Topologie Algébrique II: Problem sheet 8

Let X be a CW complex and let $\Omega_n(X)$ be the group of closed oriented n-dimensional smooth manifolds M together with a map $f\colon M \to X$, with addition by disjoint union, under the equivalence relation that $(M, f) \sim (N, g)$ if there is an $(n+1)$-manifold W with a map $F\colon W \to X$, such that $\partial W = M \sqcup -N$, with inclusion maps $i_M\colon M \to W$ and $i_N\colon N \to W$, satisfying $F \circ i_M = f\colon M \to X$ and $F \circ i_N = g\colon N \to X$. Let $\widetilde{\Omega}_n(X) = \Omega_n(X)/\Omega_n(\mathrm{pt})$. Extend the theory to all compactly generated spaces using a choice of CW approximation functor Γ.

Problem 1.* Show that $\widetilde{\Omega}_*$ is a generalised homology theory. You may use theorems from differential topology, like transversality.

Problem 2. Show that $\Sigma^n X$ is n-connected for every space X.

Problem 3. Let $\{T_n\}$ be a spectrum with T_n an $(n-1)$-connected space that is homotopy equivalent to a CW complex. Show that $E_q(X) = \operatorname{colim} \pi_{q+n}(X \wedge T_n)$ is a generalised homology theory.

Problem 4. Let $\mathbb{X} = \{\Sigma^n X\}_{n \geq 0}$ be a suspension spectrum. Write the associated generalised homology theory by $\widetilde{E}_q(-) = \widetilde{\widetilde{H}}_q(-; \mathbb{X})$. Compute the homology groups $\widetilde{H}_q(Y; \mathbb{X})$ for $q < 0$, for any space Y.

Problem 5. What is $\pi_4^S((S^1)^5)$? You can use that $\pi_q^S(\mathrm{pt})$ is isomorphic to $\mathbb{Z}/2$, $\mathbb{Z}/2$, $\mathbb{Z}/24$, 0, 0 for $q = 1, 2, 3, 4$ and 5 respectively.

Problem 6. Deduce the Mayer Vietoris theorem from the axioms of generalised unreduced homology theories.