constitutive of the human world they live. Knowing-with about edges is only possible because Tobin, Tara, and Jennifer are responsible to one another, and they take on that responsibility for the saying of the other by responding to it. This linkage of human to human is “the groundwork of all ethics as a reflection of the legitimacy of the presence of the others” (Maturana & Varela, 1987, p. 247) as it manifests itself in concrete actions. That is, coexisting demands to see the other and his/her doings as legitimate and valid. From an ethical, relational perspective, what students say express their coordination with the world and the other. However undesirable they may seem, students’ saying and ways of knowing reach out and call upon us, and for that reason demand to be acknowledged

as legitimate and valid.

This responsibility is not a-historical and culturally independent. It arises in part because we live, with and for one another, a *particular* culture of teaching and learning mathematics, of student-teacher mathematical conversation. Today's practices origin in close history (in North America: the development of scientific education in reaction to sputnik), find their inspiration all the way back to – at it seems – the so-called Greek miracle which precisely consist in breaking from mythical (dogmatic, but also, I would say, *organic*) world-views (Vernant, 1962) where self and other, past and present, expert and apprentice, are not conceptualized in isolation from one another. Luckily, this culture also provides us with means to overcome this situation, to take on responsibility by actually allowing us to reflect upon ourselves and become aware of this artificial separation, so that we can enact, embody, that awareness, that sensitivity for the other.

Embracing the legitimacy of students' voice as integral to teacher-student communication still allows us, of course, to question the *desirability* of what they say or do. Actually, as soon as we understand knowing as always already knowing-with, we move beyond questions of knowledge and its "assimilation" precisely to engage with the appropriateness of knowing in various ways. Geometry is a unique, invaluable way of knowing, an historically and culturally developed network of practices by which students can expand their experience of the world, themselves, and others: like a skin, is a part of ourselves by which we can come into contact, affecting and being affected by others and the world around us. Inasmuch, we certainly want to teach geometry, and specify with and for the students what constitute (or not) geometrical knowings. But this *also* means trying to disrupt mathematic education practices in which ethical responsibility for the other wears out (Neyland, 2001). As in the case of Tara and Tobin, erosion seems to correlate with those moments in which teachers and students experience tension

between specific expectations and solicitude to the child. Strictly setting up, and in advance, what is legitimate and valid proceeds from an understandable attempt to get the students to “get it right.” But it also, for that very reason, leads to reject alternative ways of knowing.

Alternatively, the relationality of teacher-student communication interest us in how one and other’s actions can occasion (Kieren, 1995) geometrical activity (rather than trying to *cause* specific learning outcomes) and, thereby, examine their desirability for teachers’ and students’ geometrically knowing-with one another.

### **Opening: The Relational Dimension of Communicating Research**

In this chapter, I articulate how teachers and students geometrical knowing conversationally brings about mathematical idea (such as in the making of distinction about edges), but also certain ways of doing geometry. Developing knowing geometrically as knowing-with, I explain how a relational approach to teacher-student communication places its ethical dimension at the heart of teaching and learning mathematics, and draw some consequences the praxis of mathematics education.

My final word, however, must step back from those observations and reflect on what the reader and I just did in the very process of writing and reading this chapter. As I noted, it is generally agreed that communication is of fundamental importance for mathematics education. That is true not only for teachers’ and students’ knowing, but also for my own knowing of that knowing. Looking at teacher-student communication, analyzing it, and making sense of those observations in integral part of mathematics education as whole, and directly concern us as researchers or educators who wants to contribute to mathematics teaching and learning in our own way.

Undeniably, the moment I opened this chapter, I am already in an ethical relation with my reader. From the beginning, I am responsible for the production of a textual utterance that takes place in a chain of similar utterances (Bakhtin, 1986), and in mathematics educational research as a network of conversations. This responsibility is precisely one of knowing-with, through and through conversational and permeated by ethics. It is a responsibility that is even prior to the beginning itself, as my textual utterance is also, already, a response to others, and what they said before me. Because our use of language grounds me and the reader as the basis of *con*-sciousness (viz. knowing-with), I take on this ethical responsibility for the reader by unfolding my research observations thoroughly make visible not only what I think, but also how I work with data to achieve my conceptualizations. Meticulously providing concrete illustrations for what I contend, but also explaining *why* I believe this particular way of knowing about knowing is desirable, I hope to bring about discussions that enable mathematics education to embrace mathematics as an activity in students and teachers do with and for one another. Particularly, when discussing the ethical importance of recognizing students' contribution by contrasting it with Tara approach, what I do here is not to judge it, let alone to call it "wrong." Tara and the students are perfectly legitimate in what they do, also realizing what appears to them as an appropriated way to do mathematics (education). In line with the ethics briefly introduced in this chapter, my suggestion is to examine the desirability of such practices, just as I question the desirability of framing mathematical cognition in terms of mental representations. In both respects, my position is that a relational approach to mathematics and communication invite us to go beyond abstract knowledge (e.g. about what student should know, or what teachers should or should not do) to consider how actual communicative actions opens up for knowing-with.

## CHAPTER 6

### Researching-in-the-Middle: How researching creates learning opportunities for teachers and students

#### Preface

On a sunny afternoon, a physicist, a biologist and educational researcher were sharing a bottle of good old Cuban rum.

- You know, it's funny, begins the physicist. In quantum physics, a key principle is that there is no observation that does not *change the observed phenomenon*. But when people just look at the result of our research, most of them believe that we do objectively describe phenomena.
- Very funny, continues the biologist. When we look at cognition from a biological perspective, we come to a point where we have to say that there is no observation that does not *change the observer!*
- I really wonder, says the educational researcher, why, then, it is so hard to *bring any sort of change* in education!

Research in education is generally directed toward improving teaching and learning (e.g. Van der Maren, 1999; Sauv e & Godmaire, 2004) whereas, over the last 30 years, the existence of a gap between teachers and researchers in education has been an inexhaustible object of discussion (e.g. Threadgold, 1985; Margolinas, 1998). I have long been wondering about the impression that the knowledge research produce does not address the practical aspects of teachers and students' everyday life in school (e.g. Sumara & Davis, 1997). Can research be any more than that?

With this question in mind and the various observations I had made in the previous studies, I went back to the data we had collected to give a special attention to what was happening there

in terms of teaching and learning *because* of the presence of the research. I then began to realize that there was, indeed, a possibility to think research and knowing in a very different way, more consistent with the relational epistemology (knowing as knowing-with) I had outlined. Again thoroughly working with my co-author/supervisor in clarifying that hunch, sustaining and developing it from the data, I engaged in the process of writing the following study. All and all, it took me over two years, recursively coming back to this piece while working on other texts, before I was finally able to articulate in a satisfactory way the idea, at the heart of this chapter, of researching as taking place in the middle of teaching and learning. And, for myself, get at grip at how change in education *is happening* with/in us, teachers, and students, on a day-to-day, moment-to-moment basis. Again, I offer here a map of my journey through this chapter, identifying some of the road I open, and how the visited themes are herein rendered.

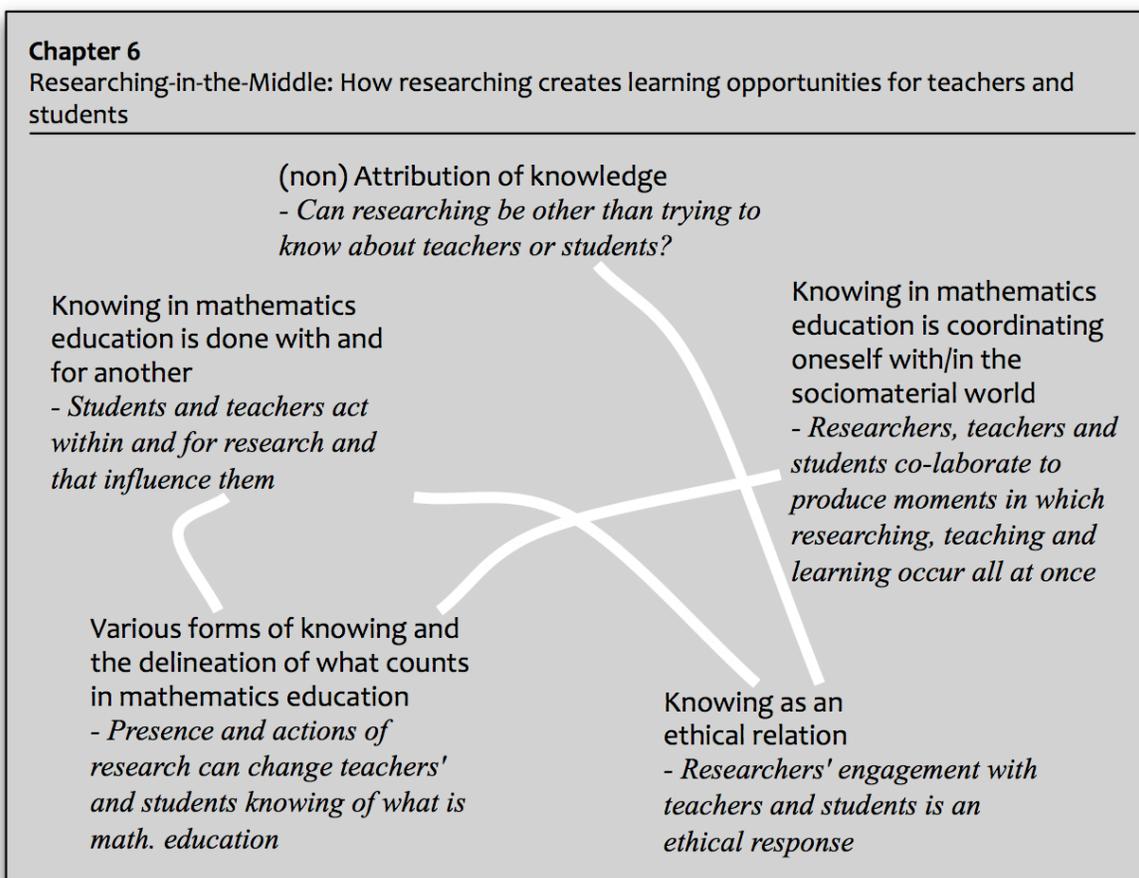


Figure 6.1: A map of my walk in “researching in the middle of teaching and learning”

A different version of this study is currently under review.

## **Researching-in-the-Middle: How researching creates learning opportunities for teachers and students**

### **Why Research?**

In the past, research activity on teaching and learning tended to be conceived in terms of objective processes that came to record events in classrooms. More recently, however, this positivist view of educational research has come into disrepute in part for neglecting the observer's influence on the contexts of knowledge production (Denzin, Lincoln & Giardina, 2006). Especially in the classroom, the researcher is not like fly on the wall whose presence does not influence what happens in the situation observed. In fact, recent work even suggests that new opportunities for teaching and learning arise from the very presence of researchers in the classroom (e.g. Roth et al., 2002). This work shows that researchers are learning together with students and teachers and not only about the situation, but also about content and content pedagogy.

The purpose of this chapter is to articulate how classroom researching takes place "in the middle" of teaching and learning rather than as a separate activity. Framed around the question of "why research," I present a moment-to-moment analysis of excerpts from a classroom data-collection around elementary children mathematical activity (e.g. Roth & Thom, 2009a, 2009b) to (a) articulate the inseparability of researching, teaching and learning when researchers are physically present in the classroom. Addressing the general claim that participating in research may benefit teachers and students, I then (b) examine how researching-in-the-middle enable us to

investigate the kind of learning that arise for students and teachers from this situation.<sup>17</sup> I finally (c) develop a reflection on the need and possibility of a new ethics of researching, one that encourages researchers to maximize their transactions with those who participate in their studies.

Why do we “research” in education? Answers to this question generally take on scientific models of knowledge production. Doing research aims at better understanding what is taking place in educational settings, what are the effects of teaching or learning in such or such a way, and why. Mathematics educational research is quite exemplar in that matter, as it tends to try and understand “the nature of mathematical thinking, teaching, and learning” (Schoenfeld, 2000, p. 641). Implicit here is an epistemology in which research is separate from teaching and learning. Researching takes place behind the scene(s), and as if researchers’ observations were not fundamentally affected by the fact that what they see is produced for and within a research project.

Alternatively, educational research literature presents some approaches alluding to different ways to think about “why research.” One possibility is to take advantage of the positive effects that researching tend to have on the situation under study (the Hawthorne effect) instead of focusing on scientific control issues (Brown, 1992). The researcher’s attention to the welfare of the teachers and students triggers a certain cognitive activity that naturally correlates with the improvements she observes. In the course of research, researchers and teachers can thus enter in co-learning agreement (Wagner, 1997) and see educational research as an opportunity to directly influence classrooms practices. At the heart of such approaches is a claim that participating into

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<sup>17</sup> Although a similar study could clearly be done regarding the researchers’ learning, I only examine, in this chapter, the influence of researching on teachers and students.

research can benefit teachers and students (e.g. Bednarz, 2008; Zembylas, 2005) and that researching can be a strong developmental tool (Jaworski, 2003). Moreover, the process of academic knowledge production seen as an opportunity for teachers and students to engage with researchers actualizes a strong link between researching, teaching and learning; the results depend on whether students, teachers and researchers “get a good learning experience” from the research (Pontecorvo, 2007). Producing those learning experiences hence becomes part of the purpose of research:

I argue that, redefined in this way, the Hawthorne effect is exactly what I am aiming for in my classrooms ... A ‘Hawthorne effect’ is what I want: improved cognitive productivity under the control of the learners, eventually with minimal expense, and with a theoretical rationale for why things work. (Brown, 1992, pp. 165-167)

I note, however, that those alternative approaches to the question of “why research” still a separation between researching and teaching and learning in the moment of collaboration. Most of the time, teachers are responsible for testing the activities with their students, whereas members of the research team first simply act as camera operators and then interview the teachers or analyze their talk and actions. Certainly, distinctions between researching, teaching and learning are useful, for example to put to the foreground different type of viewpoints, or contributions. But these qualities by which researching, teaching and learning differ do not necessarily entail a divide. An integrated understanding of the processes of learning, teaching, and researching activities is critical if we want to avoid blindly contribute to the “conventional obsession with doing more effectively whatever one is doing” (Begg, Davis & Bramald, 2003, p. 630) rather than with trying to make sense of researching, teaching, and learning. Hence, would it not the beneficial effect of researching be better understood as resulting from the mutuality by

which researching, teaching and learning affect one another, rather than from the action of separate, isolable activities? In the next section, I take on such perspective and show how researching can be seen “in the middle” of teaching and learning, and not in the backstage, or separated from those activities.

### **Researching in the Middle of Teaching and Learning**

Researching, teaching, and learning are certainly different activities, but their differences do not necessarily entail a divide, and especially not when researchers are physically present in the classroom and occasion the research. One way to see this is in the specific relationship between teaching and learning. Scholars increasingly become acquainted with the idea that although teaching and learning might appear as independent, non-reciprocal activities, a comprehensive examination of what actually happens in the classroom reveals that this is not the case. Teaching is better understood as taking place not in separation but “in the middle of the classroom activity or in-the-middle of any other form of learning/teaching discourse” (Kieren, 1995, p. 11). In the classroom, student actions always occur in a sphere of possibilities continuously shaped by the teacher, but also by the children themselves, who co-determine the situation in and through what they do (Roth, 2006a). Occasioning one another in a dynamic, responsive way, teaching and learning do not occur in a simple “back and forth,” but realize themselves *in-the-middle* of one another, through the complex web of individual and collective actions.

In this section, I use an episode from an ongoing research in mathematics education to show and explain how, when researchers are physically present in the classroom, we can conceive researching-in-the-middle, that is as not separated from teaching and learning events.

To do so, I introduce a first episode from a lesson involving 28 second-graders, their regular teacher, and various members of the research team.<sup>18</sup> From that episode, I show the inseparability of researching, teaching and learning by articulating how researchers, teachers and students not only affect one another, but also contribute to one another's activity. These observations then serve me to discuss how research is in-the-middle of teaching and learning, although maintaining the distinction between researching, teaching, and learning.

*Researching in the Classroom: Co-Introducing a Lesson?*

In the classroom on that day, along with the students and Tara, the regular teacher, Jennifer Thom (primary investigator) and two other research members are also present: Michael Roth (coPI) and Miki (research-assistant). The children sit on the floor, facing Tara (Ta), while Jennifer (Je) is in the back (Figure 6.2). On both sides, Michael (MR) and Miki (MK) are video-recording the session. As the lesson begins, we see Michael silently gesturing to Miki who quietly circumvents the group (note the different angles from figure 1a and b) as Tara starts talking. She presents the upcoming activity in which the students will have to deduce from the shadows it produces what is the 3D figure sitting (concealed) on an overhead projector:

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<sup>18</sup> Details concerning the research project and methods of analysis can be found in previous publication, for example, Roth and Thom (2009a, b).

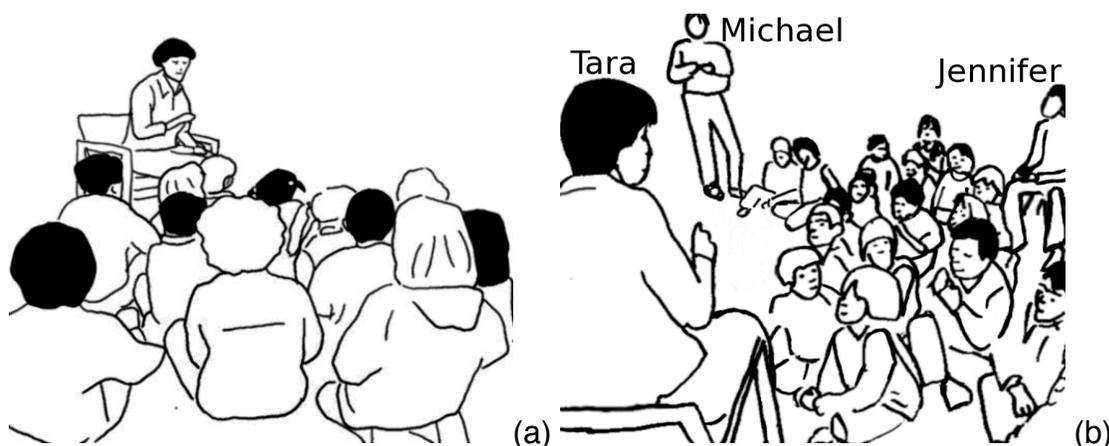


Figure 6.2: Two views from Miki's camera.

- Ta: We have a new game for you today. Some of you are going to stay at your tables and some, I am going to ask you to move your chairs over, because we are going to use the overhead projector and we are going to project a shape up on to the screen and we will ask you to think about what that shape is. Put your hand up and tell us what that shape is... Anything else we need to tell them before we start?
- Je: No. You are just going to be doing something and, you remember that when we work, we are working in partners and group today, group first, and then partners, and we want to make sure that when you think you are thinking...? Inside or out loud?
- Students: Out loud.
- Je: Out loud because your group members and your partner have to be able to hear what you are thinking so you can think together.
- Ta: Right. Does out loud means shouting? No it doesn't. We still want to hear your voice. It could be a little bit louder because we are trying to do a bit of filming for that, but it cannot be shouting, ok.

Classroom research does not generally give attention to moments like these, while at the same time, it appears “common sense” to say that in such moments, students, researchers, and teachers are closely *working together* and *into each other's hands*. The lesson opens with Tara introducing a game: After a shape is shown, the students will be asked “think,” “put their hand up,” and “tell” what they see. There is a brief pause in Tara's talk, and she completes her utterance by asking if anything else needs to be said. Jennifer picks up the question with a “No” (as if nothing needed to be added), nevertheless addressing the students and elaborated on the

way in which they work with one another and each other: The students are now to “work in partners and group” and expected to “think together.” In reaction to Jennifer’s rephrasing, Tara utters an agreement (“right”). She then engages with the students, also reframing Jennifer’s presentation (“Does out loud means shouting?”), and concluding with some conciliating arguments: Not shouting, the students will be “a little bit louder” because of the data collection (“we are trying to do a bit of filming for that”).

In the meantime, Miki has responded to Michael’s gesture, changing position (from Jennifer’s left to Tara’ right), maintaining a certain distance with the students who nevertheless visibly noticed what was going on. Rendering “strange” these familiar, common sense observations permit us to better understand how such situations are actually taking place. An attentive analysis of the excerpt in terms of the co-action of students, teachers and researchers reveals that what we observe is produced *for* researching as much as *for* teaching and learning. Although classroom research would generally report on such episode with something like “the teacher(s) introduced the lesson,” it is in fact better thought of as a “co-introduction” with/in the research. This co-action is seldom put forth in educational research, as we tend to neglect the fact that when researchers engage as such with/in the classroom, researching, teaching and learning are mutually implicated (from the Latin *implicare*, ‘fold in, entwine, entangle’). In the following section, I develop this analysis to define the meaning of researching-in-the-middle and some of its ramifications.

### *The Inseparability of Researching, Teaching, and Learning*

We know by common sense that people’s presence alone suffices to affect, and is affected, by the situation in which they find themselves. That presence in time and space inescapably

contributes to the collective realization of a given event. In this classroom where a lesson is introduced, we notice the presence not only of the teacher(s) and the students, but also that of Miki and Michael. Looking and analyzing their actions underline the importance of the nested nature of teaching, learning, and data collecting. In the fragment above, we see that when Miki changes position, she chooses to circumvent the group, hence not interrupting Tara's talk.

However, some students, noticing her movement, look in her direction before turning their sight back to Tara. Researching is noticeable, and it stretches the students' attention between Tara's talk and Miki's movement. The students see Miki silently walking with her camera pointing in their direction, they hear Tara addressing them, and have to interpret those two sets of concurrent actions as the production of a unitary moment whereto they also contribute. Moreover, this researching is also a reaction, a response to the teaching and learning currently taking place. Miki's presence is visible, but also her attention to what is going on in the classroom, and the students respond to those unfamiliar presence and attention by looking at Miki, and then turning back to Tara. Clearly, researchers, teachers and students are affecting and affected by one another, placing research in-the-middle of teaching and learning.

But being in the middle is more than affecting: it is *contributing* to one another's activity. In this situation, researching itself, in its current enactment, is part of what brings back the students to the teacher's talk. This is because doing research holds against (is in physical contact with) teaching and learning, and in return also is supports them. Miki's presence and attention bring forth the teaching and learning as the 'object' with which she enters in transactions. That process is visible to the children, reflects on them, and supports them in orienting themselves to what Tara is explaining. As a result, students can be even more attentive to what their teacher is saying, and in the end perhaps learn more, or better, thanks to the special consideration they give

to what is expected from them in the upcoming lesson. This is fundamental to what researching means, as one might realize by comparing it to making a movie or taking geodesic measurements, because all those activities are culturally, historically produced in a different manner; hence producing different outcomes. Not noticing the generative effect of researching on teaching and learning signifies that we neglect that (a) people holding camera in the classroom are always visible for teachers and students, that (b) they are not part of a “normal” classroom situation, but that (c) they are not totally alien to it either. Through the researching, there is a reification of the classroom activity. What teachers and students do becomes a research artifact in both sense of the term: a human made object of cultural and historical interest, and something that occurs as a result of the researchers investigation. Researchers, teachers and students all contribute in the production of this event in which Miki quietly moves, and the students, after following her, turn back to Tara.

This collective dimension is not visible only in presence, but also in people’s talk. On that level, the fragment from the dataset makes particularly clear how teaching and learning goes hand in hand with the research. In her first utterance, Tara finishes by asking: “Anything else we need to tell them before we start?” Jennifer picks up the question, hence not only completing it as such (it is not impossible that Tara was aloud thinking to herself), but also clearly assigning meaning to the “we” Tara uses throughout her speech. In so doing, Jennifer makes salient the fact that many people are presently in the room, looking and listening to what the teacher and the students are doing. Speaking from the back of the classroom, Jennifer literally adds another pole to the frontal “teaching center,” *and* another voice to what comes from the teacher. This new voice called upon by Tara, who explicitly combines the research team – with whom she prepared the lesson the day before – and herself, within her “we.” Tara’s call setups Jennifer to respond to

her utterance, and thus reify that she is observing. Asking whether something needs to be added, Tara shows that what happens here is not only her and students' affair. In that "we," she also marks concern for the researching itself, which is even more blatant in the final remark: "we are trying to do a bit of filming for that." Because Jennifer and the others equally have interest for what she and the students do, Tara gives opportunity for the researchers to develop other aspects of the situation. Simultaneously, she opens her own speech to be responded to in terms of what is expected from her and the students as part of the research. Tara explicitly recognizes, and thus realizes, the collective dimension of the enterprise, and Jennifer immediately follows her on this: "when we work ... we want you to ..." and so on.

The plurality of Tara's and Jennifer's "we" is not only that of co-teachers more or less driven to become like one another, but one that maintains the distinction between researching, teaching and learning. Tara's last utterance is clear on that point, as she asserts that "we" want the students to be a little bit louder because "we are trying to do a bit of filming." Clearly, there is something uncommon happening in the classroom, something other than teaching and learning. Tara suggests here that this something, researching, has its own requirements heard through Jennifer's voice: students will think "out loud," hence making the classroom louder, at least in part because of the recording of the session.<sup>19</sup> Tara thereby makes visible that introducing/realizing the classroom lesson is done with and for both the research team and her. In this, she reveals another aspect of how researching entwines with teaching and learning. We recognize in this the etymological roots of the *collective* dimension of *collecting* data, from the

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<sup>19</sup> And importantly, here is Tara's interpretation of what Jennifer said. It is not Jennifer who required this for the data recording. In fact, she later explained to me: "My intent in saying this was to occasion co-emergence and perhaps collectivity of ideas or conceptions."

Latin *col-* ‘together’ + *legere* ‘gather’, as a symbiotic (from Greek *sumbioun* ‘live together’), and collaborative (the same Latin *col-* + *laborare* ‘to work’) process.

The visibility of the mutual/reciprocal occurring of collecting data and teaching and learning mathematics contributes to everybody’s orientation in the activity. It supports and at the same time is supported by what students, teachers and researchers do when they are in presence of one another. Being observed and recorded for their contribution in teaching and learning, students are situating and situated by the teaching and the researching. Because researching, teaching and learning conserve their own ethos, and realize different historically defined activities, they preserve their significant differences: Researching *is not* teaching, just as teaching *is not* learning, although they clearly are co-emergent and mutually defining of *and* beneficial to one another’s realization when the researchers are physically present and occasion the research.

Thinking in this way puts teachers, students, and researchers *in the middle* of what is happening, a formulation highlighting how, in and for the realization of the event, teaching, learning, and researching. Better thought as reciprocal in nature, all three merge and at the same time co-emerge. Hence, for example, Tara and the students (and Jennifer as well!) contribute to the researching by the teaching and learning they enact. The research team accepts this by quietly continuing the data-collection, thereby also contributing to the unfolding of teaching and learning. And so on. In the unity of the moment-to-moment, researchers’, teachers’, and students’ actions are for researching, teaching, and learning all at once, as something they do together, with and for one another.

### **Researching-in-the-Middle and Learning Opportunities**

Concerning the culture of inservice teacher education programs, Dawson (1999) makes an

observation that applies to most educational research: This culture, he writes, seems to be based on the principle that “there is something wrong” with teaching and learning, and that we, as researchers, “must fix it.” That is, we think about classroom research within an epistemology where researching is obviously separated from teaching and learning, and therefore find ourselves caught in a paradigm in which we seem to face countless obstacles and frustrations in relation to research’s influence on teachers’ practices (Begg, Davis & Bramald, 2003). As an alternative, and to develop a deeper understanding of how researchers and practitioners produce change in the teaching and learning – and in the educational system as a whole – Dawson presents an approach of “inservice-in-the-middle.” In this approach, the scholars’ role is to support all those involved in education (including themselves) to become aware of what they do, and explore possibilities instead of focusing on making judgments about what is right or wrong, works or does not work, and so on.

Dawson’s view brings to mind a reconceptualization of educational research that sees change as continuous, local, and ongoing. What develops here is an opportunity to fully recognize that individuals and collectives change together, learn together: with, from and for one another. In doing classroom research as much as anywhere else, when people learn, the collective they form necessarily changes, and reciprocally: changes in the collective activity affect all the individuals in it (Davis & Sumara, 2005). Hence, cognitive scientists explain how people’s actions are fully determined by their lived history (or structure), whereas those actions co-emerge with socio-material possibilities, thereby producing new experiences to draw on (Varela, 1999). From there, we call learning that continuous transformation of the person and his/her (social, material) environment. This means that when teachers or students are implicated in certain ways of doing, they engage with these practices, make sense of them, which in return

changes the way they see what they are doing. The sense-making process opens new action and conceptual possibilities: they can act and think differently. Researchers, teachers, and students are all at once in-the-middle of researching, teaching, and learning, and in that way transform themselves and one another as they work together.

The same process contributes to the production and the re-production of those practices at the collective, cultural, and historical levels. In and through the bodily transactions that realize and transform them as individuals, (mathematics) education is also realized and transformed by them (e.g. Cheville, 2005; Roth & Thom, 2009b). If researching can do this, we see how its most powerful outcomes are indeed “new ways of thinking about what we [researchers and teachers] do and how we do it [...] leading to new ways of thinking about joint activity and work with pupils” (Bjuland & Jaworski, 2009, p. 37). And from this angle, we see researching-the-middle as a true locus of educational change in its continuous, local, and ongoing aspect.

### *Engaging with Teachers: Researching Creates Teaching Opportunities*

The presence of researchers may give rise to many opportunities for learning to teach (Tobin & Roth, 2006). From the perspective of researching-in-the-middle, occasions for the teacher to learn are traceable even in actions occurring over a few minutes. The simple lesson introduction excerpt presented above exemplify how researching can expose teachers to different ways of teaching. When Tara introduces the lesson and Jennifer responds, two perspectives come into contact. Tara sits in the front of the classroom and monologizes the introduction of a teacher-led game in which the students individually look, think, and tell what they see. Jennifer reframes the lesson in a dialogue with the students, and organizes the game around the students’ conversations. Such contrast takes its source in researchers’ and practitioners’ different

appreciations or views of a given activity, something scholars often see as rooted in a distinction between theory and practice, or between scientific and action knowledge (e.g. Wolfe, 1996; Bednarz, 2008; Krumer-Nevo, 2009).

Developing teaching methods or activities, researchers generally base their approach on previous studies and conceptual analyses, whereas teachers naturally draw on their experiences in the specific context of their work. Here, it makes sense for Jennifer to change the game so that it maximizes the students' activity and communication: As a researcher, her interest in part goes to the very "collective nature of thinking and learning" (excerpt from the research proposal). Multiplying students' transactional opportunities, Jennifer is grounded in her expertise regarding what might best support teaching and learning, and she also serves her data-collection purpose. We have little doubt that Tara too moves toward the creation of a learning environment which, in her view, best contributes to students' learning. There is value in an orderly conducted classroom conversation where students speak one at the time, and listen to one another. Similarly, maintaining a low sound level in the classroom can help student's concentrate, makes them feel safe, allow the teacher to effectively gain everyone's attention, and so on.

Looking across the data collection reveals that moments like these do provide teachers with learning opportunities. When I look across the data collection, I notice a progressive transformation of Tara's way of introducing and conducting lessons with the students. She does not adopt Jennifer's approach, but progressively uses some of its aspects as part of her repertoire, hence expanding her action possibilities and room to maneuver. For instance, two weeks after this lesson, Tara operated a transition between two activities. This time, she introduced the task, to use tangrams to fill an outlined shape, by asking a student (Brandon) to model with her, and emphasizing the collaborative and reflexive dimensions of the lesson:

- Ta: Notice that Brandon tried a couple of things and he still haven't figured out the space. So what does it tell you? I know what it's telling me, what is it telling you? You try those pieces and they are just not working. What does it tells you?
- Sonia: Try other pieces.
- Ta: Absolutely, try other pieces. It's about experimenting and seeing witch one's work. Now when you work on yours, with your partners, you need to be talking about why the piece works, or why it doesn't work

Without Jennifer's intervention, Tara naturally involves the students and emphasizes the collaborative and reflexive dimensions of the activity. In her own way, she asks a student to explain his understanding of the task, and stresses her expectation to see the students talk with their partners. Working with Jennifer in the course of the research Tara becomes acquainted with new, now very manageable possibilities for teaching and learning. She develops those action resources not because she attends to a special teacher-training program, or is lectured by Jennifer. Her way of seeing and doing mathematics teaching and learning changed because researching-in-the-middle places her in a situation to experience those ways of thinking and realizing mathematics education. Importantly, this is not to say that researchers' version of teaching and learning mathematics is better than that of the teachers, but to illustrate how differences articulated in the classroom can become opportunities to learn (Roth et al., 2002).

One might wonder if this change will last. Without the researching dimension to what happens in the classroom, will Tara still make use of those new actions possibilities? But fundamentally, when we see learning as continuous transformation of the person and his/her environment, we realize that the sole existence of those new possibilities for Tara already necessarily and irreversibly transforms her teaching by transforming what she knows to be a teaching possibility. Tara's situation changes with her, as it now affords new alternatives, different courses of action. It is like walking a path for the hundredth time, and suddenly noticing

the presence of a building, or a sidetrack, that probably was there the whole time, but now only is present for us. Once noticed, that presence can no more be erased. Throughout the data-collection, Tara, not acting only for teaching and learning but for researching as well, repeatedly turned to Jennifer and with a look, a nod, or a brief “right?” checked with her that the lesson was going fine. That is, the presence and the reaction of the researcher lead Tara to examine (consciously or not) her own praxis of teaching through the eyes of the researchers. Inasmuch, that researching-in-the-middle offers a significant and concrete contribution to change in education in what we framed as its continuous, local and ongoing aspect.

*Discussing with Students: Researching In or By the Way?*

Before conceptualizing forward the kind of effect we see taking place with researching-in-the-middle and draw some conclusions, let’s look at another fragment of the classroom session. Some 20 minutes into the activity, Miki (MK), the research-assistant, is standing close to 3 students, Eugene (Eu), Collin (Co) and Chichi (Ch), while Jennifer operates the overhead projector, revealing a new shadow (a circle):

Ta:           Ok now, take a look at this. Look first before you start talking. No no, voices down. Tell your partner.  
 Je:           What you see and what you think it is.  
 Eu:           I think it's a cylinder.  
 Co:           I think it's a cylinder too!  
 Eu & Co:    Cy-lin-der! Cy-lin-der! Cy-lin-der! (banging their table in rhythm)  
 MK:           Shush shush!  
 Ch:           I think it's a cylinder.  
 MK:           *(Turning her camera to Chichi, who is already looking at her)*  
               Why do you think it's a cylinder?  
 Ch:           Because it's like this (showing with her hands, Fig. 5.3a and b) and its round like that (pointing to the front of the classroom) and you can turn it over and they both have flat sides.  
 Eu:           Here's mine. *(With a lot of emphasize and gestures:)* I also think it's a cylinder because it has that part on it that can be a cylinder. If you look at it you will see a circle and if you turn it over it's still a circle.  
 Ch:           Heu Eugene, she is actually videotaping you.

Eu: Yes I know that.

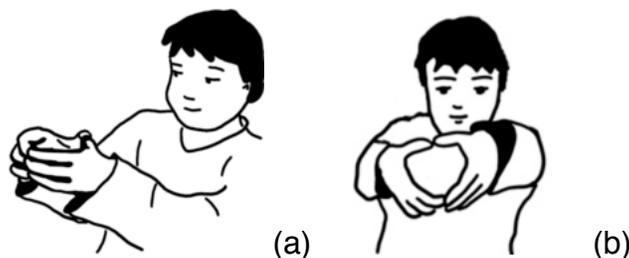


Figure 6.3: Chi-Chi gesturing for herself (a), but also for Miki and the camera (b)

In this fragment, Tara and Jennifer invite the students to look at the shape on the screen and discuss. Miki's camera then moves closer to Eugene and Collin, who quickly engage in telling one another what they see and think about the object ("I think it's a cylinder") and then become noisy, banging on their table. Miki steps in and keeps the students quiet. Chichi then turns to her and also explains what she thinks and why ("its round like that and you can turn it over and they both have flat sizes"). Next to her, Eugene also goes on explaining his thinking ("it has that part on it that can be a cylinder ...") with much vitality (gestures, high pitch and sound level). Chichi then reminds her peer that Miki is recording him, which he affirms to know ("I know that").

Unlike what happens during the introduction, Miki here is not discreet at all. Just as Jennifer finishes talking, she steps toward the students. They begin to talk, and maybe "act out" in response to the presence of the camera by banging on their table. Miki then signals them to be quiet. Her presence strongly affects the students' behavior: Chichi turns to her to share her thinking (turn 14), and Eugene want his part of the conversation too ("Here's mine ...," turn 17). We can also hear Michael, at the other side of the room, asking a student: "And tell me why you think that?" Researching here brings a lot of very noticeable people in the room, busy with their recording devices and probing questions, something the reader must now easily see as being "in

the middle” of teaching and learning. Traditional “scientific based” approaches would probably call this “bad research,” for too much perturbing the so-called “natural” unfolding of the activity, and the children's “normal” reactions. Even worse: the students themselves explicitly take on aspects of the researching, as Chichi seems to rebuke her partner for being too agitated in front of the camera. Is not researching-in-the-middle becoming an obstacle for teaching and learning? Or is it still promoting it? Is researching in or by the way of teaching and learning? Answering this question takes us back to what I suggest: Examine the special opportunities for learning that arise because of the researching activity.

#### *Engaging with Students: Researching Learning Possibilities*

Engaging with students, what are the special opportunities for learning that arise because of the researching activity? On that level, a close analysis of the video excerpt is quite revealing. As Miki moves closer to Eugene and Collin, the two boys formulate their answer for one another. This is not to say that Miki’s change of position is causing the students to talk, but certainly somehow supports it. Visible to the students is a recording device that comes closer: it is associated with anticipation (if not an expectation) that they do something worth recording. Interestingly, Chichi, who is then out the frame, remains silent during that time. We note that Eugene and Collin quickly settle for a relatively narrowed answer to the teachers’ invitation to discuss with their partner and, noisily banging the three syllables of the word cylinder on their table, they (and Chichi) are quite far from “thinking together” and “not shouting.” Miki’s intervention silences Eugene and Collin and also provides Chichi with the opportunity to formulate her idea, hence echoing both Tara’s and Jennifer’s concerns for the learning environment. By getting closer and addressing the students, Miki manifests herself as a potential

partner, someone to talk to, and being responded by. Chichi tells her what she thinks the object is, and Miki answers to her suggestion. She inquires about Chichi's thinking. The student then justifies her response not only articulating why she believes the object is a cylinder, but articulating it for the camera as Figure 6.3 illustrates. With this explanation, Chichi becomes the researching center of attention, and it is easy to hear Eugene's "Here's mine" as a will to gain that attention too (Miki does turn the camera towards him).

Chichi and Eugene take this central position precisely by "thinking out loud" and making their partner(s) "hear what [they] are thinking so that [they all] can think together." It is not possible to know to what extent the students explain their thinking because they wanted to proudly show how much they know, become the center of attention, feel coerced to say something, and so on. Nor is it possible to know what might have happened if Miki had chosen to stand beside another group, or if the research did not take place at all. But those are observations one has to make regarding *any* kind of educational research, should it be for ethnographic observations, in interviews, when using questionnaires, or any other research method: research always affects the situation. From analyzing the entire dataset, I know that students become increasingly keen and fluent in expressing their mathematical thinking. In fact, the students' "out loud thinking" visible in this excerpt, including the use of gestures as communicative resources, already constitutes a major change when compared with what they did during the first lessons. When the students were asked to collectively sort geometrical object according to criteria other than color and size, Tara and Jennifer had to continuously ask "*Why* do you think it's the same [/different]" (see Roth & Thom, 2009b). During the final sessions of this research project, however, the students naturally made their thinking available to one another. For example, in response to Jennifer showing them a poster with figures and simply

asking “Who thinks they are rectangles or squares?,” students spontaneously began to explain:

Collin: They both have four vertices and they both have equal size.  
 Je: Eugene you have something to add?  
 Eu: Yes. They are squares and rectangles because they have sides that are like heu (*moving his hands up and down*)... straight.

In words and gestures, the students too developed a new repertoire of ways of doing mathematics, of knowing and describing objects and situations mathematically, and of communicating their reasoning process. Clearly, this comes for them from learning “in the middle” of teaching and researching.

Chichi’s last observation to Eugene about the videotaping and his response (“she is actually videotaping you,” “Yes I know that”) might still be puzzling, but in fact invaluable make explicit the special contribution to students’ learning that comes from the researchers activity *as such*. Just as Tara’s and Jennifer’s inclusive “we” in the first excerpt legitimates what one is doing while opening possibilities for the other to act the children here manifest ownership for the situation as a whole. It is not that researching makes students responsible over what we normally/acceptably expect from them. Whether we are aware of it or not, students (like all of us) are always answerable for the realization of the situation in which they find themselves (Roth, 2006a). Children are always responsible for listening in the classroom, for answering teachers’ questions, for thinking mathematically and for learning by trying to work out whatever problem we pose them. The difference here is that this responsibility becomes visible, and, most importantly, open to reflection on the mathematics (learning) activity itself. Reminding Eugene that Miki videotapes him, Chichi somehow questions the relevance of his utterances to the situation. This is particularly sensible in the light of Miki first intervention to calm down the banging. Here, Eugene responds that he is aware of Miki’s recording, hence suggesting that he

considers his previous speech and gesture suitable for researching and, therefore, for teaching and learning (since this is what the researching is about). It turns out that Eugene is right, since Miki, after his “I know that,” simply continues filming (there are a few seconds of silence before Tara again addresses the classroom). She does not reprimand him for his actions, nor tells Chichi not to be concerned by the research. Unlike the banging, Miki makes acceptable this set of affirmations/transactions. Both Eugene’s explanation and confidence in being adequate, and Chichi’s questioning of what constitutes a reasonable action, also leads to new ways, for the students, to think about what they are doing. In the middle of teaching and researching, such conversations define actions possibilities, enriching what it means to learn (here by doing mathematics) for the students.

### **Toward a New Ethics of Researching**

In the previous sections I discuss how researching can be something researchers, teachers, and students do with and for one another. Along the way, I make visible in a moment-to-moment fashion how participating into research may benefit teachers and students. I now take this reasoning one step further and suggest a reflection around a central aspect of educational research: *ethics*.

Ethics is a major concern for research in education because our actions as researchers inherently involve, and thus affect other human beings. Research ethics boards tend to theorize this situation in terms of acceptable (minimal) risk, informed consent, or issues such as confidentiality, and address it by the application of codes and rules. Researchers need to carefully prepare documents for “ethical approval,” often struggling to convey the relevance and acceptability of what they do to people who know little about the actual circumstances and

requirements of the specific contexts of their work. This aspect of ethics is important in the orderly functioning of society, and proceeds in the tradition of reason to correctly adjudicate particular situations from a morality based on prescriptive principles. As researchers, we are held responsible by our peers and the community at large for the possible repercussions of our researching on the teachers and students collaborating with us, and thereby expected to cautiously consider our actions.

There is, however, another dimension to ethics that merits attention: one that situates responsibility “in the middle” of the face-to-face encounter with another. Ethics is a responsibility that we all have as “beings” because we owe our own human self to others (Levinas, 1986). As a self, I am indebted to the other whose responsive presence recognizes me as a fellow human being. And I am also simultaneously and undeniably “called upon” by that other who needs me for the same reason. This ethics makes us responsible *prior* to any action or intention summons us to respond immediately and spontaneously to some situation (Varela, 1999). That is, there is an ethics that does not separate itself from the actual life of human beings to ponder before or after the fact, or on the backstage of the human drama. Responsibility for the other in the face-to-face encounter is an ethics in-the-middle, and, hence, one that directly concerns those moments in researching where students, teachers and researchers affect one another in and through their transactions. In collecting data, we are responsible for our actions in the face of the teachers and students we meet, and we are compelled to respond to their presence and actions, to return their gazes, and answer their questions. To put it in another way: “the field of data is the whole bloody living world, turned over like a new baby birthing [... and] dealing with such realities well might not be only an institutional problem or a problem of policy or a problem of which research methodology one anonymously adopts” (Jardine, 1997, pp. 164-165).

In the moment-to-moment of researching, it is our ethical responsibility to recognize teachers and students as fellow human beings above seeing them as subjects of research. This is a responsibility from which we embodied in and through that actions that realize our very relationships with the teachers and students (Moje, 2000): Not one from which we can free ourselves by using the proper “ethics release form.”

On that basis, researching-in-the-middle finds a very strong foundation: an ethical one. Our responsibility for the other exists whether we assume or deny it (Roth, 2006a). Assuming our responsibility means embracing the observation that we always affect one another. In that view, researching-in-the-middle signifies its ethical orientation and commitment. In the middle, researchers’ involvement (from Latin *in-* ‘into’ + *volvere* ‘to roll’) enables us to be fully present, fully responsive, and fully agential. If a teacher introduction of an activity does not seem to best serve the students’ learning (and, hence, the researching and teaching), it is our possibility *and* responsibility to step in and respond, and do so in such a way that the teacher too is recognized for who she is and for what she does. If, from the perspective of a research-assistant, students seem not fully engaged in thinking aloud together, it is out of a spontaneous wisdom of what is good that she engages with them. Again, this is not to say that teachers’ and students’ have to adopt researchers’ version of teaching and learning. Rather, I stress the fact that researchers (like any individual) read situations in their own ways, and from that unique place and sensitivity, hear the call of the other in such or such manner and respond to it. For the researchers, being in-the-middle of teaching and learning, engaging with students and teachers, is in fact an ethical response to the situation in which researching takes place.

Moving us toward a new ethics of researching, these reflections interrogate the appropriateness of scientific models of knowledge production in mathematics educational

research. Traditionally supporting most of our research endeavors, is a “will to know” about specific phenomena, an endeavor to produce knowledge about students, teachers, or situations that often comes in the way of the ethical encounter with the other (Bhattacharya, 2009). With such ethics, researching-in-the-middle finds means and meaning to move from an acquisitional epistemology to a participatory world view in which we value researchers’, teachers’, and students’ co-production of the researching, teaching, and learning situation. This means letting go of any will to know and respond to the need of the other, even when our actions risk “contaminating the data.” However provocative that this may sound, this ethical take on researching conceives jeopardizing our chances to learn “exactly” what we were looking for, to make ourselves more present and, therefore, responsive to the other (with, on, for, and/or about whom we research). Instead of following protocols, and limiting the direct effect of our presence on the teachers and the students for the sake of (scientific-like) *knowledge*, the ethics of researching-in-the-middle values active, dynamic, dialogical forms of *knowing-with* teachers and students at the very instant of researching, teaching, and learning in the classroom. In this, we can expect transformation in what knowledge our activity will produce, but this change itself might just be for the best, giving us opportunities to learn more, or learn more truly: after all, what we do as researchers is not merely to *find* out whatever we are interested in, but *research*...

### **Final Remarks: The End of Educational Knowledge as We Know it?**

My observations and conceptualizations of researching-in-the-middle are still at an early stage. At this moment, clearly appears for me a need to also articulate the kind of learning that arises for the researchers themselves. Beside that, of particular interest for future research is the observation of different research settings, especially some that might not necessarily be thought

of as “participative,” or where the researchers do not make their presence so salient. In this chapter, however, I use two short fragments from a data-collection process as examples to show how researching, teaching, and learning are co-produced, while investigating the kind of learning that arises for students and teachers from this situation. From concrete, moment-to-moment observations, we see that researching-in-the-middle supports the general claim that participating in a research positively affects teachers and students. We find that researcher’s implication with teachers and students exposes them to new ways of doing and thinking mathematics teaching and learning. We observe that this happens because researchers enact new forms of doing (mathematics) education with teachers and students.

Looking across the dataset, I note that the students and the teacher expand their room to maneuver, something we can describe as a change in their knowing of what constitutes mathematics teaching and learning. I show this phenomenon by examining the actions and relations that bind teachers and students together with the researchers, hence focusing on research as creating conditions by means of which emerge new understandings of who they are and what they do. Pursuing further these observations about teachers’ and students’ opportunities to learn and how they *coincide* with researchers allowing them to be more than camera holders, we finally embrace a dynamical and ethical perspective on the co-production of researching, teaching and learning. An awareness of the “with” of researching, of being in the middle, leads us to change the way we think about troublesome or side effects related to classroom research. I suggest this affecting of students and teachers as, in fact, an ethical response to the situation in which researching takes place, which leads us to favor researchers’ transactions with those who participate in their studies, even if it jeopardizes, or transforms, what they first intended to learn about them.

Does maximizing researchers' transactions with those who participate in their studies signify the end of the production of educational knowledge as we know it? Certainly not. As a significant part of our work, interest, sensitivity, and orientation to teachers and students, we are connected with a community of researchers with and from whom our researching wants and needs to contribute, and this will remain. Furthermore, collecting data is but a moment in researching. Analyzing those data and telling those stories, certainly allows for the production of invaluable (although perhaps slightly different) knowledge. But if we really want to consider "modes of scholarship that can most powerfully impact" the people, practices or causes to which we, as researchers, are committed (Demerath, 2006, p. 110), we need go beyond working in the context of "scientific research." Identifying relevant research questions, designing modes of investigation, collecting and analyzing data, writing papers and presenting observations to our colleagues, all these can embody what we consider to be the spirit, the ethos of educational research. But at the same time, they can offer crucial opportunity for teachers and students to learn, that is, present themselves as other ways of researching-in-the-middle.

## CHAPTER 7

### **Rethinking Knowing: The Threefold Nature of Relationality**

#### **Preface**

Since I began this doctoral research, many have asked me what is so special about mathematics, and/or knowing from a mathematics education perspective, to what I articulate in my different studies. Could we not say the same thing about any forms of knowing? The question is puzzling and, in my view, *not* to be answered. On the contrary, leaving it calls for discussion, and further investigations. Indeed, it is part of my own questioning on that matter that I develop the following chapter. As a starting point, I noticed that although rationality is often thought of as the keystone of mathematical thinking, rationality is, as Bakhtin (1993) argues, necessary in any type of thinking: there are many rational domains that we can live (Maturana, 1988), and mathematics is but one of them. Inherent to mathematical activity and, thus, to mathematical knowledge, is a process in which the world and our experiences are simultaneously expanded and reduced. A mathematical perspective expands our understanding of objects or phenomena, but also reduces them to what can be mathematically rendered. This contradictory, dialogical process (although still not specific of mathematical thinking) is at the heart of what contributes to make mathematics the specific domain in which we operate during a mathematical activity.

Rationality, however, is at the basis of dominant theories/epistemology in mathematics education, including constructivism. One explanation for that is found in the fact that mathematics as reason has quite a long history in western culture. The Enlightenment conviction

that humans are rational animals, but that reason needs to be nurtured, leads to the importance of mathematics teaching for modeling and practicing reason (Richards, 2001). Hence, the Platonist tradition, through Cartesians and realist epistemologies continue in the constructivist idea that reason leads to knowledge and that mathematical objects are products of a mind that works in isolation but according to the laws of logic (Radford, 2006).

Clearly, this was not what I could observe in the students' mathematical activity. To contrast such view, develop a different understanding of knowing in mathematics, and articulate where rationality can be situated, I decided to further develop the concept of relationality introduced in chapter 5. In this study, I then offer the concept of relationality as an alternative to going conceptualizations of mathematical knowing to overcome reductions that tend to separate the knower from others, the world they know, and themselves. To do so, I discuss an episode in which two second graders look for shapes in their environment, and eventually produce a geometrical model of a flower. I articulate knowing geometrically as it arises from and is constitutive of a threefold relationality between knowing subject and: (a) the material world, (b) others, and (c) with itself. I then conclude with a reflection on how relationality frames knowing-with-in as an essential condition for rationality to take place. Connecting back with Bakhtin (1993), I follow his observation that theoretical knowledge, such as mathematics, needs to be *answerably* known. That is, I open to a reflection on ethics and responsibility for oneself, for one another, and for the more-than-human world in the moment of knowing mathematically. The following figure maps out this part of my journey:

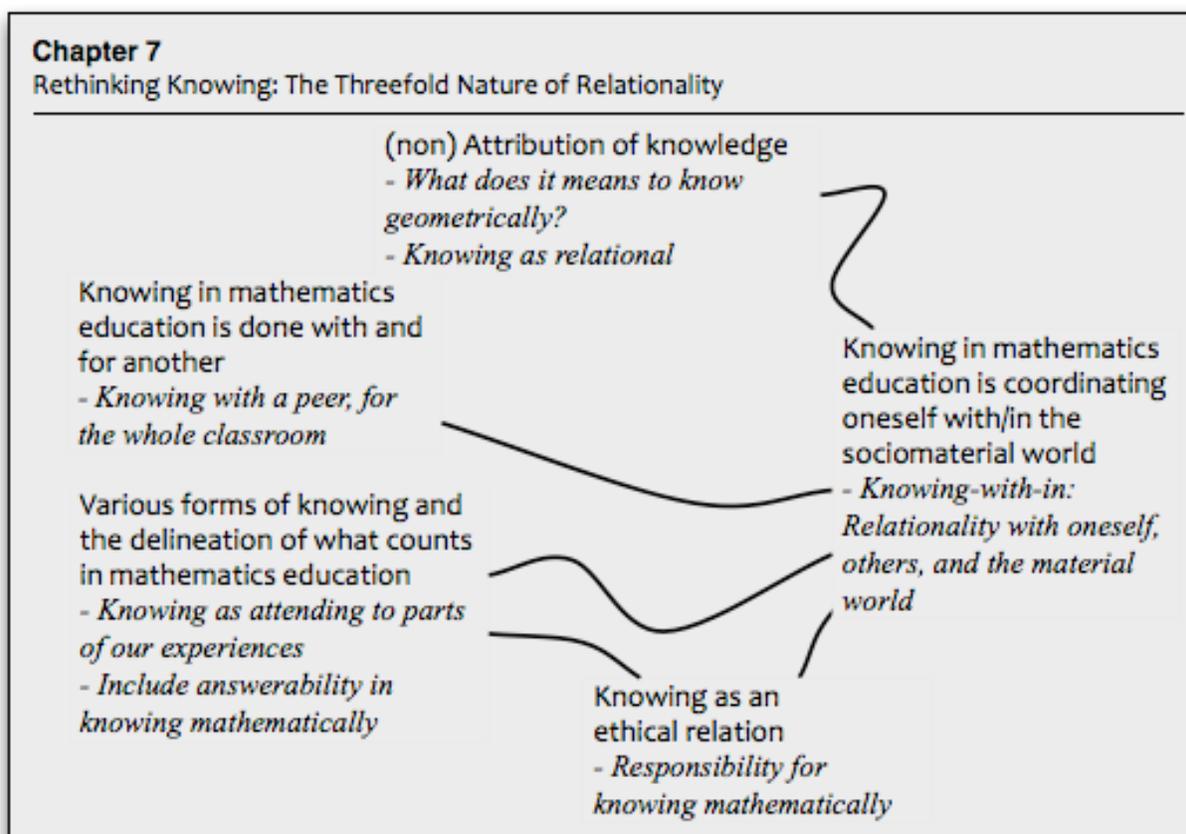


Figure 7.1: A map of my walk in “rethinking knowing”

A different version of this chapter is under review at the journal *For the Learning of Mathematics*.<sup>20</sup>

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## **Rethinking Knowing: The Threefold Nature of Relationality**

### **On Relationality**

Mathematical knowing tends to be thought and theorized in (neo-) Kantian terms of mental representation that individual knowers subjectively construct in their minds. In this chapter, I offer the concept of *relationality* as an alternative to going conceptualizations of mathematical knowing to overcome reductions that tend to separate the knower from others, the world they know, and themselves. To conceptualize the meaning of relationality in the context of the mathematics education and in terms of students' understandings, my work up to here focuses on the ways in which students' mundane, bodily actions contribute (or not) to their geometrical sense-making. I consider students' mathematical knowing to emerge from the interplay of the cultural-historical, sociogenetical and ontogenetical systems of which they are members, and articulate students' cognizing acts as the very events by which (their) mathematics recursively arises and evolves. The purpose of this chapter is to take up the notion of geometrical knowing as a form of relationality and consider how relationality then frames the distinctions we (can) make in the observation of two second grade students—Nadia and Nate—as they search their schoolyard for geometrical shapes. The episode I discuss took place in a series of activities designed to orient the students' focus on geometrical concepts, and occasion further conceptualizations of them. Using this episode, I present a moment-to-moment analysis by means of which I articulate how geometrical knowing arises from and is constitutive of a threefold relationality between (a) the subject and the material world, (b) the subject and others,

and (c) the subject and itself. In the following sections, I present each of these three aspects of knowing and then conclude with a reflection on how relationality frames *knowing-with-in* as an essential condition for “rationality” to take place.

### **First Fold: Relationality with the Material World, or Nadia Sees a Circle**

The ability to identify geometrical figures in the environment is commonly stated as a prescribed learning outcome in many K–4 mathematics curricula. What concretely happens so that the students actually make those geometrical distinctions? In this section, I show how students’ geometrical knowing arises not merely from “rational,” objective observations, but from an irreducible relationship between the knowing subject and the material world. This aspect of relationality means the absence of radical separation between observing subject and what it observes in the material world. In the moment of making an observation, we are always in relationship with an object, mediated by our knowledge, and with our situation as a whole (Freudenthal, 1983). Knowing geometrically requires the subject and the object to be in relationship because actual separation would signify the end of the differentiation process and, hence, of knowing. That is, relationality is a condition of knowing (geometrically).

To discuss and exemplify what relationality means in the context of mathematics education, let’s examine a first excerpt in which Nadia looks at the ground, then suddenly runs to a bottle lid that lies in the grass and squats down next to it. She says:

01 Nadia: I see a circle right here! I’m gonna put down a circle ((she draws a circle on her clipboard)).



Figure 7.2: Nadia squats, pointing to the “circle” she found

Nadia describes the *objet trouvé* geometrically. Using words and drawings, she bodily produces geometrical readings and writings of the world. For Plato, such actions are the very fact of reason, of human rationality, because they embody the recognition of the true, ideal circularity in the object beyond its material and mundane existence. In this view, human rationality acts on the presence of approximated shapes in the environment to deduce their pure existence. Nowadays, dominant frameworks in mathematics education contrast with the Platonist perspective and rather consider moments like these as instances of subjective experience of a knower who represents the world for him/herself and acts on the basis of those representations. From that perspective, children rationally develop intuitive theories from evidence by construing, defining, relating the material of sense perception (Glaserfeld, 1995). However, what is visible on that videotape fragment does not resemble a student thinking in accordance to logic, making definitions, deducing or constructing “circle” from the observation that a very large number of objects in her environment “strive after circularity.” We see Nadia looking for shapes, suddenly seeing a circle, and drawing another one on her clipboard. I ask, in the spirit of Freudenthal (1983): Mathematical rationality may well develop from such experiences, but what is it that makes them possible in the first place?

Observing the circular shape of the object is not a mere reaction to raw stimuli (the

physical world). Nadia walks about the school ground and recognizes something particular about a bottle lid on the ground (its circular shape). She has been walking in the area for several minutes without saying anything, although she could have seen the lid. Her sudden observation, “I see a circle right here,” shows that the presence of something does not necessarily correlate with its being there for the observer. The bottle lid only exists for Nadia, and exists geometrically, because of her way of making sense and interpreting the raw material offered to her senses. When Nadia searches for shapes in the schoolyard, she is in conditions that influence but do not determine what she observes, and what she experiences. Hence, conditions for geometrical rationality to emerge cannot be reduced to the presence of shapes in the environment on which a pre-existing rationality applies.

Nadia gives a description of the bottle lid and simultaneously defines herself as an observer making geometrical distinctions about her material world. Perception is an ongoing process of coordination between sensory patterns and motor activity in the transactions of the organism with the environment (e.g., Varela, Thompson & Rosch, 1991). This would explain why the bottle lid can be seen geometrically (or not) when Nadia engages with it. It is not sufficient to be in the presence of an object to perceive it: we need to have a certain orientation to it. To make her observation, Nadia’s attention needs to be oriented toward the ground, to the objects she might see there, and to the geometrical shapes that she might recognize in those objects. What is observed is contingent on the observer’s means and intentions, and only certain objects are visible from a second grader’s vantage point. Within that range, when Nadia touches the bottle lid with her pen and says “I see a circle right here!,” she identifies the presence of “something” on the ground, and at the same time makes a geometrical observation about it. A “surplus signification” arises as what she encounters is valued or not, acted upon or not. It is in

that sense too that the world is not something given to us, but something we engage in by moving, touching, breathing, and so on. In the episode, Nadia attends to one aspect of all that is around her. In doing so, she produces herself as an observer with that particular orientation, both for the object of observation (looking at the bottle lid) and what is observed about it (attending to its shape, rather than focusing on its colors, its material, or the fact that it lies in the grass, etc.). What we see is not simply a construing of the material world, but the bringing forth of an object within its geometrical context. To know geometrically, students' need to attend to objects in a certain way, and this act of observing is not distinct from the perceived object and its features.

That is, there is a mutual determination of the observer and what is observed, and that mutuality is a very first condition for geometrical distinctions to take place. Perception of shapes in the environment depends on the correlation, the ongoing relation, between the material world and the cognitive activity that constitutes us as observers. Observing the circular shape of the object is the ongoing structuring of an experience. In knowing, subject and object (observer/observed) are not independent entities, but co-emerge and co-evolve. It is by these operations that a person brings into the realm of attention components of the surroundings. Hence, knowing is not simply about rationally making connections, but assumes that we are already connected, always in relationships. Geometrically knowing a bottle lid proceeds from Nadia's dynamic existence in the material world, with/in a material body, and, through this relation, not separated from her environment.

These observations put forth a first understanding of what relationality means in the context of mathematic education. More than from reasoning about objects, knowing geometrically arises from the process of coordination by which bringing forth an object coalesces with its geometrical knowing. When Nadia distinguishes a shape in her environment,

she co-exists with her findings, and in so doing she realizes a relationship with the material world. Here, objects in the environment exist for Nadia as she recognizes them, and exist as the locus of geometrical shapes because Nadia attends to them in the particular context of *this* lesson. Moreover, she can learn about the circular nature of the shape of an everyday object, bring it into the realm of mathematical thinking, only because she is already in dynamical relationship with this material object, because she already knows it, and thus can direct her attention to it.

Inasmuch, Nadia's geometrical knowings does not simply result in her being in relationship with/in the material world: relationality actually is a condition of knowing, but only a knowledge-mediated relation makes this a geometrical relationship. This knowledge-mediation of the sensuous experience, however, precisely constitutes the pedagogical intention, the very object of learning. Actually, sending students to search for geometrical shapes is only reasonable if we believe (a) in the possibility for them to find those shapes, and also (b) that they do not generally observe them (otherwise there would be nothing for them to learn). Although she looks for shapes, Nadia only finds what she intends as she intends what she finds. Searching for shapes and suddenly saying "I see a circle right here!" result of a subject-object relationality whereby Nadia's "understanding [and consequent knowing] of self is not abstracted from the world which contains it but, rather, is the world" (Davis, Sumara & Kieren, 1996, p.154). The geometrical knowing proceeds from the full situation of Nadia as subject/organism, and the bottle lid as object in its environment: both emerge within knowing, while the knowing itself emerges from them. To put it another way, Nadia does not exist as an independent and autonomous knower, but as an implicated (from Latin *implicatus*, 'folded in') subject knowing with-and-in the

material world. The sense-date that our biological, physical body produce comes from a materiality, a biotic compatibility that allows us to enter in the back and forth dialectic of interpreting that “ontological knowing” in (cultural-historically defined) mathematical terms, and give raise to the world – notice and attend to objects – in the “conceptual” knowledge-mediation. It is because she is “of the same wood” (or better: *of the same tree*), in continuous relationship with a material world, that Nadia can emerge with the world in the act of knowing.

### **Second Fold: Relationality with Others, or Nadia and Nate Find another Circle**

Relationship with the material world clearly grounds geometrically thinking with/in that world, but can this relationality suffice to the emergence of mathematical rationality? Dominant approaches to mathematics education suggest that students (have to) construct their own mathematical reality. In the last decade, observations around the importance of the social in mathematical activity attempted to include the constitutive role of others in knowing. From that perspective, coming across objects and noticing structures or properties, one gradually organizes all his/her experiences by validating them in a community of knowers with whom this organization is negotiated on a rational basis (e.g. Ernest, 1993). Others adopt a different perspective, and attempt to start from the very collective – even social, or cultural – nature of knowing (so, in some cases, as to also consider its historical development). In such views, knowledge tend to be conceptualized as “produced by cognizing subjects who are, in their productive endeavours, subsumed in historically constituted traditions of thinking ... a subject that thinks within a cultural background” (Radford, 2008c, p. 11). Ongoing practices then become the locus of knowing (as they continue the tradition of thinking the world in a certain way, suggest ways of perceiving it) in place of apparently a-historical and a-cultural negotiation

of meaning.

In this section, I go beyond perspectives that somehow remain distant from actual encounters and attend to the constitutive role of the other in concrete act of knowing. I do this by considering that other not only as (rational) opponent or cultural mediator, but as a knower of the very same nature (so to speak), who enact and incarnate knowing as communality. That is, I see how knowing is inherently *knowing-with*, hence arising not only from rationality or cultural practice but from relations with others who is present. From this view, any personal knowing geometrically is a concrete realization of collective possibilities of knowing enabled by the continuously evolving cultural history of mathematics in which another is constitutive. Hence, relational conceptualization of knowing departs from perspectives that either locate the individual as the centre from which sense-making occurs, or contend that it arises from rational negotiation of meaning. But it also offers a change in focus from sociocultural perspectives in which “knowledge is not seen as something coming from *within* (a kind of private or subjective construction endlessly seeking to reach a culturally-objective piece of knowledge) but from *without*” (idem, p.12) to the very prefix *with*.

In relations with others, knowing geometrically arises from those concrete communicative actions where one’s observations call upon and respond to another’s observations, producing alignment to certain (geometrical) distinctions thereby realized as knowing.

To develop my observations on relationality with others, I return to the lesson in which students were sent to find shapes in their environment. Immediately after drawing a circle on her clipboard to record the observation, Nadia takes a few more steps, followed by her partner Nate, still looking at the ground. Upon stopping again, she places her hands on her hips as she makes the following observations:

02 Nadia: Flowers have some sort in it ((pointing to the ground with her pencil))

03 Nate: Circle! ((looking down))

04 Nadia: Circle ((rotating her pencil, still pointing with it))

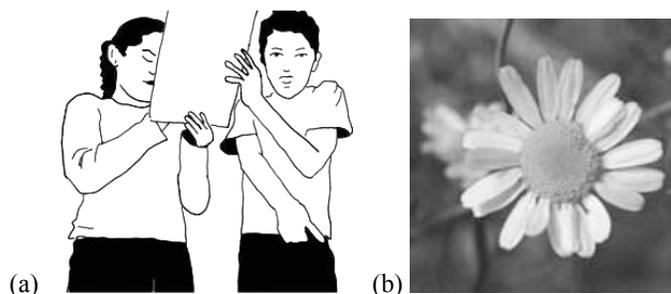


Figure 7.3: (a) The students find shapes and record their observation; (b) a flower similar to the ones they observed

Pointing to flowers or a bottle lid, using words and gestures, Nadia and Nate speak and gesture for themselves and for one another. Nadia names and categorizes the natural objects as “flowers” and the presence of something in them. Nate responds by stating “circle!” In uttering the same word in constative manner, Nadia confirms Nate’s hearing of her own turn as making the observation of a circle in the flowers. She further affirms and confirms Nate’s hearing by means of a hand gesture. Making their visual observations and conceptualizations (e.g. flowers and the presence of shapes in it) available to one another, the two students make common – i.e., *com-municate* – what can be seen and thought.

By addressing and responding another, the students are not only thinking, but thinking and being aware together. Consciousness, as the etymology of the word suggests, from the Latin *con-* “with” + *scire* “to know,” is possible only thanks to the presence and existence of others; others are the condition for knowing anything at all (Leont’ev, 1978). We become (knowledgeable, rational) human beings *with* others and because we are in relationship with them, and for that reason, knowing mathematically is necessarily knowing-with others (e.g. Radford, 2008b), because mathematical understandings always occur through concrete actions of human beings in

a social world. Mobilizing communicative resources presupposed to be intelligible, the students here name and gesture the concept of “circle,” and that intelligibility entirely depend on the relationality with others. Thinking geometrically may be something we do for ourselves, “but this thinking is done, at least in anticipation of, communicating with others and acting in a community of others interested in mathematics” (Kieren, 1995, p.7). In the case of Nate and Nadia, we see the two students mutually orienting themselves to the flowers and to geometrical shapes. The presence of circles in the flowers is brought into awareness in and through Nate’s confirmation of Nadia’s observation and Nadia’s confirmation of Nate’s interpretation. Both affirm to one another that something is visible, and relevant. From the complexity of visual stimulus coming from the material world, the students align themselves on what they see, which becomes what they see because of this alignment. Knowing about circles and flowers is something they do together, relationally.

That mutual alignment, however, is not merely the fact of some rational negotiation of meaning. The students make distinctions about aspects of the flower in and through their conversation. From their co-elaboration results a shared visibility of something that is not simply “out there” and about which the students develop explanations, elaborations or justifications to come to some agreement. Before anything like this can happen, distinctions are *made*, the students brought forth those distinctions in their recursive and responsive articulations. Nate, picking up on Nadia’s utterance, creates with her the experience of allowing geometrical words to accrue to a world always already shot through with (infinite possible) significance, including flowers. Nadia similarly contributes by specifying with an iconic gesture what the index word “circle” denotes. Here, then, the students’ knowing about the circular shape of flowers is entirely intertwined in their coordinated communications. This requires the words they use to be true

bridges that constitute the speaker and listener, signatory and counter-signatory, as an irreducible unit. That is, their knowing is not “negotiated” nor “taken-as-shared,” but rests on the always already existing common understanding required by the use of words. And it is also a contribution, an offering, the meaning of which irreducibly depends on the response of another.

An important aspect of the relationality with others of students’ knowing thus rests in the re/production of geometry as a culturally enabled way of knowing. Finding shapes in their environment, Nadia and Nate contribute in the production of a geometry lesson, but in this again concretely enact relationships with the other students and with their teacher, *as* with society (or culture) as a whole. In that sense, knowing the flower mathematically is also knowing with others. Nadia and Nate observe for/with their teacher and peers, and, to count as knowing, their actions must correlate with what others count as geometrical.

Rational discussions involving teacher and students to delineate what count as knowing geometrically are possible, because of the actual presence of others, and because talking geometry is always already rooted in a commonality of what geometry can be, otherwise there could be no such discussions. Such commonality arises from students’ collective participation in activities that come to be geometrical by dint of the contributions of the teacher (Roth & Thom, 2009b) who herself learned to talk geometry with others, and so on. Nadia and Nate’s perception of the flower takes place because geometry exists in the form of historically developed and preserved ongoing conversations between people. A geometrical orientation to the world consists in the production and the reproduction of a cultural way of knowing, and thus implies the presence of others, but also necessitate a presence to acknowledge it as such. To perceive the flower (mathematically), Nadia and Nate draw on this collective dimension to determine the content, the meaning and the signification of their bodily experiences. The other-dimension of

relationality is a condition for knowing geometrically because Nadia and Nate can attend to the flower in (what we recognize as) a culturally relevant mathematical way. Serving them only a few minutes later to share observations with the teacher and the other students, the Nadia and Nate's knowings and doings are collective because they take place in, and contribute to, their classroom mathematical search for geometrical shapes in the environment.

### **Third Fold: Relationality with Oneself, or Nadia (as Nate) Creates a Geometrical Model**

Being in relationship with the material world and with others is constitutive of Nadia's and Nate's geometrical knowing. There is, however, another aspect in the emergence of mathematical rationality that is not clearly rendered from this perspective: the relationship of the knowing subject to itself. There are moments where, as it seems, I do not have another in my proximity, where I have the conviction of thinking mathematically for and by myself only. That self who seems to be ever present to me, has in fact the very texture of another with whom I do not coincide (Ricoeur, 1992), but converse. It is from this relation, which requires the objectification of the subject, who appropriates the result of the previous process by means of subjectification (Hegel, 1977). The self-relation of the knowing subject is a condition for the knowing consciousness and its development. Hence, this third dimension of relationality offers an alternative to perspectives that substantialize students as subjects whose knowing changed under the exercise of rationality. Rather, relationality suggests attending to how students recursively delimit the field of their experiences in recursive processes of objectification and subjectification (Radford, 2009a, b) to bring mathematical understandings into being and, in that process, develop that particular kind of rationality.

I return to Nadia and Nate and concretely examine what relationality with oneself tells us

about students' geometrical activity. Having together come to see the presence of circles in the flowers, Nadia and Nate continue observing them. In the last part of the episode, we see them not only expanding their recognition of a shape but also noticing other geometrical aspects in the flowers, and producing a model:

05 Nadia: And then they have the little roly things petals ((gesturing an oblong in the air, and then moving to the clipboard, where she starts drawing))

06 Nate: There are little circles in it ((looking down at the flower, then joining Nadia in completing the drawing, see Figure 7.4))

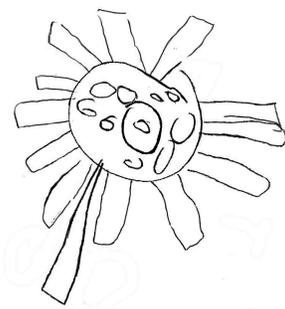


Figure 7.4: The students' model of a flower

Nadia and Nate now make new observations, calling the petals “little roly things” (Nadia later explains them to be cylinders) and mentioning “little circles” (in what turns out to be the flower heads). Both students then join in the production of a drawing (Figure 7.4) in which we recognize a flower. In doing this, the students attend to the flowers in a particular way, making observations that connect everyday experiences with mathematics/geometry. Their search for shapes in the environment now looks more like an investigation, a systematic inquiry that could be interpreted in terms of a rational, methodical self. Their “thinking” transforms itself in the process of objectification (by means of speaking, gesturing, drawing) into material entities and in the concurrent reverse process of subjectification of these material entities. It is precisely in this concurrent double movement that thinking and conceptual development take place (Vygotsky, 1986). By finding shapes in their environment, Nadia and Nate enable the two processes to occur

and their thinking to develop. Looking for shapes, the students identify one, and then another. At some point, Nadia finds a circle (bottle lid) and then observes that flowers too “have some sort in it,” and now she also notices cylindrical shapes she calls “little roly things.” Through Nadia’s history of coordination with the material world, with Nate, but also *with herself*, a certain form of awareness develops. In the never-ending flow of her experiences, a pattern of thought and communicative actions emerges, a coherence in which we recognize the expression of rationality. For Nadia and Nate, there is recursion in the search for geometrical figures in the environment as the process of bringing forth a world of geometrical significance keeps repeating itself. When they observe the flowers, they distinguish circles, and keep making further geometrical distinctions in the same flowers, and, thereby, in their environment. In and through Nadia and Nate’s thinking and observing, patterns emerge in what comes about among all that is offered to the senses (e.g. not the color or the size of the flowers): External stimuli call upon one of the possible paths for their development as individual organisms in the space defined by their own internal dynamical structure.

Hence, it is significant to observe that Nadia and Nate do not simply make observations but relate these observations and produce a model that expands their experience of the flowers. When Nadia and Nate notice and attend to the flowers in the grass, they distinguish circles in the flower heads, petals as cylindrical, and produce a sketch (see Figure 7.4) which in return, expands their experience of modeling about the flowers. Their modeling activity enables Nadia and Nate to make observations in particular geometrical ways. The circles and cylinders that they distinguish affect how the two students make sense of the flowers: For Nadia and Nate, making such distinctions specifies a criterion of distinction which indicates what they express to one another, and how they constrain the object of their observation. When Nadia and Nate create a

model of the flower, not only once but twice they focus on what can be rendered from a (specific) geometrical perspective, and in emergently different ways. Nadia and Nate repeatedly describe the flower in geometrical terms, each repetition constituting a difference, as it occurs against a background that includes all previous iterations. In this manner, the two students as perceivers are in relation with their own perceiving activity, continuously determine what counts as relevant. Nadia observes a circle and then notices “little roly things petals.” Narrowing the field of the experience to the geometrical shapes of the flower expands the range of experience, and this expansion is possible because the students continually draw on their geometrical framework to focus and shape what they attend to. Nadia’s thinking develops as it relates recursively with itself.

Inasmuch as we follow students walking the schoolyard and finding shapes in their environment, we recognize in the emergence of a pattern of thought and communicative actions. In and through those patterns, knowing takes place as a part of students’ ontogenesis (i.e., development as organism). Recursively, Nadia’s knowing provides itself the conditions and the means by which further knowings emerge as she makes distinctions (and distinctions of distinctions) that delineate the observational posture that she enacts; and this is the same for Nate. Hence, we do not consider students’ knowing to merely rationally organize itself as it progressively organizes the world, but articulate students’ relationship with their own selves as the conservation of an operational congruence between the nervous systems and the medium(s) in which it exist. Through Nadia’s and Nate’s history of transactions with the socio-material world, they develop their ability to relate geometrical ideas and experiences, make relevant distinctions to the purpose at hand, and delimit the object of their observation. A mathematical/geometrical form of rationality then is what actually results in the form of a

consistent pattern of thought and communicative actions, which lies in the relationality with oneself constitutive nature in knowing.

### **Knowing-With-In: An Enfolding Relationality**

In this chapter, I articulate *relationality* as a different way of thinking about knowing. Relationality, as I present it here, contains three folds, three forms of relations that are integrally related. This threefold nature of relationality is captured in the second concept that I offer here: *knowing-with-in*. Thus, knowing is always *knowing-with* others, always *knowing-in* a context, and always *knowing-within* the subject of knowing. Describing flowers or a lid, knowing and doing mathematically is something in which Nadia and Nate are together, and with the material world. It is in and through the same relationality of *knowing-with-in* that recursive coupling takes place between the students, with their material environment, and within themselves. This coupling enables and sustains the process by which features of the material world become geometrically meaningful to them: relationality with the material world, others, and oneself are inseparably intertwined, mutually influential, and simultaneously brought forth. Relationality we *unfold* actually *enfolds*, by three times enveloping, the students' geometrical knowing.

Over the last 15 years or so, mathematics education consistently moves toward broader understanding of what it means to know mathematically. Theoretically speaking, key to those new perspectives is the inseparability of knowing with doing, but also with the nature of human existence in the interplay of the cultural-historical, sociogenetic and ontogenetic levels: “all knowing is doing, is being” (Davis, Kieren & Sumara, 1996, p. 155). Through this association with *being*, the relationality of knowing introduces a crucial move, as it reflects one of the most fundamental advances in the philosophy of ontology. Thoroughly pursuing Heidegger's work on

the nature of Being, philosopher Jean-Luc Nancy (2000) articulates how human existence is always at the same time Being-there (*Dasein*) and Being-with (*Mitsein*), that is: Being-there-with (*Mitdasein*). Nancy develops the irreducible co-essentiality of the two dimensions and shows that it leads to an entire reformulation of the sense of Being enfolding both its internal and external nature: Being *singular* as an essential property of the individual, and being *plural* as co-existence with other human beings. The relationality of knowing I articulate in this chapter offers a similar reformulation of the sense of Knowing. Theorizing the irreducibility of Knowing-with-in, relationality emphasizes how knowing is always knowing *with* an other, and *within* a situation. Just like being singular plural goes beyond commonality in the sense of individual and collective cooperation to consider co-propriation, *viz.* the essential conjunction by which individual and collective are united and mutually define one another, the relationality of knowing goes beyond the scope of, and in fact is conditional to, individual and collective rational deliberation.

In a more “practical” manner, the threefold nature of relationality has important methodological repercussions for mathematics educational research. It means, for example, giving attention not merely to whatever we find to be present in students’ environment, but to distinctively examine what the students themselves articulate as resources for their mathematical knowing. It also means analyzing mathematical communication not from the perspective of individual expressing “conceptions” and the like, but as co-production of mathematical understandings. Pointing to but another one of those consequences, the relationality of knowing demands to give attention to the *process* by means of which certain forms of knowing are produced, as oppose to try and make assumptions regarding what students know or do not know.

In this study, I offer *relationality* and *knowing-with-in* as alternatives to discourses that tend to focus on rationality through rules and norms, atomizing the classroom, and reproducing

the individualist tradition that sees knowledge as something personally “constructed” by rational, autonomous and culturally free individuals. Relationality and knowing-with-in express situated and situating knowing in action, and articulates an essential condition for any kind of knowing to take place.

### **Answerability in Knowing Mathematically**

I often wonder what it is that we do, when we do mathematics, and only recently understood why the question is so important to me. It is Russian philosopher M. Bakhtin who explains, in a essay that only partially survived the 1920 Soviet Russia regime, that theoretical knowledge, such as mathematics, needs to be *answerably* known (Bakhtin, 1993). What Bakhtin sketches is the need for awareness to what we do by knowing things in a certain way.

Answerability, then, demands an acknowledgment of our responsibility for doing so in the very act of knowing:

...this answerability [is] accomplished by everyone who cognizes, insofar as he accepts answerability for every integral act of his cognition, that is, insofar as the act of cognition as *my* deed is included, along with all its content, in the unity of my answerability, in which and by virtue of which I actually live – perform deeds. (p.12)

Knowing-with-in opens to answerability in the three-fold nature of relationality in knowing. Acknowledging our responsibility precisely goes to the recognition that knowing mathematically enfolds oneself, others and the material environment and proceeds from one’s relationship with those three dimensions. The “integral act of cognition” composes ethical responsibility for oneself, for others, and for the more-than-human world. That is, we have to develop awareness, an ethical *know-how* (Varela, 1991) to respond to our answerability in the

very concrete acts in and through which we know mathematically. When recognizing shapes in the environment, when naming that experience and orienting others to it, when producing oneself as a geometrical knower.

Bakhtin (1993) writes: “The world of content/sense is infinite and self-sufficient; its being valid in itself makes me myself useless, and my acts of deeds are fortuitous from its standpoint” (p. 43) and the only way out of that indifference is the “actual acknowledgment of [one’s] actual participation” in knowing mathematically. Mathematical activity takes place in concrete acts, (which includes talk, but also thoughts and emotions) and as such is something they do with and for one another and material world. With Bakhtin, theoretical knowledge must become *knowledge for me*. For me, for myself and for the more-than-human world, I am asked to acknowledge (Bakhtin uses the term *uznanie*, “to become conscious”) the fact that it is *me* who subjects the flower or the lid to mathematical thinking, that I do this with and for others, producing the world as I produce myself with/in it. In the absence of such answerability, I ignore (or even deny) my unique participation in the world. If knowledge is not my knowing, I make myself instrumental to it and act *as if* I was possessed by some immanent necessity in my material environment (“this is the way things are”), in some domain of culture (“this is what others impose on me”) or within myself (“that’s just the way I am”). Only when I recognize knowing as my deed do I find my unique, necessary place with-in the sociomaterial world.

When I know a flower mathematically, this act is a deed, something in which I am answerable even before the knowing act itself. This is because knowing is already being in relationship (with oneself, others, and the material world). The three-fold relationality of knowing is conditional to the act, while at the same time realizes in it. For that reason, I can simultaneously *be* with/in the world, myself and others and *become* with/in them; I can respond

to the sociomaterial world before me, and setup possibilities for what will come next.

Concretely, I can recognize a circle in a flower, and in that open up for more geometrical observations, but this is also how I see a circle in a bottle lid and *move on* to try and find other shapes and objects without picking up the trash.

Answerability in knowing is not to take place theoretically, once and for all, but to become part of all acts of knowing, of all our mathematical doings: It needs to be nourished and nurtured through a journey in which it comes to be an embodied, spontaneous attitude. Like Nadia and Nate, we rarely act, in concrete situation, out of deliberation or conscious appreciation, and even then, we are subject/subjected to the specific relations we create by knowing in a certain way. To put it differently, as Dewey (1929) once affirmed, “the more sure one is that the world which encompasses human life is of such and such a character (no matter what his definition), the more one is committed to try to direct the conduct of life, that of others as well as of himself, upon the basis of the character assigned to the world.” (p. 414). Looking at the bottle lid or the flower, the students are able to quickly move in a space from which they apprehend them in terms of shapes. As students grow mathematically, such readings of the world subtly permeate their overall orientation to the world, because in learning mathematics, they *become* mathematical, and not simply “acquire” knowledge and competences they can use or not as they please.

For mathematics to be answerably known, it is crucial that these aspects become an integral part of the teachers’ and students’ activities when thinking and acting mathematically. Indeed, we acquire ethical behavior like any others, as we grow up in the society, learning what we are supposed to be in order to be accepted as participants in our communities (Varela, 1999). And of course, it is our responsibility, as teachers and educators, to do the same: Embody answerability in mathematical knowledge in our encounters with pre-service and in-service

teachers, and with our colleagues. Ethical living “is based on the pragmatics of transformation that demand nothing less than a moment-to-moment awareness of the virtual nature of our selves” (p. 75), that is: Resistance the temptation to see oneself, others, and the material world as finite, independent entities. Rethinking knowing with ethics, and away from Kantian categorical imperatives which attempts to settle ethical ought once and for all (“Always act in such a way that [...] the maxim of your action should become a universal law”), we see that even children’s geometrical knowings (like Nadia and Nate’s) already calls for answerability. From here, knowing what we know about knowing mathematically, making this part of the production and reproduction of mathematically knowing with-in, is simply... up to us:

*The knowledge of knowledge compels. .... It compels us to realize that the world everyone sees is not the world but a world which we bring forth with others. It compels us to see that the world will be different only if we live differently. It compels us because, when we know that we know, we cannot deny (to ourselves or to others) that we know ... It is not knowledge, but the knowledge of knowledge, that compels. It is not the knowledge that a bomb kills, but what we want to do with the bomb, that determines whether or not we use it. (Maturana & Varela, 1987, p. 245-247)*

## CHAPTER 8

### Going Back: There's No Place like Home

I have come a long way and, I hope, so did the readers who followed me, from chapter to chapter. If this dissertation (as a whole) is, as the title suggests, *An Epistemological Journey in the Day-to-day, Moment-to-Moment of Researching, Teaching and Learning in Mathematics Education*, then the Journey, somehow, must come to an end. Or does it? As Maturana (2002) did when he was studying colour vision in 1965, writing this chapter made me realize that I need to ask the question “what happens to me?,” as an observer, when I say things about researching, teaching and learning in the day-to-day, moment-to-moment of mathematics education. Reflecting on my situation as a researcher researching research, I offer yet another contribution to the understanding of “how we know” in mathematics education, here from the perspective of a graduate student. Interestingly, doing so also makes this dissertation “as a whole” an integral part of that whole. I do this, in this chapter, by eluding from writing a new summary of my work, drawing implications, and taking the dissertation to an end. However, I do address the questions of summarizing, implying and concluding, but creating a “loop/hole” to where it all began: right in the middle of this place I call “home,” a place I realize I never left.

#### How (Not) to (Not) Conclude a Dissertation

From:	Jean-Francois Maheux <maheuxjf@uvic.ca>
To:	Wolff-Michael Roth <mroth@uvic.ca>
Subject:	Last chapter!
<p>Hi Michael!</p> <p>You must be on your way back to Victoria? Hope the flight is not too painful! :-)</p> <p>I am sending you here the first version of my last chapter. It's pretty short (under 2k). I personally like that: so much as already been said... I wrote something along the lines of our discussions. If you can take a moment to have a look and tell me what you think, that would be awesome!</p> <p>Cheers, JF</p>	

## Chapter 8

### ~~Conclusion~~Opening

What have we learned from this wondering and wandering around “How we know” in the day-to-day, moment-to-moment, of mathematics education? In this dissertation, I presented five studies that investigated researching, teaching and learning. Together, these studies show the interdependence of students’, teachers’ and researchers’ actions in/as they (re)produce mathematics education. Weaving together the five emerging themes I “mapped out” for the reader several time, this dissertation renders in many ways the question of “knowledge” as a concrete act constitutive of and constituted by human experience.

That is, I ask very broad and fundamental questions. I wonder how knowing (geometrically) is at the same time cultural and embodied. I explore how, as teachers or researchers, we can then attend and contribute to knowing in mathematics education. I examine what it means for knowing to be always and already knowing “with,” and ask similar questions about researching, teaching and learning. Each and every time, I do this by looking at concrete acts in their minute details. That is, I endeavor to bring about the complexity of ideas and situations not by reducing them to something simple, but by providing entry points and a way to navigate that complexity. I try to use little things and speak to big issues, convinced that:

The *summum* of the art, in the social sciences, is ... to be capable of engaging very high "theoretical stakes" by means of very precise and often apparently very mundane, if not derisory, empirical objects (Bourdieu & Wacquant, 1992, p. 220)

From that perspective, I realize in writing this last chapter, it would make little sense to “conclude” in the way dissertations and academic writings typically do. On the one hand, I certainly do not want to bring my work to an end, or arrive at a final judgment, that is “to shut completely” (conclude comes from the Latin *con-* ‘completely’ + *cludere* ‘to shut’). I do not want to move on, or allow the reader to move on, to something else. I want us to keep going! I want this piece of work to be a stepping-stone, not a brick in the wall. All in all, reporting here ready-to-use “findings,” chewing up my work into little bits of knowledge would be contrary the whole spirit of what I have discussed page after page. It would be saying one thing, the irreducibility of the students’, teachers’ and researchers’ actions I observed, and doing another: ‘essentializing’ the walk instead of keeping the focus on the process itself.

There is, of course what Bourdieu describes as a “bureaucratic state demand for social knowledge” (p. 237). As a graduate student, I am expected make a clear contribution to my field of research, to be “scholarly.” In most cases, this is heard as a requirement to exhibit new knowledge: A very modern epistemological inflection to explain the world so as to conquer and transform it (Reiss, 1982). Drawing “implications” and telling the reader what he/she should take out of one’s academic labour is the common way to demonstrate that considerable intellectual work has been done so as to ensure the conferral of a doctorate. In this (re)producing a certain way of doing mathematics education in its “research” dimension.

Certainly, if I am but a little successful in my effort to trigger change in my reader’s gaze throughout the whole of this epistemological journey in the day-to-day of mathematics education, clearly it appears that we need something different. Can we think about research and scholarship in terms that break from our western obsession with knowledge, our attempts to make people, objects and situations some “things” to be understood? A break from a view where knowledge is essentially and fundamentally a form of assimilation, appropriation and control leading to the

suppression of alterity (Levinas, 1985)? Is there a way not to conclude, but instead open a dialogue?

We are now familiar with the observation that utterances are necessarily produced with/in a particular context that is always social, or, to put it another way, that an utterance is always addressed to *someone* (should it be oneself) and that, in this, the listener is an integral part of the speech-act. In the philosophy of language, we describe this as the “social evaluation” of the utterance (e.g., Voloshinov, 1973). What I am offering here as a dissertation constitutes such an utterance, both as to its content-sense, and as a communicative act responding to other utterances, and creating demand for others to answer. As a whole, the text in and of itself—including the present discussion—is already part of a dialogue. It does not need (and actually cannot really offer) a self-containing “conclusion” to finalize it. The completion of the utterance belongs not to the utterer, but instead to the reader/listener who realizes it as such. Whether this piece of text counts as a dissertation, a “good enough” moment of researching in mathematics education, does not belong to me. It belongs to you, the reader, to complete my utterance, providing the social validation that will give it this special value. This dissertation is not only mine, but also a social phenomenon in the same way that what a teacher and a student says in a conversations cannot be attributed to isolated individuals.

Is this to say that I do not have responsibility in this final chapter? On the contrary: we are always accountable for our speech-act; responsible to answer what was said before, and to set up the next turns in the conversation. In conjunction with its function in the unity of social life, my text will count as a dissertation depending on how we (want to) define mathematics education (research) as a community: culturally, historically, here and now. As the producer of the utterance, it is in my power to try and set up the reader so that he/she will read me in a certain way. I can buy into a traditional model of concluding in educational research, or I can be consistent with myself

and instead insist on the irreducibility of what I have written. I have the possibility/opportunity to argue that the dissertation as a whole cannot be explained in a short conclusion, just as the complexity of researching, teaching and learning cannot be pinned down in any framework, or study. I can recall that the map is not the territory, and that when one wants to offers a way to look at “how we know” in mathematics education that breaks with dominant epistemology, reifying this “new way” in the manner of traditional knowledge is nonsensical. That is, I have the alternative of doing just what I am attempting to do here.

When I do this, I am not simply refusing to conclude, but actually explicitly offering my work as a “turn taking” in the ongoing conversations in and through which we produce and reproduce, form and transform, commutate (from *com-* ‘altogether’ + *mutare* ‘to change’) and conserve (from *con-* ‘together’ + *servare* ‘to keep’) mathematics education. I open the dialogue not by summarizing what I have said, but making clear that I have spoken. I demand to be heard in the whole of what I have offered: A journey in which *engaging* with high theoretical stakes by means of mundane empirical objects changes the way we look at the social world. As part of this turn-taking, I respond to many other utterances, thereby situating myself within others’ scholarships. My response accepts some ideas and refuses others, and now demands the reader to do the same: questioning, taking up and accepting, or even rejecting what I have done. Therefore, he/she will acknowledge the dissertation in/as present, and allow the dialogue to continue.

Before finally opening the conversation to the reader, I want my final words to go to another salient aspect of my work, which includes the role of metaphors in my writing. Both in my reflection here and in the central chapters themselves, metaphors offer a special way of thinking in mathematics education. Many times I have called this dissertation a “Journey,” and even entitled it that way. I know of no better way to communicate the importance of metaphors than this excerpt

of a conversation between Fritjof Capra and Gregory Bateson (Capra, 1988):

“Logic is a very elegant tool,” [Bateson] said, “and we’ve got a lot of mileage out of it for two thousand years or so. The trouble is, you know, when you apply it to crabs and porpoises, and butterflies and habit formation” – his voice trailed off, and he added after a pause, looking out over the ocean – “you know, to all those pretty things” – and now, looking straight at [Capra] – “logic won’t quite do ... because that whole fabric of living things is not put together by logic. You see when you get circular trains of causation, as you always do in the living world, the use of logic will make you walk into paradoxes.”

He stopped again, and at that moment [Capra] suddenly had an insight, making a connection to something [he] had been interested in for a long time. [He] got very excited and said with a provocative smile: “Heraclitus knew that! ... And so did Lao Tzu.”

“Yes, indeed; and so do the trees over there. Logic won’t do for them.”

“So what do they use instead?”

“Metaphor.”

“Metaphor?”

“Yes, metaphor. That’s how the whole fabric of mental interconnections holds together. Metaphor is right at the bottom of being alive.” (p. 76-77)

I began this dissertation with an exergue making another metaphor: That of a road in the country which comes into being as many people use it. There was never a road, but as we start walking, some routes commence to exist, while others may be abandoned. In the last three years, and with/in this dissertation, I have laid out paths in walking. I now offer these trails as a whole, as a way to walk the walk of researching, teaching and learning in mathematics education. As a unitary, irreducible utterance, I propose here to the reader an opportunity to elaborate or correct, refine or expand, reject or redraw those paths and what I have said. That is, I provide an active link between previous turns in mathematics educational research and those that will follow. A path that may become a road... provided that we choose to walk on it.

From:	Wolff-Michael Roth <mroth@uvic.ca>
To:	Jean-Francois Maheux <maheuxjf@uvic.ca>
Subject:	RE: Last chapter!
<p>I took a quick look. This is one part of the final chapter . . . because you do your PhD in education, all questions will be about "so what . . .?" What recommendations do you have for mathematics education? What should we change? etc.</p> <p>Have a subsection on briefly summarizing, one on implication, one on conclusion   opening, because, according to Bakhtin, there is a continuous death and birth, ending and beginning, etc. What you are doing right now is negating the preceding convention, and this does not sublimate the opposition of conclusion and beginning, you want to sublimate (overcome and retain) the opposition, which you would in the proposed manner.</p> <p>By the way, if you feel you can creatively use our exchanges, please feel free to do so, in the way you use quote or however you make the attribution. We have had very much a conversation, and if you want to conclude   end with conversational materials, please feel free to do so. :-) Cheers, Michael</p>	

From:	Jean-Francois Maheux <maheuxjf@uvic.ca>
To:	Wolff-Michael Roth <mroth@uvic.ca>
Subject:	RE: RE: Last chapter!
<p>Ok, I will try and "sublate" the opposition, and bring in some "recommendations," but I'm really not comfortable with that. Telling people what to do, what to retain... I would really prefer to address the "so what?" question up front, of "implications"... :-/ Well, I'll do my best to come up with something I'll be comfortable with..</p> <p>About the conversation between us, you'll notice that I actually used a lot of your observations within the text already. But yes, I would like to have it more conversational... And actually, this part here about the "requirement" for the final chapter is very interesting... Probably not enough to "sublate" though: you will still ask me to provide "recommendations," ahah!</p> <p>Cheers, JF</p>	

From:	Wolff-Michael Roth <mroth@uvic.ca>
To:	Jean-Francois Maheux <maheuxjf@uvic.ca>
Subject:	RE: RE: RE: Last chapter!
<p>Come up with something interesting. If you don't want to do implications, somehow you got to get yourself out of this question, which is one that is likely going to be asked, everyone of my students was asked that even though I don't care asking it. I am just getting you ready for possible questions and for possible amendments that you can anticipate in this manner. Doesn't matter to me, I am just helping you to get in shape and ready and prepared.</p> <p>If you have access to this book: The Reflexive Thesis, by Malcolm Ashmore, it is his</p>	

dissertation with 1 chapter added. Very funny, because it is so self-referential. Chapter 7 is the thesis defense . . . "The Fiction of the Candidate" perhaps you can link what you want to say to the observer in Maturana/Varela, and to the one you already figure in the researching paper, and perhaps in the two first chapters. Starting p.199 is the "edited and annotated transcript of oral examination." I always thought it was so funny. Cheers, Michael

From:	Jean-Francois Maheux <maheuxjf@uvic.ca>
To:	Wolff-Michael Roth <mroth@uvic.ca>
Subject:	RE: RE: RE: RE: Last chapter!
<p>Yes, I know you are asking so that I am prepared to answer!          And I do need to do a good job on that in the chapter first, because the text also needs to be accepted: not just some verbal response!</p> <p>But yes, I'd prefer a "loophole," so to say, to get out of the question. There is something important there. It's a big question. Maybe a bit "too big" for a final chapter, but I don't think this should stop me at this point ;-)</p> <p>I just began reading Ashmore. Love it! I feel so much at home...</p> <p>Cheers,          JF</p>	

## Back Cover

From:	Jean-Francois Maheux <maheuxjf@uvic.ca>
To:	Wolff-Michael Roth <mroth@uvic.ca>
Subject:	RE: RE: RE: RE: Last chapter!
<p>Hi Michael</p> <p>Here I am, with a new version of the chapter!          (Sorry for this long email, but you'll understand why: I want to use it in the chapter to bring/explain some ideas. It's actually already in the document (p. 213), so you can read it there if you prefer... If you want to make comments, you can either email me like we usually do, or "wright" in the text and openly as another voice, which could also be an interesting idea!)</p> <p>So...</p> <p>I had some fun playing with our conversation, and a little bit in the spirit of Ashmore's thesis. Actually, there is so much interesting things there I could use! About <u>summarizing</u>, I like when he says that he doesn't want his conclusion to be a retrospective introduction, and have the 'external' say "I see very little point in you coming here and just repeating your thesis at us" (p. 213). However, I do get the point that despite the dissertation having an abstract, and a section is Chapter 2 (Paths Laid Down in Walking), and the maps everywhere... it can be useful to give again a quick</p>	

overview, at god's eye. So what I did is just to use the abstract again! It's nice in a way because now it makes it part of the text, and reflect the fact that in researching, the abstract generally comes together only at the very end :-)

Now, regarding the "recommendations" stuff... There's a few very nice quotes/ideas I'd like to use:

Loathe as I am to allow, sorry, to trust, to saddle you with this author's task, it is *you* who will now have to sort my wheat from my chaff, collect my findings, and recommend the ways in which my work may fruitfully be built upon by others (p. 196)

...you are invited to make (wright) the text rather than to consume it (p. 197)

There is nothing wrong with a report on the world which admits/shows/celebrates its status as a construction, that is as *not* a report (p. 198)

It seemed to me that to make bold, fact-like assertions [...] would entirely undermine the whole spirit of the project. (At that time I saw the solution to the problem [...] as a simple matter of not making claims. Now I know that it is neither as simple nor as difficult as that!) (p. 211)

When the *form* of the text *is* the argument, such a format would *destroy* the argument. I want my conclusion, in contrast, to advance the argument – to be a climax rather than a collection (p.219)

It's nice to see I am not the first one who relinquishes to dry out implications from my work. Like Ashmore, I did not do it in the chapters, and I don't want to do it here. For me, the *one* point I am making with the dissertation as a whole is that one needs to "walk the walk." The implications, if any, are personal ones: the reader should find them in the mirror, not in my words (that would have been another funny way to go).

At the same time, I do see the "risk" in letting the reader makes his/her own conclusions: (a) fails me as a PhD Candidate; (b) misinterprets my work; (c) do not find any repercussions; and so on. On the other hand, this is largely balanced by the problems that come with actually making recommendations, such as: (a) creating a contradiction with myself and the dissertation; (b) shifting attention from the actuality of my dissertation as a speech-act to its content-sense; (c) contributing in the reproduction of a certain way of doing mathematics educational research I am trying to break with; etc.

Of course, "implications" means that people want to look forward. They ask "so what?" but the real question is more "what's next?" The "what" of "so what" *is* the dissertation, but the "what" of "what's next" is not. And of course, I have a whole lot to say about what's next. For example about *relationality* and what I just found out (and *that* might be the most anticipated "finding" of the dissertation as a whole): that the dissertation is actually *not* a Journey. This is why I changed the title of the last chapter: I am not going back home at the end of a journey. The journey *is* my home, and that of the reader. There is no such place as a home I am going back to. But this is also a comforting observation, precisely because "*there is no place like home*" :-)

For me, this is what many still miss: We are not people in relation, like nodes in a

network. We *are* relations: fluid, dynamical networks re/producing evolving patterns and so on. Relationships are not things-in-themselves (this would be going back to the old epistemology), but they are themselves relational...

But this is the kind of things we really need to talk about, and I mean: to “talk the talk.” Not just mentions, but conversations. Then it takes me back to what you were referring to: The observer/participant in researching. Such conversations are researching researching, or researching in the middle of itself (wink to Ashmore again) and they involve (I should say “implicate!”) individuals in their knowing of knowing. Their knowing acts of knowing, that is what they actually do with/as/when knowing in the day-to-day, moment-to-moment, of researching, teaching and learning in mathematics education...

Alright, I somehow already said too much, more that I thought I would. But “ce qui est dit est dit.” To make my “loop/hole,” I added in the chapter a section called “Back Cover” and included this email. Then comes the “Abstract,” and I also included a “Coda.” As you will see, I made it so that its meaning/function is pretty clear, and similar to what it does in musical sheets: sending the reader/player back to the beginning. I like the idea of connecting it with Bateson (again!) (it’s the chapter “The Last Lecture” of *A Sacred Unity: Further steps to an Ecology of the Mind*, 1991) when he plays with a quote from T. S. Eliot:

The end of all our exploring will be to arrive where we started and know the place for the first time

So this is it!

I hope you’ll have a moment to look and tell me what you think! Because I also want to have it English-proof before sending it to the committee.

Many, many thanks!

Cheers,

JF

PS: Reading Ashmore: what a temptation! What if I would actually do for the defense what he wrote he presumably did? Fiction becomes reality, which is, if I understood him well, just as much fiction, but one that does not recognize itself... In many ways, just as “concluding” doesn’t make sense to me, “defending” is also contradictive. Plus, we all know (and he shows that so well!) it is not really his work the candidate is defending, but himself! Now, since I am in a situation of “self-defense,” I should be granted to move first. Which is also what I’m trying to do in the last chapter... and by saying this here!

Reading this in the chapter, everybody should expect “something” unusual to happen for the oral. We were talking about “continuing the dialogue” (Bakhtin, etc.), and I like that. It is much less antagonistic than a “defense” (from *de-* ‘off’ + *-fendere* ‘to strike’). But then again, I don’t know what I/we will do: I guess a lot is going to depends on their response to this chapter. Hence already making it a conversation. That would be a good start. Dialogue stops when nobody responds...

**Supervisory Committee**

Dr. Wolff-Michael Roth, Co-Supervisor  
(Department of Curriculum and Instruction)

Dr. Jennifer S. Thom, Co-Supervisor  
(Department of Curriculum and Instruction)

Dr. Luis Radford, Outside Member  
(École des sciences de l'éducation, Laurentian University)

**ABSTRACT**

In this dissertation, I offer an epistemological journey in the day-to-day, moment-to-moment of mathematics education. Drawing on enaction and cultural historical activity theory, I examine various episodes from research involving children in second and third grade doing geometry with their regular teacher and a research team using tools from the tradition of interaction and conversation analysis.

My interest is to go beyond interpreting teachers' and students' mathematical activity to explore the question of "How do we know" in mathematics education, including a reflection upon the researcher's own actions. I want to better understand how the actions of researchers, teachers and students intertwine to co-produce mathematics education in its actual form and, from that angle, articulate some of the aspects by which mathematics education becomes a (more) meaningful undertaking for all of us.

In total, I present five studies from a travel journal (first written as book chapters or journal articles) that came to fruition from this journey. The first one looks at how geometrical knowings came into being in a second grade classroom, and articulates the

interdependence of abstract, concrete, cultural and bodily mathematical knowings. The second takes a more critical look at the analysis of classroom episodes from video data to produce such “knowledge” about students’ knowing. The third study examines student-teacher communication. It articulates the irreducible, dynamical nature of mathematical knowing through communicative activity that is always knowing-with another and therefore constitutes an ethical relation. The fourth study takes yet another look at the role of researchers and that of knowledge production to appreciate how, in collecting data, research can create learning opportunities for both teachers and their students. The final study returns to the first one, and presents a more elaborated understanding of what it means to know geometrically from the students’ perspective. Rethinking knowing through relationality with oneself, others, and the material world, it concludes with a reflection on the ethical responsibility that comes with knowing mathematically.

As a whole, the dissertation presents itself like a single (textual) utterance, a “turn taking” in our ongoing conversations about researching, teaching and learning in the field of mathematics education. Running through the studies themselves and the reflections surrounding them, metaphors (such as that of a journey across a landscape of theories, methods, and concrete observations in the day-to-day, moment-to-moment of mathematics education) invite the reader to “walk the walk” of thinking differently about “how we know.” In the last chapter, I call upon the reader to join the conversation by questioning, taking up and accepting, or even rejecting what has been done, hence acknowledging it in/as present, so that the dialogue is furthered.

**Coda**

Is this really the end?
Certainly not: There is always another sentence coming
And another one
And see:

Another one
The only way, I guess is to make a loop
A loophole out of this never-ending story
Which at the same time would be a hole going straight to its center

Because, I finally realize, this was <i>not</i> a Journey
There is no going back, no place like “home”
This <i>is</i> home, this is it!
Looking towards an end

I can only roll up my sleeves, and go one step further
I keep arriving
I keep arriving
In new places in which, it seems to me

It is a becoming, a coming to be	
And this coming is also my doing, my being	•
And this somewhere I am taking within me:	•
I am coming from somewhere	

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