# Topologie Algébrique II: Problem sheet 5

**Problem 1.** Recall from class that for $1 \leq k \leq n$, $V_{k,n}$ is the Stiefel manifold of $k$ frames in $\mathbb{R}^n$, and $G_{k,n}$ is the Grassmannian of $k$-dimensional subspaces of $\mathbb{R}^n$. Show that $V_{k,k} \to V_{k,n} \xrightarrow{\pi} G_{k,n}$ is a fibre bundle, where $\pi$ is a forgetful map. Suggested steps:

(i) Show that $\pi$ is an open mapping.

(ii) Show that $G_{k,n} \cong \big(O(n)/O(n-k) \cdot O'(k)\big)$.

(iii) Recall that $V_{k,n} \cong O(n)/O(n-k)$. We want to show that $\pi$ has a local section at the subspace $\langle e_{n-k+1}, \ldots, e_n \rangle$.

(iv) Let $U$ be the set of $k$ dimensional subspaces $W$ of $\mathbb{R}^n$ such that $W \cap \langle e_1, \ldots, e_n \rangle = \{0\}$ Show that $\pi^{-1}(U)$ is open, and deduce that $U$ is open from (i).

(v) By projecting the last $k$ standard basis vectors onto a subspace $W$ and applying Gram Schmidt, define a local section of $\pi$.

**Problem 2.** Suppose that $X \simeq X'$. Show that the homotopy mapping sets $[X, Y] \cong [X', Y]$ are isomorphic, and also that $[Y, X] \cong [Y, X']$ are isomorphic.

**Problem 3.** Prove the 5 lemma without looking in a book. Given a commutative diagram of abelian groups with exact rows, and the vertical maps as shown (left hand surjective, second and fourth isomorphisms, right hand map injective), prove that the middle vertical map is an isomorphism.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$