A CLASSROOM SITUATION CONFRONTING EXPERIMENTATION AND PROOF IN SOLID GEOMETRY

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We want to bring into the debate a classroom situation submitted to preservice teachers in France and in Quebec, situation in which experimentation should play a central role to solve the problem, without being sufficient to establish the result, a theoretical proof being necessary. Such a situation may lead to a better understanding of how the different phases are linked: exploration, conjectures, argumentation and proof. However, we will witness the difficulties of the students in acknowledging the necessity for a theoretical proof, and how experimentation, in conjunction with some usual classroom contracts about proof, may sometimes contribute to move students away from the relevant reasonings and proving.

1. TWO APPROACHES IN OPPOSITION

As it shows in the Discussion Document of ICMI Study 19, educational research on teaching and learning proof in mathematics is subject to a tension between two approaches in opposition. Very globally, the underlying problem derives from the clash between different forms of communication :

- everyday life communication, through which we use natural language to *convince* our interlocutor, without the quest for truth being necessarily the main concern, but where truth may well be at stake — this form of communication being now referred to as *argumentation* by math-education researchers;
- mathematical communication, where natural language and formal languages are combined, and which relies on validation mechanisms of its own, *formal proof* being the ultimate among them.

1.1. Focusing on the operational feature of proof

Some researchers, such as Duval (1992-93), are convinced that there is a deep gap between argumentation and formal proof (the French 'démonstration') :

> *The development of argumentation, even in its most elaborated forms, does not open up a way towards formal proof. A specific and independent apprenticeship is needed as regards deductive reasoning.* (op. cit., p. 60, our translation)

… and it may go as far as prescribing separation between heuristic tasks and working on proof. Indeed, relatively accurate and sophisticated teaching devices are proposed, where an apprenticeship of the keen deductive structure is targeted : propositional graphs (Duval & Egret, 1989; Tanguay, 2005, 2007), identification of premises (Noirfoilise, 1997-98; Houdebine & al., 2004), resorting to sheets and files (Gaud & Guichard, 1984), etc. Whether their designers adhere to Duval's dichotomic standpoint or not[1], such devices clearly aim at acquiring competences which are particularly drawn on in proof : not using the thesis as an argument, distinguishing an implication from its converse, correctly using a definition, keeping a minimal control on the logical/formal structure of a mathematical writing, etc.

1.2. Focusing on the meaning of proof

At the other end of the spectrum, influenced by the works of Pólya, Mason and Lakatos, stand researchers who estimate that proof should not be the object of an isolated teaching : by laying emphasis on abstract logical mechanisms, independently of the construction of concepts and results to which they are linked, one would reflect a distorted image of the mathematical activity, with proof being at the center, as the goal to achieve, rather than as a tool allowing a better understanding of meanings. The teaching here promulgated is conveyed through activities of exploration, experimentation, search for examples and counterexamples, enunciation of conjectures… It is forecast that the necessity of proving will naturally stem from the process — to validate a conjecture as well as to understand 'why it works' — and that one can expect a relatively smooth transition from argumentation to formal proof (Grenier & Payan, 1998; Grenier, 2001; Mariotti, 2001; Godot & Grenier, 2004...)

1.3. Transposition into the teaching context

The implementation, in the classroom or in the curricula, of the findings of any didactical study almost always goes with alterations, even distortions. The caricatured transposition of the approach described in § 1.1 could take the following form : *its teaching being confined to Euclidean geometry, proof should be produced according to a specific format*[2]*, in isolation from exploring and constructing activities. Statements to be proven are declared to be true ('Show that…') or are quasi-obvious from the figure*. Research has pointed out the pitfalls of this curricular trend. Indeed, according to several studies (Chazan, 1993; Hanna & Jahnke, 1993; Wu, 1996…), from such an approach stems, in students' mind, a strongly ritualistic conception of proving (Harel & Sowder, 1998), proof remaining meaningless and purposeless in students' understanding. Tanguay (2005, §3; 2007) diagnoses this ritualistic scheme as the psychological result of the student being vaguely conscious that he or she remains unable to unravel the terms of the contract : he or she thinks that proof is about truth of the called in propositions, while it is in fact about validity of the deductive chainings.
But on the other hand, Hoyles (1997) warns against what could be side effects from the second approach, against a too drastic shift from the first (§1.1) to the

[1] Designers' motivation may simply derive from the good old pedagogical precept according to which 'difficulties should be tackled one at a time'.
[2] For example, the two columns format in America, the three columns format, or the 'on sait que, … or, … donc' format in France. We reiterate that these are alterations from what has been proposed by research. Duval, for one, insists that there should not be any specific format imposed to the student when he or she is asked to write a proof.

second (§1.2) in the curricula, with social argumentation leaving no room to reasoning and scaffolding genuinely deductive in nature :

> *Students* […] *are deficient in ways not observed before the* [recent UK] *reforms :* [they] *have little sense of mathematics; they think it is about measuring, estimating, induction from individual cases, rather than rational scientific process.* […] *Given that there are so few definitions in the* [new] *curriculum, it would hardly be surprising if students are unable to distinguish premises and to reason from these to any conclusion.* (op. cit., p. 10)

The aim of the present contribution to ICMI 19 is to report on an experimentation which partly support this warning.

2. THE EXPERIMENT

2.1. Didactical hypothesis

Our starting hypothesis is that understanding the process of proof in its entirety requires that students regularly be placed in the situation of experimenting, defining, modelling, formulating conjectures and proving, with formal proof thus appearing as a **requirement** in establishing the truth of the proposed conjectures. Solid geometry, a field where basic properties are not obvious, strikes us as a source of problems in which the conditions mentioned above may be combined. Indeed, the situation proposed here relates to the activities of defining (Phase 1; see § 2.2 below), of exploring via concrete constructions and manipulation (Phase 2), and to the necessity of resorting to proof in order both to validate the constructions done and to ascertain that no others are possible (Phase 2 and Phase 3).

2.2. The situation

The situation was explored in an experiment with students in the third year of a four-year teacher-training programme at UQAM, who are studying to become high school math teachers, and, in a second terrain, with pre-service math teachers in their third year of the *Licence de mathématiques* at UJF. The following three tasks were given to students. *Phase 1 :* Describe and define regular polyhedra. *Phase 2 :* Produce them with specific materials. *Phase 3 :* Prove that the previously established list is valid and complete. The tasks were detailed in a document given to the students. The researchers were both present at the UQAM session, but only one was there at UJF. The students worked in teams of three or four. At UQAM, two teams were filmed. We collected working notes from teams at both universities. More complete reports on the experiment will be available through two articles, still in process of evaluation. We will here focus on findings linked with the particular issue at stake.

3. ANALYSES AND FINDINGS

3.1. The definition phase

The first regularity property which spontaneously came to the fore is the congruence and regularity of the faces of the polyhedron : “The faces are all the same", "It's everywhere the same regular polygon", etc. Depending on teams, other criteria are added : convexity, closure (in the sense that it encloses a finite volume), inscribability in a sphere (one team) or its more fuzzy version : "The more sides it has, the more it looks like a ball". Neither of symmetry, congruence of dihedral angles or equality of degrees[3] is stated. Filmed Team 1 try to find a property about edges, and proposes the formula[4]

Nr of edges = (*Nr of faces* x *Nr of sides per face*)/2,

without realizing that it is true for any polyhedron. It stands out from this phase that students have great difficulty in conceptualizing the dihedral angle[5]. It is more or less surprising, considering that the only 'visibly represented' angles are those between two incident edges : "The angles, there is no need talking about it because, … because it's the polygons that form the angles". It will require the debate episode, when the researcher ask students to decide whether the following polyhedra are regular — the one formed by gluing two tetrahedra, and the star polyhedron (with dihedral angles greater than 180º) formed by gluing square-based pyramids on the faces of a cube — to bring the discussion on and a solution to the issues of equality of degrees and congruence of dihedral angles, dihedral angles being somewhat clarified.

3.2. The construction phase

It should be mentioned from the start that according to the presentation document, students were not to stay confined to construction, but were directed towards arguments relevant to the proving phase, which was to be next :

> *Which geometrical properties (number of edges, of faces, type of faces, angles) should be verified to ensure having a regular polyhedron? For instance, what occurs at a given vertex? Justify your answers. Try to construct as much regular polyhedra as possible…* (Presentation document)

The most striking misconception, shared by several teams (including the two filmed teams), and which explicitly reveals itself in Phase 2, is the following : regular polyhedra constitute an infinite family, with one polyhedron per type of (regular) polygon for the faces, with the number of faces increasing with n, the number of sides of these faces, and a resulting polyhedron closer and closer to the sphere (as n goes to infinity). This conception will bring most of the teams into attempts to construct a polyhedron with hexagons[6]. Filmed Team 1, for instance, constructs quickly the tetrahedron and the cube with Plasticine and woodsticks, and then manages to construct the dodecahedron, despite the instability of the construction. Eventually being supplied with jointed plastic hexagons (*Polydron*), the three teammates assemble about ten of these — all lying flatly on the desktop from the start — and then try to raise the ones on the fringe. They blame the rigidity of the material for not being able to do so and to

[3] The *degree* of a vertex is the number of edges (equivalently, the number of faces) incident to this vertex.
[4] We will see in Phase 2 to what extent this focusing on formulae will get.
[5] Yet all the observed students had already taken a course in Linear Algebra, in which the dihedral angle between two secant planes is defined and computed from the plane equations.
[6] The same phenomenon has been observed from primary-school teachers: see Dias & Durand-Guerrier (2005).

construct effectively the expected polyhedron. They formulate the conjectures according to which the polyhedron with hexagonal faces will have one more 'level' (the French 'étage') than the dodecahedron (1+5+5+1), that it will have 26 faces (1+6+12+6+1) and that more generally, a level should be added each time n is increased by one.

Having constructed the tetrahedron, the cube and the dodecahedron, without its purpose being quite clear, Filmed Team 2 search for a formula which would allow to compute the number of edges knowing the number n of sides for each face. One of the student proposes $n+n(n-2)$, while pointing out on the cube to what corresponds each term of the sum. She notices that the formula does work for the tetrahedron, but not for the dodecahedron. She then proposes $n+n(n-2)+n(n-3)$, and none of the three teammates acknowledges that the formula does not work with the cube! They then assemble some hexagons, trying to understand the decomposition in levels of the hypothesized polyhedron. They formulate the conjecture that it has 20 hexagonal faces (1+6+6+6+1 : three levels plus two 'caps') and that in general, regular polyhedra have two levels when n is odd, and three levels when n is even. Without anymore manipulation done whatsoever, a formula pursuit ensues, bringing the team to propose the following computation : the number of edges is given by $n+n(n-2)+n(n-3)$ and to get the number of faces, one must divide by n (each face has n edges) and multiply by two (each edge touches two faces). The teammates are now giving 16 as the number of faces for the polyhedron with hexagonal faces, without showing annoyance that it does not fit with their previous level decomposition.

3.3. Proving phase and conclusion

None of the teams has been able to produce a satisfactory argumentation that there exists only five regular polyhedra, and that they are indeed the one exhibited in the debate episode of Phase 2. It should be said that the presentation document proposed steps based on the production of graphs (Schlegel's diagrams), and it may have complexified the access to proof. But nevertheless, elementary arguments (the faces around a vertex must not cover 360 degrees of angle or more) were accessible right from Phase 2, but were recognized by only two or three students, who won't be able to use them afterwards.

We still support the hypothesis that the situation contains all relevant elements for building a relationship between experimentation and proof, while discerning the role and status of each. Our concern here is about meaning and purpose allocated to the experimental process by a majority among the students. For them, the quest for regularity and formulae has overshadowed any other form of reasoning or judgement; to the point that, for example, members of Team 2 would admit, without batting an eyelid, two distinct and incompatible formulae (cf. § 3.2) to 'justify' the terms of a sequence of three integers! The prevalence of algebra in the curricula and of formulae associated to proving in class, the difficulties linked to dihedral angles (cf. § 3.1), the notion of cognitive unity proposed by Mariotti (2001), none of these are sufficient in our view to explain such a downswing, to understand why the potentiality offered by the experimentation have contributed so little in the commitment of the students to the proving process. We argue that there is here a matter of investigation for educational research, and it may start by being debated at ICMI 19.

REFERENCES

Chazan, D. (1993). High School Geometry Students' Justification for their Views of Empirical Evidence and Mathematical Proof. *Educational Studies in Mathematics*, 24, 359-387.

Dias, T. & Durand-Guerrier, V. (2005). Expérimenter pour apprendre en mathématiques. *Repères IREM*, 60, 61-78.

Duval, R. (1992-93). Argumenter, démontrer, expliquer : continuité ou rupture cognitive ? *Petit x*, 31, 37-61.

Duval, R. & Egret, M. A. (1989). L'organisation déductive du discours. *Annales de didactique et de sciences cognitives*, 2, 25-40.

Gaud, D. & Guichard, J.-P. (1984). Apprentissage de la démonstration. *Petit x*, 4, 5-25.

Hanna, G. & Jahnke, H. N. (1993). Proof and Application. *Educational Studies in Mathematics*, 24, 421-438.

Harel, G. & Sowder, L. (1998). Students' proof schemes : Results from Exploratory studies. *Research in collegiate maths education III*, 234-283.

Godot, K. & Grenier, D. (2004). Research Situations for teaching : a modelization proposal and examples. Proceedings of the *10th International Congress for Mathematics Education*, ICME 10, Copenhague.

Grenier, D. (2001). Learning proof and modeling. Inventory of fixtures and new problems. Proceedings of the *9th International Congress for Mathematics Education*, ICME 9, Tokyo.

Grenier, D. & Payan, C. (1998). Spécificités de la preuve et de la modélisation en mathématiques discrètes. *Recherches en didactiques des mathématiques*, 18 (1), 59-99.

Houdebine, J., Paugam, A., Mary, M.-N., Alagroute, F. (2004). *La démonstration au collège : quelles tâches ? quels outils ?* IREM de Rennes.

Hoyles, C. (1997). The Curricular Shaping of Students' Approaches to Proof. *For the Learning of Mathematics*, 17, 7-16.

Mariotti, M. A. (2001). La preuve en mathématique. *Canadian Journal of Science, Mathematics and Technology Education*, 1 (4), 437-458.

Noirfoilise, R. (1997-98). Une étude sur le maniement d'énoncés dans une démonstration. *Petit x*, 46, 5-17.

Tanguay, D. (2007). Learning Proof : from Truth towards Validity. Proceedings of the Xth *Conference on Research in Undergraduate Mathematics Education* (RUME), San Diego State University, San Diego, California. On the Web.

Tanguay, D. (2005). Apprentissage de la démonstration et graphes orientés. *Annales de didactique et de sciences cognitives*, 10, 55-93.

Wu, H.-H. (1996). The Role of Euclidean Geometry in High School. *Journal of Mathematical Behavior*, 15, 221-237.